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The Fractal Structure of Space Entails Origin of Pauli's Equation[#]

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Abstract—This study links the fractal structure of physical space-time to quantum-mechanical laws. It is shown that primitive distortions of the pregeometric surface, a fractal cell of 3D space, gives birth to a condition eliminating the metric defect while providing "eternal validity" of the exclusive algebras (of real, complex, and quaternion numbers). Written in the physical units typical for the micro-world entities, this condition acquires the precise form of the Pauli equation describing mechanics of the quantum electron with spin.

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1. INTRODUCTION

The recent decade is marked with a mathematical embodiment of Wheeler's ideas of pregeometry [1], an unobserved entity regulating the quantum mechanical laws. Ref. [2] strongly supported the mathematical development of this idea in the framework of three exclusive algebras, demonstrating the existence of an "interior" structure of 3D space (possibly, the physical space) represented by a fractal surface of dimensionality 1/2 having a clear geometric, in fact, pregeometric, image and a strict mathematical description. A closer inspection of this entity gave birth to a new visual model of a complex number, of an original version of the theory of relativity [3]; it was a base for new mathematical algorithms controlling space flight of objects subject to recently revealed GR effects [4]. However, the most impressive adhesion relates the space fractal structures with quantum physics [5].

The equation describing the mechanics of a quantum particle, heuristically introduced by Schrödinger, surprisingly admits a derivation from pure mathematical considerations involving fundamental mathematical structures and matrix tools. In brief, the derivation course is as follows. Non-contradictory, square non-degenerate simple matrices may perfectly represent the units of (at least) three exclusive algebras of real, complex and quaternion numbers. The spectral theorem (see, e.g., [6]) reveals a composite structure of these matrices built as square products of vectors of a single biorthogonal basis. In the simplest case, the basis forms a 2D complex-number-valued surface, fractal (of dimension 1/2) with respect to the space attributed to the algebras' units. Two primitive deformations of the fractal surface (oscillation and stretching) ruin the algebras' basic binary operation (product). Introduction of a normalization integral cures the multiplication, while the stability condition (constancy of the integral with respect to a free parameter) makes the algebras "eternally valid" and gives birth to a propagation vector which can be chosen. In the simplest case, when this vector is the gradient of an oscillation phase, the algebra stability condition fractalizes and, in the micro-world with the physical units, acquires precise the form of the Schrödinger equation. This approach, when logically continued, leads to classical and relativistic mechanics [7].

In this study, we consider an extended form of the propagation vector comprising, apart from the phase gradient, an arbitrary 3D vector field. In Section 2, we analyze the respective algebra stability condition, written in a scalar format. In Section 3, we apply to the obtained equation the Bohm procedure (separation of real and imaginary parts) and recollect the complex-number-valued operator for a fractal (spinor) function. In Section 4, the unitless fractal equation written in the physical micro-world units takes the form of the Pauli equation.

We use the following mathematical objects and notations. The full set of nondegenerate simple unit matrices is

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q}_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

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$$\mathbf{q}_2 = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \mathbf{q}_3 = -i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The vector matrices $\mathbf{q}_n$ form a metric triad in 3D space and obey the product law

$$\mathbf{q}_m\mathbf{q}_n = -\delta_{mn} + \varepsilon_{mnj}\mathbf{q}_j,$$

where δ_{mn} and ε_{mnj} are the Kronecker and Levi-Civita symbols, respectively, and summing in repeated indices is implied. The vectors (covectors) of the respective biorthogonal basis generating the fractal surface are

$$\psi^+ = \begin{pmatrix}0\\ 1\end{pmatrix}, \quad \psi^- = \begin{pmatrix}1\\ 0\end{pmatrix};$$

$$\phi^+ = (0 \quad 1), \quad \phi^- = (1 \quad 0),$$

so that

$$\mathbf{1} = \psi^+\phi^+ + \psi^-\phi^-,$$
$$\mathbf{q}_1 = -i(\psi^+\phi^- + \psi^-\phi^+),$$
$$\mathbf{q}_2 = \psi^+\phi^- - \psi^-\phi^+,$$
$$\mathbf{q}_3 = i(\psi^+\phi^+ - \psi^-\phi^-).$$

Below we omit the parity indicators $\pm$ meaning that both vectors (covectors) fit to the equalities.

