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Velocities of Distant Objects in General Relativity Revisited

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Abstract—We consider two most popular definitions of velocities of remote objects in General Relativity. Our work has two motivations. From a research point of view, we generalize the formula connecting these two velocities in FRW metrics found by Chodorowski to arbitrary synchronous spherically symmetric metrics. From a methodological point of view, our goal is to outline certain counter-intuitive properties of the definitions in question, which would allow us to use them when it is reasonable and to avoid incorrect statements, based on inappropriate use of the intuition.

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1. INTRODUCTION

The problem of interpretation of recession velocities has a long story. A seemingly trivial question often led to mistakes and wrong interpretations, both in scientific and pedagogical literature. A lot of striking examples have been collected in the paper [2], including incorrect statements of such prominent researchers as R. Feynman, W. Rindler, S. Weinberg and others. The greater part of mistakes has been connected with the fact that recession velocities can exceed the speed of light, which is now considered as a well established property following from the nonlocal nature of the velocities of distant objects.

However, the approach of [2, 3] is not unique. In fact, it used the Hubble law to define the velocities. Apart from this definition (we will consider it in detail later), there are other proposals, which lead to totally different properties of the recession velocities. In particular, the approach intensively popularized by Synge in [4] never leads to superluminal velocities. This may be considered as an unambiguous advantage of such a definition, and its supporters usually stress this point [1]. It was also stated that, unlike cosmological recession velocities, this approach (which is also explained later in our paper) can be applied to arbitrary space-times.

We should say, however, that Synge's definition of the velocity has its own features which can be considered as disadvantages (or, at least, counter-intuitive properties). As for the claimed universality, the situation is more interesting. In the recent methodological paper [7] it was shown that almost all "cosmological" concepts (namely, those which do not require spatial homogeneity) can work successfully for static space-times (for example, black hole metrics) if we work in a synchronous coordinate system (a system connected with a body freely falling into a black hole). In particular, the Hubble law and the recession velocities have their analogs in this picture and are not bound to only cosmology.

The goal of the present paper is not to discuss which definition is better. As both are mathematically correct, both of them can be used in calculations. Our goal is to discuss some features of these definitions which can be a source of errors since they look "strange" as compared to the properties of the velocity known from special relativity. In principle, there exists a viewpoint (shared by some researchers in extragalactic astronomy) that it is in general meaningless to think about such immeasurable entities as the distance and velocities of remote galaxies, and we should use only measurable quantities like the redshift. We think, however, that it is impossible to forbid the question of how remote distant galaxies are moving, as well as on their velocities relative to us (this is especially important for students studying general relativity), and it is better to consider correctly different definitions of these entities with their properties (sometime unexpected). We hope that the present paper may help in achieving this goal.

The structure of the paper is as follows: in Section 2 we remind a reader on the definitions of the velocity used in [2] and [1, 4]. In Section 3 we show that these two velocities are connected by rather a simple formula in any spherically symmetric metric. This result generalizes the relation found for FRW metrics in [1]. In Section 4 we write down explicit relations for cosmological and spherically symmetric black hole space-times, showing the counter-intuitive features of the velocities in question (note that for different space-times different properties may be considered to be "counter-intuitive"). In Section 5 we summarize our results and conclusions.

2. TWO DIFFERENT WAYS FOR VELOCITIES OF REMOTE OBJECTS

We begin our consideration of the first definition of the velocity in a cosmological context, where this definition appears naturally. The FRW metric written in the usual form

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2)$$

(where r is the comoving radial coordinate, and $d\Omega$ is the angular element), admits a particular foliation of space-time by the hypersurfaces $t = \mathrm{const}$ such that the corresponding spatial slices are homogeneous. This property is so convenient for this foliation to be used everywhere where possible, often without a particular attention. Note, however, that a realistic observer which moves with respect to the frame of constant r would see an inhomogeneous Universe, as it is for Earth-based astronomers, observing a dipole anisotropy of the Cosmic Microwave Background. Nevertheless, it is much easier to work in the frame with homogeneous spatial slices, passing to other frames only when it is absolutely necessary.

Having this foliation, it is natural to introduce the proper distance to a remote object defined as $d = ar$ (assuming that the observer is located at the origin, $r = 0$) and the corresponding velocity as $v = \dot{d} = \dot{a}r$ for an object being at rest with respect to the frame considered. These two simple formulas give us the Hubble law $v = Hd$, which in the FRW Universe is an exact relation. For other proposals for FRW cosmology see, e.g., [5, 6].

This definition of the velocity looks rather natural, however, we can immediately see some unusual features. Since the Hubble law $v = Hd$ in the FRW metrics is exact, we can apply it to remote objects with arbitrary large d. To justify that, we need to consider a universe without a particle horizon. Such models exist, and in these models the unboundedness of d for objects seen by an observer at $r = 0$ ultimately leads to the unboundedness of the velocity v.

If we consider objects with nonzero peculiar velocities, we can see a further deviation from things familiar from special relativity. Indeed, it is reasonable to define $v = d(ar)/dt = \dot{a}r + \dot{r}a$. This means that if we introduce the recession velocity $v_r = \dot{a}r$, existing due to the nonstationary nature of the metrics, and the peculiar velocity $v_p = \dot{r}a$ existing due to motion of the object with respect to the FRW frame, we get $v = v_p + v_r$, the classical Galilean law, independently of the values of the velocities. This leads to relations like $c/2 + c/2 = c$ (or, even $2c/3 + 2c/3 = 4c/3$!) which can shock any person familiar with special relativity.

On the other hand, recession velocities have a "natural" additivity property. Namely, suppose we have an observer located at $r = 0$, the first emitter located at r_1 and the second emitter located at r_2. It is evident from the definition that the recession velocity of the second emitter v_2 is equal to the recession velocity of the first emitter v_1 plus the recession velocity v_{12} of the second emitter with respect to the first one (since the coordinate r is an additive variable, and the scale factor is fixed as it corresponds to a fixed time). We will see that such a "natural" property does not hold for other definitions of the velocity.