2. THE ALGEBRA STABILITY CONDITION AND THE EXTENDED ANSATZ FOR THE PROPAGATOR

We deform the fractal surface twice: (i) we make it oscillate, so that each its vector (covector) pumps over its length from the real sector to the imaginary one, $\psi' = e^{i\alpha}\psi$, $\phi' = e^{-i\alpha}\phi$, (ii) we stretch the oscillating basis

$$\Psi \equiv \sigma e^{i\alpha}\psi \equiv \lambda\psi,$$
$$\Phi \equiv \sigma e^{-i\alpha}\phi = \lambda^*\phi; \quad \sigma \neq 0, \quad \sigma \neq 1, \qquad (1)$$

where the phase and the conformal factor may be real functions of a parameter and the 3D coordinates, $\alpha(\theta, \xi_n)$, $\sigma(\theta, \xi_n)$. In fact, we just multiplied a constant 2D unit vector ψ by a complex function of four real variables $\lambda(\theta, \xi_n)$.

The factor σ introduces a 3D metric defect since the triad loses its unitarity property, $|\sigma^2\mathbf{q}'_k| \neq 1$. To save the algebra, we confine the metric defect to a point of 3D space, $f(\theta) = \int_{V_\xi}\Phi\Psi dV_\xi = 1$. Then the triad $\mathbf{q}''_n = f\mathbf{q}'_n$ restores its property of units of the algebra, its "eternal" stability (with respect to θ) provided by the condition $\partial_\theta\int_{V_\xi}\Phi\Psi dV_\xi = 0$. This stability condition produces the differential continuity-type equation

$$\partial_\theta(\Phi\Psi) + \partial_n(\Phi\Psi k_n) = 0. \qquad (2)$$

As mentioned above, if the free vector k_n ("propagator") is chosen to be the gradient of a phase, then in the micro-world physical units

$$x_n = (\hbar/mc)\xi_n, \quad t = (\hbar/mc^2)\theta \qquad (3)$$

(where m is the electron mass, c is the vacuum speed of light) the fractalized equation becomes the Schrödinger equation for the electron in a potential field.

Here we consider an extended version of the propagator comprising (apart from the phase gradient) an arbitrary 3D vector field,

$$k_n = \partial_n\alpha(\theta, \xi) + A_n(\theta, \xi). \qquad (4)$$

We stress that the phase gradient describes the direction and rate of "propagation" of the 2D vector Ψ (and the triad $\mathbf{q}'_n$), while the vector $A_n(\theta, \xi)$ may unify the characteristics of Ψ and 3D space. We assume the space to be Euclidean, though possessing "hidden" vector properties, a Clifford-type 3D metric tensor well fits to this condition,

$$\delta_{mn} = (\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)/2, \qquad (5)$$

where $\mathbf{p}_n \equiv i\mathbf{q}_n$ are three Pauli-type matrices obeying the multiplication rule

$$\mathbf{p}_m\mathbf{p}_n = \delta_{mn} + i\varepsilon_{mnj}\mathbf{p}_j.$$

With Eqs. (1), (4), (5), the algebra stability condition (2) acquires the form

$$\partial_\theta(\phi\lambda^*\lambda\psi) + \frac{1}{2}(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)$$
$$\times\, \partial_m[\phi\lambda^*\lambda\psi(\partial_n\alpha + A_n)] = 0. \qquad (6)$$

The next purely technical stage is to be exposed in details. Since the 2D vectors (covectors) are constant, the first term in Eq. (6) is

$$\phi\partial_\theta\lambda^*\cdot\lambda\psi + \phi\lambda^*\cdot\partial_\theta\lambda\psi. \qquad (7)$$

The phase gradient in the last term can be represented as

$$\partial_n\alpha = \frac{i}{2}\partial_n(\lambda^*/\lambda) = \frac{i}{2}\left(\frac{\partial_n\lambda^*}{\lambda^*} - \frac{\partial_n\lambda}{\lambda}\right), \qquad (8)$$

so that the square brackets in Eq. (6) are

$$\phi\lambda^*\cdot\lambda\psi\,(\partial_n\alpha + A_n) =$$
$$= \frac{i}{2}(\phi\partial_n\lambda^*\lambda\psi - \phi\lambda^*\cdot\partial_n\lambda\psi) + A_n\phi\lambda^*\lambda\psi, \qquad (9a)$$

and the second term in Eq. (6) is rearranged as

$$\partial_n\Big[\frac{i}{2}(\phi\partial_n\lambda^*\cdot\lambda\psi - \phi\lambda^*\cdot\partial_n\lambda\psi)$$
$$+ \frac{1}{2}(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)\,\phi\lambda^*\cdot\lambda\psi A_n\Big]$$
$$= \frac{i}{2}(\phi\partial_n\partial_n\lambda^*\cdot\lambda\psi - \phi\lambda^*\cdot\partial_n\partial_n\lambda\psi)$$

$$+ A_n\phi\partial_n\lambda^* \cdot \lambda\psi + A_n\phi\lambda^* \cdot \partial_n\lambda\psi$$
$$+ \phi\lambda^*\lambda\psi \cdot \frac{1}{2}(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)\partial_m A_n, \quad (9b)$$

the Clifford metric, meaningful only with asymmetric free-vector's space derivative

$$\frac{1}{2}\left(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m\right)\partial_m A_n$$
$$= \partial_n A_n + \frac{i}{2}\varepsilon_{mnj}\mathbf{p}_j\partial_m A_n + \frac{i}{2}\varepsilon_{nmj}\mathbf{p}_j\partial_m A_n. \quad (9)$$

Now we unite the terms from Eqs. (7), and (9b) in two brackets, one with the (left) common factor $\lambda^*\phi = \Phi$, the other with the (right) factor $\lambda\psi = \Psi$:

$$\Phi \cdot \Big(\partial_\theta - \frac{i}{2}\partial_n\partial_n + A_n\partial_n + \frac{1}{2}\partial_n A_n$$
$$+ \frac{i}{2}A_nA_n - \frac{i}{2}\varepsilon_{mnj}\mathbf{p}_j\partial_m A_n\Big)\Psi$$
$$+ \Big[\Phi\Big(\overleftarrow{\partial}_\theta + \frac{i}{2}\overleftarrow{\partial}_n\overleftarrow{\partial}_n + A_n\overleftarrow{\partial}_n + \frac{1}{2}\overleftarrow{\partial}_n A_n$$
$$- \frac{i}{2}A_nA_n + \frac{i}{2}\varepsilon_{mnj}\mathbf{p}_j\partial_m A_n\Big)\Big] \cdot \Psi = 0. \quad (10)$$

It is necessary to note that in Eq. (11) we put in each bracket a half of the "symmetric" term $\partial_n A_n(\Phi\Psi)/2$, and added the terms $\pm A_nA_n(\Phi\Psi)/2$ in the sum annihilating each other. With these additional terms, we can replace the sum of four operators with a square of one two-term operator

$$-\frac{i}{2}\partial_n\partial_n + A_n\partial_n + \frac{1}{2}\partial_n A_n + \frac{i}{2}A_nA_n$$
$$= \frac{i}{2}(-i\partial_n + A_n)(-i\partial_n + A_n); \quad (11)$$

Taking this into account and endowing Eq. (11) with the factor $-i$, we rewrite it in the form

$$\Phi(D\Psi) + (\Phi\overleftarrow{D}^\dagger)\Psi = 0 \quad (12)$$

with the "covariant derivative" operator defined as

$$D \equiv \partial_\theta + \frac{i}{2}(-i\partial_n + A_n)(-i\partial_n + A_n)$$
$$- \frac{i}{2}\varepsilon_{mnj}\mathbf{p}_j\partial_m A_n, \quad (13)$$

the dagger meaning Hermitian conjugation. Since $\Phi = \Psi^\dagger$, the spinor equation

$$D\Psi = 0 \ \Rightarrow \ (D\Psi)^\dagger = \Phi\overleftarrow{D}^\dagger = 0 \quad (14)$$

converts the scalar equation (13) into an identity as a sufficient condition. To analyze the "necessary-condition solutions," we address to the Bohm procedure.

3. BOHM-TYPE PROCEDURE APPLIED TO THE SCALAR STABILITY CONDITION

Here we, first, apply the Bohm procedure to the scalar equation (13) in order to present a detailed form of action of the operators onto the involved scalar functions. Second, we use the obtained equalities to build a "complex-number" equation for a spinor function; and we expect that the result will confirm the sufficient condition given by Eqs. (15).