As is shown in [7], the definition considered here can be easily generalized to non-cosmological situations, if we use a synchronous coordinate system. In the absence of homogeneity, it is not in general convenient to put an observer at the origin of the coordinate system, so let the position of an observer be at r_1 and the emitter at r_2. It is known that in a synchronous system the coordinate lines are geodesics, so considering a frame in which local observers have a fixed radial coordinate, we can define

$$d = \int_{r_1}^{r_2} \sqrt{g_{rr}}\, dr \qquad (1)$$

and

$$v = \frac{d}{dt}\int_{r_1}^{r_2} \sqrt{g_{rr}}\, dr = v_{\text{fl}} + v_{\text{loc}}. \qquad (2)$$

The velocity naturally decomposes to a part originating from the fact that the metric is not stationary (an analog of the recession velocity with the Hubble flow),

$$v_{\text{fl}} = \int_{r_1}^{r_2} \frac{d}{dt}\sqrt{g_{rr}}\, dr, \qquad (3)$$

and a part originating from changes of the radial coordinate (an analog of the peculiar velocity in cosmology)

$$v_{\text{loc}} = \sqrt{g_{rr}(r_2)}\, dr_2/dt. \qquad (4)$$

(The above was written for the situation when the observer has a smaller radial coordinate than the emitter. This is natural in cosmology, where the observer is usually considered to be located at the origin of the coordinate system. However, in black hole space-times we can meet the opposite situation where the observer is located on the upper limit r_2, then we define $v_{\text{loc}} = \sqrt{g_{rr}(r_1)}\, dr_1/dt$ and so,

$$v = \frac{d}{dt}\int_{r_1}^{r_2} \sqrt{g_{rr}}\, dr = v_{\text{fl}} - v_{\text{loc}}$$

in order to set $v_{\rm loc}$ to a positive value if it is directed towards increasing r.)

We now turn to another proposal for the velocity, which uses completely different ideas. Intuitively, let us "transfer" the velocity from a distant point to the location of an observer. The parallel transport on a Riemannian manifold is a well-defined mathematical operation. However, we cannot apply it directly to velocities since they are 3-dimensional objects. We can make the parallel transport of 4-vector using the appropriate connections (general relativity in its standard form uses Levi-Civita connections, we use them in the present paper and will later comment another choice), so, to start the procedure, we take the 4-velocity of a distant object and transport it to the observer point. Then, we restore the 3-velocity using the transported 4-velocity and the 4-velocity of the observer.

One property of such a definition is clear: any 3-velocity obtained by this procedure is subluminal (a hypothetical superluminal 3-velocity would correspond to an imaginary 4-velocity vector which cannot be a result of parallel transport of any real 4-velocity vector). However, the procedure is still not fixed completely since for the Levi-Civita connection the result of parallel transport depends on the path. Which path is better to specify? One proposal is to choose a null geodesic between the emitter and the observer. This proposal does not require any additional structures like a particular foliation of space-time. In this sense, it can be applied to any space-time. Moreover, the 3-velocity defined in this way is exactly the velocity which produces *in flat space-time* the same redshift as the observer sees in curved space-time—this rather simple fact can be given to students for an exercise.

(Informal hint: let us express the redshift through the energy ratio of emitted and observed photons, $1+z=(k_\mu U^\mu)_e/(k_\mu U^\mu)_o$. Then transport the emitter values to the point of observation. The scalar product does not change, as for individual meaning of the variables, note that the wave vector at the point of emission $(k_\mu)e$ is transported along null geodesics and thus gives the wave vector at the point of observation $(k_\mu)o$. As for the 4-velocity of the emitter, it gives some transported value $\tilde{U}^\mu$. After that, the standard formula expressing z through the 3-velocity $\tilde{V}$ can be obtained precisely in the same way as in special relativity.)

There are, however, arguments against this choice. Usually in physically interesting situations we assume some foliation by hypersurfaces of constant time. The emitter sent the light at some time t_1 which is earlier than the time when the observer received it t_2. This means that the velocity obtained by parallel transport along the light path has a meaning of an average (in some sense) velocity in between t_1 and t_2 (see [1]). To construct a velocity *at particular time t,* we need to transfer the 4-velocity along the line $t=\text{const}$ —if we consider only radial motion, the line of parallel transport is fully specified. Explicit calculations of such a velocity for a FRW Universe have been performed in [1], where it was shown that it is connected with the Hubble law velocity (which we denoted here as $v_{\rm fl}$) by the simple formula $v=\tanh(v_{\rm fl})$. In the next section we generalize this result to any synchronous frame and nonzero peculiar velocities.

Before making calculations, we would like to point out conceptual differences between these types of velocities. If the Hubble law velocity is approximately constant and is expressed, say, in kilometers per second, this tells us that some distance changes by $v_{\rm fl}$ kilometers during one second (or, very close to this value, if we decide to be more pedantic). As for the velocity defined via parallel transport (regardless of a particular path used), no physical object covers v kilometers per second. The 4-velocity of an object being transported along any line different from the world line of this object loses any connection with it. Strictly speaking, the transported 4-velocity is not a 4-velocity of any physical object. This leads to some counterintuitive features of transported velocities, as we shall see later.

Parallel transport of the 4-velocity vector $U^\alpha = dx^\alpha/d\lambda$ along a curve parametrized by λ is defined by the differential equation

$$\frac{dU^\alpha}{d\lambda}+\Gamma^\alpha{}_{\beta\gamma}U^\beta\frac{dx^\gamma}{d\lambda}=0. \tag{5}$$

The initial conditions are the components of the emitter 4-velocity vector.

For the Levi-Civita connection the Christoffel symbols are:

$$\Gamma^\alpha{}_{\beta\gamma}=\frac{1}{2}g^{\alpha\delta}(\partial_\gamma g_{\beta\delta}+\partial_\beta g_{\gamma\delta}-\partial_\delta g_{\beta\gamma}). \tag{6}$$

We will consider here only radial motion, so $U^2=U^3=0$ and $\theta=\text{const}$, $\phi=\text{const}$.

For parallel transport along the constant time curve $t=\text{const}$, we have the following. Denoting $(x^\mu)=(x^0,x^1,x^2,x^3)$, $x^0=t=\text{const}$, $x^2=\theta=\text{const}$, $x^3=\phi=\text{const}$, the equation (5) takes the form

$$\frac{dU^\alpha}{dx^1}=-\Gamma^\alpha{}_{\beta 1}U^\beta. \tag{7}$$

Let us now consider parallel transport along a null geodesic. Time is increasing along the null geodesic, so we can write $x^0 = x^0(x^1)$ (other coordinates are constant). Then we rewrite (5):

$$dU^\alpha = -\Gamma^\alpha{}_{\beta 0} U^\beta dx^0 - \Gamma^\alpha{}_{\beta 1} U^\beta dx^1$$
$$= -\Gamma^\alpha{}_{\beta 0} \frac{dx^0}{dx^1} U^\beta dx^1 - \Gamma^\alpha{}_{\beta 1} U^\beta dx^1,$$

which gives

$$\frac{dU^\alpha}{dx^1} = -\Gamma^\alpha{}_{\beta 0} \frac{dx^0}{dx^1} U^\beta - \Gamma^\alpha{}_{\beta 1} U^\beta. \quad (8)$$

Using the condition $dS = 0$, we obtain

$$\sqrt{g_{00}}dx^0 = \pm\sqrt{-g_{11}}dx^1. \quad (9)$$

This gives

$$\frac{dU^\alpha}{dx^1} = -\left(\pm\sqrt{-\frac{g_{11}}{g_{00}}}\Gamma^\alpha{}_{\beta 0} + \Gamma^\alpha{}_{\beta 1}\right)U^\beta. \quad (10)$$

We should use "$-$" if the emitter has a larger radial coordinate r than the observer, and "$+$" in the opposite situation.

Now we need to recover the 3-velocities from the transported 4-velocity in a particular observation frame. In the present paper, we will consider only observers which are at rest with respect to the used coordinate frame, so the corresponding tetrad defining the frame is not boosted and not rotated. In our case this relation is given by

$$V = \frac{U^{1'}}{U^{0'}} = \sqrt{-\frac{g_{11}(t_0, r_0)}{g_{00}(t_0, r_0)}}\frac{U^1}{U^0}, \quad (11)$$

where

$$U^{0'} = \sqrt{g_{00}(x_0)}U^0 = \frac{1}{\sqrt{1-V^2}},$$
$$U^{1'} = \sqrt{-g_{11}(x_0)}U^1 = \frac{V}{\sqrt{1-V^2}}. \quad (12)$$

are the tetrad components of the 4-velocity vector.

3. CONNECTING VELOCITIES

We remind the reader that several years ago Chodorowski showed that, in FRW cosmological metrics, the two velocities defined in the previous section are connected by a very simple formula if we use parallel transport along a line with $t = \text{const}$. The goal of this section is to show that this relation is still valid in any spherically symmetric metrics if we use a synchronous coordinate system.

In the spherically symmetric case and pure radial motion, the 4-velocity can be expressed in a parametric form which we will use in this section. Namely, using the condition $U_\mu U^\mu = 1$, which now gives

$$g_{00}(U^0)^2 + g_{11}(U^1)^2 = |g_{00}|(U^0)^2 - |g_{11}|(U^1)^2 = 1,$$

one can write

$$\sqrt{g_{00}}U^0 = \cosh\alpha, \quad \sqrt{|g_{11}|}U^1 = \sinh\alpha. \quad (13)$$

So, the 4-velocity vector depends now on the single parameter α, which changes in some way along the curve of 4-velocity transport; $\sqrt{g_{00}}U^0 = U^{0'}$ and $\sqrt{|g_{11}|}U^1 = U^{1'}$ are the tetrad components of the 4-velocity. Therefore, the 3-velocity in the reference frame in question is $V = \tanh(\alpha) < 1$ and never exceeds the speed of light.

If the emitter is at rest with respect to the coordinate system used, the parameter α at the beginning of the parallel transport path (we will use the subscript $(_*)$ to mark initial values) is $\alpha_* = 0$. For an emitter with a nonzero peculiar velocity $v_{\rm loc}$ we easily get from the definition of α that the initial value is equal to $\alpha_* = \tanh^{-1} v_{\rm loc}$. During the parallel transport, the vector $(U^{0'}, U^{1'})$ undergoes a hyperbolic rotation. This rotation can be written as follows:

$$\begin{pmatrix} U^{0'} \\ U^{1'} \end{pmatrix} = \begin{pmatrix} \cosh(\Delta\alpha) & \sinh(\Delta\alpha) \\ \sinh(\Delta\alpha) & \cosh(\Delta\alpha) \end{pmatrix} \begin{pmatrix} U^{0'}_* \\ U^{1'}_* \end{pmatrix}$$
$$= \begin{pmatrix} \cosh(\Delta\alpha) & \sinh(\Delta\alpha) \\ \sinh(\Delta\alpha) & \cosh(\Delta\alpha) \end{pmatrix} \begin{pmatrix} \cosh(\alpha_*) \\ \sinh(\alpha_*) \end{pmatrix}$$
$$= \begin{pmatrix} \cosh(\alpha_* + \Delta\alpha) \\ \sinh(\alpha_* + \Delta\alpha) \end{pmatrix}. \quad (14)$$

To relate α with the connection coefficients, we need the infinitesimal form of (14). An infinitesimal rotation by the angle $\delta\alpha$ can be written as

$$\begin{pmatrix} U^{0'} \\ U^{1'} \end{pmatrix} + \delta\begin{pmatrix} U^{0'} \\ U^{1'} \end{pmatrix} = \begin{pmatrix} 1 & \delta\alpha \\ \delta\alpha & 1 \end{pmatrix} \begin{pmatrix} U^{0'} \\ U^{1'} \end{pmatrix}$$
$$= \begin{pmatrix} \cosh(\alpha + \delta\alpha) \\ \sinh(\alpha + \delta\alpha) \end{pmatrix}, \quad (15)$$

or

$$\delta U^{0'} = U^{1'}\delta\alpha = U^{1'}\frac{d\alpha}{d\lambda}\delta\lambda,$$
$$\delta U^{1'} = U^{0'}\delta\alpha = U^{0'}\frac{d\alpha}{d\lambda}\delta\lambda. \quad (16)$$