The Bohm procedure [8] implies separation of real and imaginary parts in one complex-number equation. To apply it to Eq. (13), we first process the derivatives:

$$\partial_\theta\Psi \equiv \partial_\theta(\sigma e^{i\alpha})\psi = (\partial_\theta\sigma/\sigma + i\partial_\theta\alpha)\Psi,$$
$$\partial_n\Psi = (\partial_n\sigma/\sigma + i\partial_n\alpha)\Psi,$$
$$\partial_n\partial_n\Psi = (\partial_n\partial_n\sigma/\sigma + 2i\partial_n\sigma \cdot \partial_n\alpha/\sigma + i\partial_n\partial_n\alpha$$
$$- \partial_n\alpha \cdot \partial_n\alpha)\Psi,$$
$$\Phi\overleftarrow{\partial}_\theta = (\partial_\theta\Psi)^\dagger, \quad \Phi\overleftarrow{\partial}_n = (\partial_n\Psi)^\dagger,$$
$$\Phi\overleftarrow{\partial}_n\overleftarrow{\partial}_n = (\partial_n\partial_n\Psi)^\dagger. \quad (15)$$

Then we rewrite in an explicit form the first term of Eq. (13) [or (11)] (divided by $\sigma^2 \neq 0$):

$$\Phi(D\Psi)/\sigma^2 = \partial_\theta\sigma/\sigma + i\partial_\theta\alpha - \frac{i}{2}\partial_n\partial_n\sigma/\sigma$$
$$+ \partial_n\sigma \cdot \partial_n\alpha/\sigma + \frac{1}{2}\partial_n\partial_n\alpha + \frac{i}{2}\partial_n\alpha \cdot \partial_n\alpha$$
$$+ A_n\partial_n\sigma/\sigma + iA_n\partial_n\alpha + \frac{1}{2}\partial_n A_n$$
$$+ \frac{i}{2}A_nA_n - \frac{i}{2}\varepsilon_{mnj}(\phi\mathbf{p}_j\psi)\partial_m A_n \quad (17a)$$

and similar expression for the second term

$$(\Phi\overleftarrow{D})\Psi/\sigma^2 = \partial_\theta\sigma/\sigma - i\partial_\theta\alpha + \frac{i}{2}\partial_n\partial_n\sigma/\sigma$$
$$+ \partial_n\sigma/\sigma \cdot \partial_n\alpha + \frac{1}{2}\partial_n\partial_n\alpha - \frac{i}{2}\partial_n\alpha \cdot \partial_n\alpha$$
$$+ A_n\partial_n\sigma/\sigma - iA_n\partial_n\alpha + \frac{1}{2}\partial_n A_n$$
$$- \frac{i}{2}A_nA_n + \frac{i}{2}\varepsilon_{mnj}(\phi\mathbf{p}_j\psi)\partial_m A_n. \quad (17b)$$

Then the real part of Eq. (13), i.e., of the sum of Eqs. (17a) and (17b) (we take one-half of it), is

$$\partial_\theta\sigma/\sigma + \partial_n\sigma \cdot \partial_n\alpha/\sigma + \frac{1}{2}\partial_n\partial_n\alpha$$
$$+ A_n\partial_n\sigma/\sigma + \frac{1}{2}\partial_n A_n = 0, \quad (18)$$

while the imaginary part identically vanishes like $iW - iW = 0$ with the scalar W defined as

$$W = -\partial_\theta\alpha + \frac{1}{2}\partial_n\partial_n\sigma/\sigma - \frac{1}{2}\partial_n\alpha \cdot \partial_n\alpha$$

$$- A_n\partial_n\alpha - \frac{1}{2}A_nA_n + \frac{1}{2}\varepsilon_{mnj}(\phi\mathbf{p}_j\psi)\partial_m A_n. \quad (19)$$

We can regard the term W as an arbitrary function of the parameter θ and the coordinates ξ_n, so that the definition (19) may be represented as an equation:

$$\partial_\theta\alpha - \frac{1}{2}\partial_n\partial_n\sigma/\sigma + \frac{1}{2}\partial_n\alpha\cdot\partial_n\alpha$$
$$+ W + A_n\partial_n\alpha + \frac{1}{2}A_nA_n \pm \frac{1}{2}B = 0, \quad (20)$$

where we denote $\varepsilon_{mnj}\partial_m A_n \equiv B_j$, while projections of the Q-vector $\mathbf{B} \equiv \mathbf{p}_j B_j$ on the direction distinguished by the spinors (in the simplest case) are $\phi\mathbf{B}\psi = \pm B$.