Thus, we get a system of differential equations:

$$\frac{dU^{0'}}{d\lambda} = U^{1'}\frac{d\alpha}{d\lambda} = U^{1'}\frac{\partial\alpha}{\partial t}\frac{dt}{d\lambda} + U^{1'}\frac{\partial\alpha}{\partial r}\frac{dr}{d\lambda},$$
$$\frac{dU^{1'}}{d\lambda} = U^{0'}\frac{d\alpha}{d\lambda} = U^{0'}\frac{\partial\alpha}{\partial t}\frac{dt}{d\lambda} + U^{0'}\frac{\partial\alpha}{\partial r}\frac{dr}{d\lambda}. \quad (17)$$

Considering $U^{0'} = \sqrt{g_{00}}U^0$, $U^{1'} = \sqrt{|g_{11}|}U^1$, we can rewrite it in the form

$$\frac{dU^0}{d\lambda} = -\frac{1}{\sqrt{g_{00}}}\frac{\partial\sqrt{g_{00}}}{\partial t}U^0\frac{dt}{d\lambda} - \frac{1}{\sqrt{g_{00}}}\frac{\partial\sqrt{g_{00}}}{\partial r}U^0\frac{dr}{d\lambda} + \frac{\sqrt{|g_{11}|}}{\sqrt{g_{00}}}\frac{\partial\alpha}{\partial t}U^1\frac{dt}{d\lambda} + \frac{\sqrt{|g_{11}|}}{\sqrt{g_{00}}}\frac{\partial\alpha}{\partial r}U^1\frac{dr}{d\lambda},$$

$$\frac{dU^1}{d\lambda} = -\frac{1}{\sqrt{|g_{11}|}}\frac{\partial\sqrt{|g_{11}|}}{\partial t}U^1\frac{dt}{d\lambda} - \frac{1}{\sqrt{|g_{11}|}}\frac{\partial\sqrt{|g_{11}|}}{\partial r}U^1\frac{dr}{d\lambda} + \frac{\sqrt{g_{00}}}{\sqrt{|g_{11}|}}\frac{\partial\alpha}{\partial t}U^0\frac{dt}{d\lambda} + \frac{\sqrt{g_{00}}}{\sqrt{|g_{11}|}}\frac{\partial\alpha}{\partial r}U^0\frac{dr}{d\lambda}. \quad (18)$$

This system of equations coincides with the system (5), which, for spherical symmetry, simplifies and has the form

$$\frac{dU^0}{d\lambda} = -\Gamma^0{}_{00}U^0\frac{dt}{d\lambda} - \Gamma^0{}_{01}U^0\frac{dr}{d\lambda} - \Gamma^0{}_{10}U^1\frac{dt}{d\lambda} - \Gamma^0{}_{11}U^1\frac{dr}{d\lambda},$$

$$\frac{dU^1}{d\lambda} = -\Gamma^1{}_{10}U^1\frac{dt}{d\lambda} - \Gamma^1{}_{11}U^1\frac{dr}{d\lambda} - \Gamma^1{}_{00}U^0\frac{dt}{d\lambda} - \Gamma^1{}_{01}U^0\frac{dr}{d\lambda}. \quad (19)$$

Equating the same terms with $U^\mu \frac{dx^\nu}{d\lambda}$, we get the following two equations:

$$\frac{\sqrt{|g_{11}|}}{\sqrt{g_{00}}}\frac{\partial\alpha}{\partial t} = -\Gamma^0{}_{10},$$

$$\frac{\sqrt{|g_{11}|}}{\sqrt{g_{00}}}\frac{\partial\alpha}{\partial r} = -\Gamma^0{}_{11}, \quad (20)$$

and other equations are equivalent to these two (this can be easily verified using the fact that the connection we use is the metric connection).

Using these equations, one can integrate α along the curve, find $\alpha(\lambda)$ and then get the corresponding velocity:

$$V = \tanh(\alpha(\lambda)) = \tanh\left(\alpha_* + \int_{\lambda_*}^{\lambda}\frac{d\alpha}{d\lambda}d\lambda\right) = \tanh(\tanh^{-1}(\pm v_{\rm loc}) + \Delta\alpha).$$

At this point we can argue that the choice of the Weitzenböck connection (zero curvature and nonzero torsion, see details, e.g., in [8]) used for the formulation of the Teleparallel Equivalent of General Relativity [9] is not good for describing parallel transport.

We remind the reader that, unlike the Levi-Civita connection which is determined solely in terms of the metric, the definition of the Weitzenböck connection needs an additional structure in the form of a tetrad field. We have

$$\Gamma^\alpha{}_{\beta\gamma} = h_A{}^\alpha \partial_\gamma h^A{}_\beta, \quad (21)$$

where h^A_α is a tetrad field:

$$g_{\alpha\beta} = \eta_{AB}h^A{}_\alpha h^B{}_\beta, \quad (22)$$

$$\eta_{AB} = \mathrm{diag}(1, -1, -1, -1). \quad (23)$$

Usually the tetrad field $h^A{}_\alpha$ is assumed to be diagonal. However, if so, and the metric is diagonal as well, then it can be easily verified that $\Gamma^0{}_{10} = \Gamma^0{}_{11} = \Gamma^1{}_{00} = \Gamma^1{}_{01} = 0$. It means that α and then the 3-velocity always remains constant during the transport. So, it gives us that the recession velocities of distant galaxies are zero.

From this point we return to the Levi-Civita connection and consider only it. If the transport is along the curve $t = \mathrm{const}$, we need only one equation for $d\alpha/dr$. In the particular case of a synchronous metrics we can use it to get a general relation between the velocities defined by two different methods of Section 2.

In a synchronous metric $g_{00} = 1$. We denote for brevity $-g_{11} = g_1(t, r)$.

Using (6), we calculate the Christoffel symbols:

$$\Gamma^0{}_{01} = 0, \quad \Gamma^0{}_{11} = \frac{1}{2}\frac{\partial g_1}{\partial t},$$

$$\Gamma^1{}_{01} = \frac{1}{2g_1}\frac{\partial g_1}{\partial t} \quad \Gamma^1{}_{11} = \frac{1}{2g_1}\frac{\partial g_1}{\partial r}. \quad (24)$$

Choosing in (24) and (20) the equations with $-\Gamma^0{}_{11}$ and equating them, we get that

$$\frac{d\alpha}{dr} = -\frac{1}{2\sqrt{g_1}}\frac{\partial g_1}{\partial t} = -\frac{\partial\sqrt{g_1}}{\partial t}.$$

It the emitter has a larger radial coordinate r than the observer, we have

$$\alpha(r_{\rm obs}) - \alpha(r_*) = \Delta\alpha = -\int_{r_*}^{r_{\rm obs}}\frac{d\sqrt{g_1}}{dt}dr = -\int_{r_2}^{r_1}\frac{d\sqrt{g_1}}{dt}dr = \int_{r_1}^{r_2}\frac{d\sqrt{g_1}}{dt} = v_{\rm fl}. \quad (25)$$

Hence,

$$V = \tanh\left(\tanh^{-1}(v_{\rm loc}) + v_{\rm fl}\right) \quad (26)$$

—it is the velocity of recession from the observer— if the emitter is receding from the center, it is receding from the observer as well. For the particular case of the emitter with zero peculiar velocity, we have a very similar expression, relating these two velocities: $V = \tanh(v_{\text{fl}})$. This formula was obtained in [1] for the Friedman metric. Now we can see that it is valid for any synchronous spherically symmetric metric and that the result can be generalized to nonzero peculiar velocities of the emitter with the modification of this formula given by (26).

And when the emitter has a smaller radial coordinate r than the observer,

$$\alpha(r_{\text{obs}}) - \alpha(r_*) = \Delta\alpha = -\int\limits_{r_*}^{r_{\text{obs}}} \frac{d\sqrt{g_1}}{dt} dr$$

$$= -\int\limits_{r_1}^{r_2} \frac{d\sqrt{g_1}}{dt} = -v_{\text{fl}}, \tag{27}$$

and

$$V = \tanh\left(\tanh^{-1}(v_{\text{loc}}) - v_{\text{fl}}\right). \tag{28}$$

Note that this velocity V is positive if it is directed to larger values of the radial coordinate r and negative in the opposite case. If, however, we want to define the velocity of recession $V_{\text{recession}}$ seen by the observer looking "inside" in the direction of the emitter, we need to attribute a positive sign to the velocity if it is directed inward and a negative sign if it is directed outward. Thus

$$V_{\text{recession}} = -V = \tanh\left(v_{\text{fl}} - \tanh^{-1}(v_{\text{loc}})\right),$$

if the emitter is receding from the center of the coordinate system, it is approaching the observer.

Note, that if $v_{\text{loc}} = 0$, we have the same formula in both cases: $V_{\text{recession}} = \tanh(v_{\text{fl}})$.

Thus, we have obtained that in any spherically symmetric synchronous metrics the recession velocity defined through parallel transport is expressed through a hyperbolic tangent of the velocity defined as a derivative.

4. SOME PARTICULAR EXAMPLES

In this section we consider certain important metrics and show what the results of the preceding section mean for known physical situations.

4.1. Cosmological Metric

We start with FRW cosmology. Since cosmological recession velocities are very well known both from their apologists (see, e.g., [2]) and critics (see, e.g., [1]), we mostly concentrate on the properties of the transported velocity. It seems that this concept is almost totally ignored by adherents of the other proposal, so that, ironically, any critical considerations of this concept are less presented in the methodological literature than the criticisms of the "standard" recession velocities.

The FRW metric has the form

$$ds^2 = dt^2 - a^2(t)\Big(dr^2 + R_0^2 S^2(r/R_0) d\theta^2$$

$$+ R_0^2 S^2(r/R_0)\sin^2\theta d\phi^2\Big), \tag{29}$$

where $S(r) = (\sin r, r, \sinh r)$ for closed, flat and hyperbolic models, respectively.

We start with parallel transport along a $t =$ const radial line. First, consider the transport of a 4-velocity of an object in the Hubble flow (no peculiar velocity). The 4-velocity of the emitter is

$$U^* = (1\ ,\ 0\ ,\ 0\ ,\ 0). \tag{30}$$

The set of equations for the parallel transport has the form

$$\frac{dU^\alpha}{dx^1} = -\Gamma^\alpha{}_{\beta 1} U^\beta. \tag{31}$$

The first and second equations, after substitution of the corresponding Christoffel symbols, give

$$\frac{dU^0}{dx^1} = -a\dot{a}U^1, \quad \frac{dU^1}{dx^1} = -\frac{\dot{a}}{a}U^0. \tag{32}$$

The solution of this system is

$$U = \left(\cosh\left[\frac{\dot{a}}{a}d\right],\ \frac{1}{a}\sinh\left[\frac{\dot{a}}{a}d\right],\ 0\ ,\ 0\right)$$

$$= \Big(\cosh[Hd]\ ,\ \frac{1}{a}\sinh[Hd]\ ,\ 0\ ,\ 0\Big). \tag{33}$$

So that

$$V = \tanh(Hd) = \tanh v_{\text{fl}}, \tag{34}$$

where we use $d = ar$ to denote the physical distance to the emitter. This result was first obtained in [1]. From this formula we immediately see that the additivity, mentioned in the introduction, does not hold for transported velocities. If one source is located at the comoving coordinate r_1, and the second one at r_2, then recession velocity of the second object Hd_2 is equal to the recession velocity of the first object Hd_1 plus the recession velocity of the second object with respect to the first one $H(d_2 - d_1)$. As the transported velocity is connected with the recession velocity by Eq. (36), the additivity of transported velocities is absent simply because $\tanh(v)$ is not a linear function. Moreover, expressing this function in terms of exponentials, it is easy to show that these velocities should satisfy the special-relativistic rule $v = (v_1 + v_2)/(1 + v_1 v_2)$ instead of the simple Galilean rule $v = v_1 + v_2$.

Now we consider an object with nonzero peculiar velocity. The 4-velocity of the object in question is

$$U^* = \left(\frac{1}{\sqrt{1 - v_{\rm loc}^2}} \, , \, \frac{1}{a^*} \frac{v_{\rm loc}}{\sqrt{1 - v_{\rm loc}^2}} \, , \, 0 \, , \, 0 \right). \quad (35)$$

After a parallel transport we get

$$U = \left(\cosh \left[\frac{\dot{a}}{a} d + \tanh^{-1}(v_{\rm loc}) \right], \right.$$
$$\left. \frac{1}{a} \sinh \left[\frac{\dot{a}}{a} d + \tanh^{-1}(v_{\rm loc}) \right] \, , \, 0 \, , \, 0 \right)$$
$$= \left(\cosh[Hd + \tanh^{-1}(v_{\rm loc})], \right.$$
$$\left. \frac{1}{a} \sinh[Hd + \tanh^{-1}(v_{\rm loc})] \, , \, 0 \, , \, 0 \right). \quad (36)$$

For the corresponding 3-velocity we obtain

$$V = \tanh[Hd + \tanh^{-1}(v_{\rm loc})]$$
$$= \tanh[v_{\rm fl} + \tanh^{-1}(v_{\rm loc})], \quad (37)$$

as it should be.