Now we restore the "complex-number" format of the equation and endow it with the spinor form. To do so, we sum Eq. (18) (as a real part) and Eq. (20) (as an imaginary part), return back the matrices, and multiply the result by the spinor $\Psi = \sigma e^{i\alpha}\psi$:

$$\Big(\partial_\theta\sigma/\sigma + \partial_n\sigma\cdot\partial_n\alpha/\sigma + \frac{1}{2}\partial_n\partial_n\alpha + A_n\partial_n\sigma/\sigma$$
$$+ \frac{1}{2}\partial_n A_n + i\partial_\theta\alpha - \frac{i}{2}\partial_n\partial_n\sigma/\sigma$$
$$+ \frac{i}{2}\partial_n\alpha\cdot\partial_n\alpha + iW + iA_n\partial_n\alpha$$
$$+ \frac{i}{2}A_nA_n + \frac{i}{2}B_n\mathbf{p}_n\Big)\sigma e^{i\alpha}\psi = 0; \quad (21)$$

where we use the property of ψ to be an eigenvector of one matrix of the set $\mathbf{p}_n$ with the eigenvalues ± 1. With the help of Eqs. (12), (14), (16), we recollect Eq. (21) in the format "differential & matrix operator acting on a spinor":

$$\Big[- i\partial_\theta + \frac{1}{2}(-i\partial_n + A_n)(-i\partial_n + A_n)$$
$$- W - \frac{1}{2}B_n\mathbf{p}_n\Big]\Psi = 0. \quad (22)$$

Equation (22) evidently correlates with the ansatz (15), the only difference being represented by a free function W (can be zero).

4. THE PAULI EQUATION

We emphasize that Eq. (22) is abstract, i.e., its terms are not measured in any physical units. The equation emerges as a condition providing the existence and stability of an exclusive algebra. We endow it with physical units, a suitable set of equalities linking the mathematical unitless quantities (the abstract 3D coordinates and the free parameter) with micro-world spatial coordinates (length) and time which is suggested in Eq. (3). We apply them to the differential operators

$$\partial_\theta = \frac{\hbar}{mc^2}\partial_t,$$
$$\partial_n \equiv \frac{\partial}{\partial\xi_n} = \frac{\hbar}{mc}\frac{\partial}{\partial x_n} \equiv \frac{\hbar}{mc}\partial_{\tilde{n}}, \quad (23)$$

and relate the arbitrary vector to the magnetic potential $A_n \equiv \frac{q}{mc^2}A_{\tilde{n}}$, so that

$$B_n = \varepsilon_{kmn}\partial_k A_m = \frac{q\hbar}{m^2c^3}B_{\tilde{n}}, \quad (24)$$

where q is the particle's electric charge, $B_{\tilde{n}}$ is the magnetic field strength; we also denote the arbitrary potential energy as $U \equiv mc^2W$. With these notations, Eq. (22) takes the precise form of the equation heuristically suggested by Pauli to describe a non-relativistic electrically charged quantum particle with spin 1/2 (an electron) in an exterior magnetic field,

$$\Big[-i\hbar\partial_t + \frac{1}{2m}\Big(-i\hbar\partial_n + \frac{q}{c}A_n\Big)\Big(-i\hbar\partial_{\tilde{n}} + \frac{q}{c}A_{\tilde{n}}\Big)$$
$$- U - \frac{q\hbar}{2mc}B_{\tilde{n}}\mathbf{p}_n\Big]\Psi = 0. \quad (25)$$

We stress that the particle's magnetic moment (Bohr magneton) in this deduction appears automatically:

$$\mu_{\mathrm{B}} \equiv \frac{q\hbar}{2mc} \cong 0.927\ \times 10^{-20}\ \mathrm{g}^{1/2}\ \mathrm{cm}^{5/2}\ \mathrm{s}^{-1}. \quad (26)$$

Equation (25) previews two polarizations of the electron's spin (since $B_{\tilde{n}}\mathbf{p}_n\Psi = \pm B\Psi$).