If we subtract the transported Hubble flow velocity from this result, we should get a naive analog of the peculiar velocity. Note, however, that this "peculiar velocity" is not equal to $v_{\rm loc}$, and, moreover, it depends on the distance to the object. Vice versa, a peculiar velocity as an intrinsic property of an emitter is *not* equal to the difference between the transported velocities of the object and that of the Hubble flow at its location. Again, we should use here the velocity addition formula of special relativity.

Now we consider parallel transport along the light cone. Since the light equation of motion reads $dt = -adr$, the system for parallel transport is

$$\frac{dU^0}{dt} = \dot{a} U^1, \quad \frac{dU^1}{dt} = -\frac{\dot{a}}{a} U^1 + \frac{\dot{a}}{a^2} U^0. \quad (38)$$

Starting from

$$U^* = (1 \, , \, 0 \, , \, 0 \, , \, 0), \quad (39)$$

we have, using the equation of motion for light,

$$\frac{dU^0}{da} = U^1, \quad \frac{dU^1}{da} = -\frac{1}{a} U^1 + \frac{1}{a^2} U^0. \quad (40)$$

Solving this system for an object in the Hubble flow, we get

$$U = \left(\frac{a^2 + a_*^2}{2aa_*}, \, \frac{1}{2a_*} - \frac{a_*}{2a^2}, \, 0, \, 0 \right)$$
$$= \left(\cosh \left[\ln \frac{a}{a_*} \right], \, \frac{1}{a} \sinh \left[\ln \frac{a}{a_*} \right] \, , \, 0 \, , \, 0 \right) \quad (41)$$

with the corresponding 3-velocity

$$V = \frac{-a^2 + a_*^2}{a^2 + a_*^2} = \tanh \left[\frac{a}{a_*} \right]. \quad (42)$$

This 3-velocity gives the cosmological redshift coinciding with the relativistic Doppler formula

$$1 + z = \sqrt{\frac{1 + V}{1 - V}} = \frac{a}{a_*}, \quad (43)$$

as it should be for this particular definition of velocity.

For completeness, we also write down the 3-velocity of an object with peculiar motion:

$$V = \tanh[\ln \frac{a}{a_*} + \tanh^{-1} v_{\rm loc}], \quad (44)$$

which gives the redshift

$$1 + z = \sqrt{\frac{1 + V}{1 - V}} = \frac{a}{a_*} \sqrt{\frac{1 + v_{\rm loc}}{1 - v_{\rm loc}}}. \quad (45)$$

4.2. Spherically Symmetric Black Hole Metric

In this subsection we consider static, spherically symmetric black hole metrics. We start to consider this metric in stationary coordinates. This case is not covered by the general result of the preceding section (since it is valid only for synchronous coordinates), however, a stationary coordinate system is the most popular one for black hole description, so we consider it first. As the coordinate system is not synchronous, we have no natural method of define an analog of the Hubble flow. On the contrary, there is no problem in defining transported velocities.

The static spherically symmetric metric in stationary coordinates looks like

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 d\theta^2$$
$$- r^2 \sin^2 \theta d\phi^2. \quad (46)$$

The equations for parallel transport along $t = \text{const}$ are

$$\frac{dU^0}{df} = -\frac{1}{2f} U^0, \quad \frac{dU^1}{df} = \frac{1}{2f} U^1. \quad (47)$$

For an emitter at rest ($r = \text{const}$), the 4-velocity is

$$U^* = \left(\frac{1}{\sqrt{f^*}} \, , \, 0 \, , \, 0 \, , \, 0 \right). \quad (48)$$

If the emitter moves with respect to the stationary frame in a radial direction with the local velocity $v_{\rm loc}$, its 4-velocity is

$$U^* = \left(\frac{1}{\sqrt{f^*}} \frac{1}{\sqrt{1 - v_{\rm loc}^2}}, \sqrt{f^*} \frac{v_{\rm loc}}{\sqrt{1 - v_{\rm loc}^2}}, \, 0, \, 0 \right). \quad (49)$$

Solving the system, we get that in both cases

$$\sqrt{f}U^0 = \sqrt{f^*}U^0_*, \quad \frac{U^1}{\sqrt{f}} = \frac{U^1_*}{\sqrt{f^*}}, \tag{50}$$

and

$$U^{0'} = U^{0'}_* = \text{const}, \quad U^{1'} = U^{1'}_* = \text{const}, \tag{51}$$

which means that the resulting 3-velocity coincides with $v_{\rm loc}$.

This result looks like being trivial. However, if we consider transport along a light line, we get a kind of counter-intuitive result (though not unexpected, as we will soon see). Indeed, since the equation for the light propagation gives us $f dt = \pm dr$ (the upper sign is used if the observer has a larger radial coordinate r than the emitter, and the lower sign in the opposite case), and we get the following equations for parallel transport:

$$\frac{dU^0}{df} = \mp\frac{1}{2f^2}U^1 - \frac{1}{2f}U^0,$$
$$\frac{dU^1}{df} = \mp\frac{1}{2}U^0 + \frac{1}{2f}U^1. \tag{52}$$

Now, the solution for the 4-velocity of an emitter "at rest" is

$$U = \left(\frac{1}{\sqrt{f}}\cosh\left[\ln\sqrt{\frac{f}{f^*}}\right],\right.$$
$$\left.\mp\sqrt{f}\sinh\left[\ln(\sqrt{\frac{f}{f^*}})\right],\ 0,\ 0\right), \tag{53}$$

which gives us the 3-velocity

$$V = \mp\tanh\left[\ln\sqrt{\frac{f}{f^*}}\right]. \tag{54}$$

The velocity of recession from the observer is

$$V_{\rm recession} = \mp V = \tanh\left[\ln\sqrt{\frac{f}{f^*}}\right]$$

(where we should use the lower sign if the emitter has a larger radial coordinate r than the observer, and the upper sign in the opposite case).

We see that after adopting the procedure of parallel transport along a lightlike curve we get a nonzero velocity of an object being at rest with respect to a stationary coordinate system when observed by an observer which is also at rest. Obviously, the proper distance between an observer at rest and an emitter at rest in stationary coordinates does not change with time. However, the nonzero transported 3-velocity in this situation is not unexpected because, as we noted above, this velocity coincides with the velocity corresponding to the observed redshift if interpreted as a standard relativistic Doppler effect. Indeed, it is possible to show that the velocity found induces the Doppler shift equal to

$$1+z = \sqrt{\frac{1\mp V}{1\pm V}} = \sqrt{\frac{1+V_{\rm recession}}{1-V_{\rm recession}}} = \sqrt{\frac{f}{f^*}} \tag{55}$$

(with the same sign convention as above), so the usual gravitational redshift is interpreted as a Doppler shift.

The situation where two rest particles in stationary coordinates (so that nothing changes in time at all!) have a nonzero mutual velocity is a good illustration of the point mentioned in the introduction: when the velocity vector undergoes a parallel transport, it is no longer a velocity vector of some physical object, but it can rather be considered as an abstract mathematical vector.

For completeness, we note that for a particle moving in a radial direction,

$$U^* = \left(\frac{1}{\sqrt{f^*}}\frac{1}{\sqrt{1-v^2_{\rm loc}}},\right.$$
$$\left.\sqrt{f^*}\frac{v_{\rm loc}}{\sqrt{1-v^2_{\rm loc}}},\ 0,\ 0\right), \tag{56}$$

and the resulting 3-velocity is

$$V = \tanh\left[\tanh^{-1}(v_{\rm loc}) \mp \ln\sqrt{\frac{f}{f^*}}\right], \tag{57}$$

with the expected form of the redshift

$$1+z = \sqrt{\frac{1+V_{\rm recession}}{1-V_{\rm recession}}} = \sqrt{\frac{1\mp V}{1\pm V}}$$
$$= \sqrt{\frac{f}{f^*}}\sqrt{\frac{1\mp v_{\rm loc}}{1\pm v_{\rm loc}}}. \tag{58}$$

We now return to synchronous coordinates, so that we will consider particles freely falling into a black hole. The coordinates associated with such particles generalize the well-known Lemâitre coordinates for Schwarzschild space-time and have the line element

$$ds^2 = d\tau^2 - (1-f)d\rho^2 - r^2 d\theta^2$$
$$- r^2\sin^2\theta d\phi^2. \tag{59}$$

It can be derived from the static coordinate after making the coordinate transformations

$$\rho = t + \int\frac{dr}{f\sqrt{1-f}}, \tag{60}$$

$$\tau = t + \int \frac{dr}{f}\sqrt{1-f}. \quad (61)$$

The role of r in the Lemâitre metrics becomes clear if we calculate the proper distance

$$d = \int_{\rho_1}^{\rho_2} d\rho\sqrt{|g_{\rho\rho}|} = \int_{\rho_1}^{\rho_2} d\rho\sqrt{1-f} = r_2 - r_1$$

(here we assume that $r_2 > r_1$).

Thus, for the velocity of a "freely falling flow," an analog of the Hubble flow in which particles with $\rho = \text{const}$ are participating—we can try to take $v = dr/d\tau = -\sqrt{1-f}$. This velocity v reaches the speed of light at the horizon and continues to grow with decreasing r. In a Schwarzschild black hole it diverges at the singularity.

However, in contrast to the FRW cosmology, now the position of an observer is important since there is no spatial homogeneity. If the emitter is located at r_1, and the observer at r_2, then their relative velocity is equal to $|v(r_1) - v(r_2)|$. It is the relative velocity which an observer should attribute to a distant point to measure how the proper distance between him/her and the emitter changes with time. The properties of the relative velocity require special attention to avoid possible confusions. We start with the case of zero peculiar velocities—let both the observer and the emitter have constant ρ. First of all, if the emitter crosses a horizon, its velocity with respect to any observer located at final r is subluminal since $v(r_g) = 1$ and $v(r_2)$ is nonzero and positive. Remember, on the contrary, that any horizon-crossing object has the velocity *with respect to a stationary coordinate system* equal to the speed of light independently of the velocity of an observer from outside. The relative velocity $v(r_1) - v(r_2)$ reaches the speed of light somewhere inside the horizon, depending on the position of the observer r_2. On the other side, if we include peculiar velocities, the overall velocity can be superluminal even for both the observer and the emitter located outside the horizon. For example, the light beam at the point r_1 propagating inward, from the viewpoint of an observer at r_2, has the velocity $c + v(r_1) - v(r_2)$ which is superluminal for any observer with $r_2 > r_1$.

As for the transported velocities, the calculation is straightforward, and the results match with the general results obtained in the previous section. After substituting the appropriate Christoffel symbols, we get the relevant equations for a transfer along $\tau = \text{const}$ in the form

$$\frac{dU^0}{d\rho} = -\frac{1}{2}\frac{df}{d\rho}U^1, \quad \frac{dU^1}{d\rho} = \left(\frac{df}{d\rho}\right)\frac{U^1 - U^0}{2(1-f)}. \quad (62)$$

For the emitter at rest in the Lemâitre frame $U^* = (1\ ,\ 0\ ,\ 0\ ,\ 0)$, the solution is

$$U^0 = \cosh[\sqrt{1-f} - \sqrt{1-f^*}],$$
$$U^1 = \frac{\sinh[\sqrt{1-f} - \sqrt{1-f^*}]}{\sqrt{1-f}}, \quad (63)$$

which gives us the 3-velocity

$$V = \tanh[\sqrt{1-f} - \sqrt{1-f^*}] = \tanh[\mp v_{\text{fl}}]$$

in accordance with (26).