5. DISCUSSION

The Pauli equation, a specific law of physics having a complicated mathematical structure, came to Wolfgang Pauli "in a flash of genius." So, its logical derivation from hypercomplex mathematics suggested here in a precise mathematical form (25) once more (and even stronger) confirms the existence of links between the physical world and "non-material" though fundamental mathematical constructions, first noticed in a similar derivation of the simpler Schrödinger equation and of the vector version of relativity theory.

A special attention is deserved by the fact that the mathematical Pauli-type equation (22) is just a definition of the "mathematical potential" W; it was shown that the basic complex-number condition (6) leaves significant only its real part, Eqs. (18), (23), while its imaginary part identically vanishes as a subtraction of two equal "mathematical potentials." Nonetheless, it is the operator determining the "potential" (together with the freely chosen potential function itself) that acts on the fractal function thus

Table 1. Some physical constants

Magnitude	Symbol	Units	Value
Electron mass	m	g	9.1094×10^{-28}
Electron electrical charge	q	$g^{1/2}\,cm^{3/2}\,s^{-1}$	4.8030×10^{-10}
Planck constant	$\hbar$	$g\,cm^2\,s^{-1}$	1.0546×10^{-27}
Fundamental velocity	c	$cm\,s^{-1}$	2.9979×10^{10}
Bohr magneton	μ_B	$g^{1/2}\,cm^{5/2}\,s^{-1}$	9.274×10^{-21}

forming the quantum mechanical equation. A closer look prompts that it is not an odd coincidence. Using Eqs. (23) and (24), we identically rewrite the "imaginary" equation (20) in the physical units,

$$\partial_t(\hbar\alpha) + \frac{1}{2m}\left(\partial_n(\hbar\alpha) + \frac{e}{c}A_{\tilde{n}}\right)\left(\partial_n(\hbar\alpha) + \frac{e}{c}A_{\tilde{n}}\right) + U \pm \mu_B B - \frac{\hbar^2}{2m}\partial_{\tilde{n}}\partial_{\tilde{n}}\sigma/\sigma = 0. \qquad (27)$$

Let $U = U_{ex} + U_{in}$, where the functions U_{ex}, α, $A_{\tilde{n}}$ and $B_{\tilde{n}}$ comparatively slowly change at the lab scale, while U_{in} and σ change very fast. Then Eq. (27) decays into a Hamilton-Jacobi-type "exterior" classical equation (with $S \equiv \alpha\hbar$)

$$\partial_t S + \frac{1}{2m}\left(\partial_n S + \frac{e}{c}A_{\tilde{n}}\right)\left(\partial_n S + \frac{e}{c}A_{\tilde{n}}\right) + U_{ex} \pm \mu_B B = 0 \qquad (28)$$

and a Helmholtz-type "interior" equation [with $R \equiv (2m/\hbar^2)U_{in}$]

$$\partial_{\tilde{n}}\partial_{\tilde{n}}\sigma + R\sigma = 0, \qquad (29)$$

regulating the distribution of the factor σ in a limited volume of physical space. The factor σ remains to be a unitless magnitude, the conservation of its square demanded by the algebra stability condition, in particular, by Eq. (18), rearranged after multiplication with σ^2,

$$\partial_\theta\sigma^2 + \partial_n[\sigma^2(\xi_n\alpha + A_n)] = 0. \qquad (30)$$

If we, for example, endow the factor σ with the sense of a fractal relative mass density, $\sigma \equiv [\rho(t,x)/\rho_{mean}]^{1/2}$ of the particle (ρ_{mean} being an average mass density), then the respective normalizing integral written in physical units

$$\int_{V_\xi} \sigma^2 dV_\xi = \frac{1}{\rho_{mean}V}\int_V \rho(t,x)dV = 1,$$

where $V \equiv [\hbar/(mc)]^3$, determines the particle mass. Of course, it is just a version of its interpretation.

Finally, we attract attention to the fact that the empirical term $\frac{q\hbar}{2mc}B_{\tilde{n}}\mathbf{p}_n$ was earlier deduced theoretically under the assumption of an interaction between the particle's electric charge and the microstructure of the quaternion space with an asymmetric metric tensor [10].

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