In a similar way, the 4-velocity of an emitter with nonzero peculiar velocity is

$$U^* = \left(\frac{1}{\sqrt{1-v_{\text{loc}}^2}},\right.$$
$$\left.\frac{1}{\sqrt{1-f^*}}\frac{v_{\text{loc}}}{\sqrt{1-v_{\text{loc}}^2}}\ ,\ 0\ ,\ 0\right), \quad (64)$$

and the solution of the system gives

$$U^0 = \cosh[\sqrt{1-f} - \sqrt{1-f^*} + \tanh^{-1}(v_{\text{loc}})],$$
$$U^1 = \frac{\sinh[\sqrt{1-f} - \sqrt{1-f^*} + \tanh^{-1}(v_{\text{loc}})]}{\sqrt{1-f}},$$
$$V = \tanh[\sqrt{1-f} - \sqrt{1-f^*} + \tanh^{-1}(v_{\text{loc}})]$$
$$= \tanh[\mp v_{\text{fl}} + \tanh^{-1}(v_{\text{loc}})], \quad (65)$$

as it should be.

For the transport along the light line we should use $\tau = \tau(\rho)$ and $d\tau = \pm\sqrt{1-f}d\rho$, and since $f(\tau,\rho) = f(\tau(\rho),\rho)$, then

$$\frac{df}{d\rho} = \frac{\partial f}{\partial \rho} \pm \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial \rho} = \frac{\partial f}{\partial \rho}(1 \mp \sqrt{1-f})$$

(where the upper sign is for the case when the observer has a larger radial coordinate).

Now the system for parallel transport takes the form

$$(1 \mp \sqrt{1-f})\frac{dU^0}{df} = -\frac{1}{2}U^1,$$
$$(1 \mp \sqrt{1-f})\frac{dU^1}{df} = \frac{(1 \mp \sqrt{1-f})U^1 - U^0}{2(1-f)}. \quad (66)$$

The resulting 4-velocity for the emitter at rest is

$$U^0 = \cosh\left[\ln\frac{1 \mp \sqrt{1-f}}{1 \mp \sqrt{1-f^*}}\right],$$
$$U^1 = \frac{1}{\sqrt{1-f}}\sinh\left[\mp\ln\frac{1 \mp \sqrt{1-f}}{1 \mp \sqrt{1-f^*}}\right] \quad (67)$$

and the recession velocity is

$$V_{\text{recession}} = \pm V = \tanh\left[\ln\frac{1 \mp \sqrt{1-f}}{1 \mp \sqrt{1-f^*}}\right]. \quad (68)$$

The Doppler shift corresponding to this velocity is

$$1+z=\sqrt{\frac{1\mp V}{1\pm V}}=\frac{1\mp\sqrt{1-f}}{1\mp\sqrt{1-f^*}}. \qquad (69)$$

Note the difference between the redshift expressed through the function f for the observer and emitter at rest with respect to the stationary coordinate system (60) and with respect to the freely falling coordinate system (75).

For completeness, a nonzero peculiar velocity leads to the resulting 4-velocity

$$U^0=\cosh\left[\mp\ln\frac{1\mp\sqrt{1-f}}{1\mp\sqrt{1-f^*}}+\tanh^{-1}(v_{\rm loc})\right],$$

$$U^1=\frac{1}{\sqrt{1-f}}\sinh\left[\mp\ln\frac{1\mp\sqrt{1-f}}{1\mp\sqrt{1-f^*}}+\tanh^{-1}(v_{\rm loc})\right], \qquad (70)$$

to the 3-velocity

$$V=\tanh\left[\mp\ln\frac{1\mp\sqrt{1-f}}{1\mp\sqrt{1-f^*}}+\tanh^{-1}(v_{\rm loc})\right], \qquad (71)$$

and the corresponding redshift

$$1+z=\sqrt{\frac{1\mp V}{1\pm V}}=\frac{1\mp\sqrt{1-f}}{1\mp\sqrt{1-f^*}}\sqrt{\frac{1\mp v_{\rm loc}}{1\pm v_{\rm loc}}}. \qquad (72)$$

5. CONCLUSIONS

In the present paper we have considered two different definitions of the velocities of remote objects in general relativity. Since both are mathematically correct (if the foliation of space-time in question by hypersurfaces of constant time is given), it is possible to use them in appropriate situations without any problems. However, it is necessary to remember that the properties of these velocities may look strange in comparison with "usual" velocities in classical physics and even in special relativity, and using intuition instead of calculations might be dangerous.

The main feature which distinguishes the GR situation from that in special relativity in the first case considered in the present paper—the velocity defined as the derivative of the proper distance to the object with respect to the proper time of the observer—is the possibility for the velocity to be superluminal. This fact has been discussed many times in the cosmological situation, and superluminal cosmological recession velocities are now accepted by the scientific community. In the present paper, we show that this situation is not strictly connected with cosmology, but appears in any synchronous coordinate system. The FRW frame, being synchronous, is the most natural for homogeneous and isotropic cosmology, so not surprising is that this property is well known in a cosmological situation. However, the same picture appears in black hole space-times if we use the Lemâitre synchronous frame to define $t=\text{const}$ hypersurfaces. It is also curious that not only a free fall inside the event horizon can be superluminal, but even the motion of a particle located outside the horizon, but having a nonzero peculiar velocity with respect to the Lemâitre frame and directed inward can be superluminal as well. The same situation in cosmology (when the resulting velocity of a particle which has a big enough peculiar velocity directed outward can exceed the speed of light even if the particle is located within the Hubble sphere) is also possible, though it seems to look less counterintuitive (possibly because the Hubble sphere is obviously observer-dependent in contrast to the black hole horizon). The other "strange" feature of the velocity in question is that it can be subluminal for a particle crossing the event horizon and reach the speed of light somewhere inside the horizon (depending on the observer's position).

The other definition of the velocity discussed in the present paper—the velocity defined by parallel transport of the initial 4-velocity of the emitter using the Levi-Civita connection—is free from such superluminal properties by definition. Since the result of parallel transport depends on the path, the velocities defined by the transport along the line of constant time and along the light geodesic will be different. The former case matches our intuition about the velocity "now." We show that it is connected with the velocity defined as a derivative by a simple formula. As for the latter case, the velocity defined in this way allows us to interpret any redshift in the presence of gravity as a Doppler kinematic shift. This can be counter-intuitive, for example, in stationary black hole metrics, where a stationary emitter has a nonzero "velocity" with respect to a stationary observer.

We hope that our treatment of some "strange" features of the velocities will help in using these both definitions correctly in the appropriate physical situations.

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