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On the Notions of Energy Tensors in Tetrad-Affine Gravity

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Abstract—We are concerned with the precise modalities by which mathematical constructions related to energy tensors can be adapted to a tetrad-affine setting. We show that, for fairly general gauge field theories formulated in that setting, two notions of energy tensor (the canonical tensor and the stress-energy tensor) exactly coincide with no need for tweaking. Moreover, we show how both notions of energy tensor can be naturally extended to include the gravitational field itself, represented by a couple constituted by the tetrad and the spinor connection. Then we examine the on-shell divergences of these tensors in relation to the issue of local energy conservation in the presence of torsion.

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1. INTRODUCTION

In Lagrangian field theory [1–6] one has a precise mathematical construction yielding a "canonical energy-tensor" associated with each field sector. Such tensors are related to conservation laws by generalizations of the classical Noether theorem, which constitutes the basis for physical interpretation. When the considered field theory is formulated over a curved Lorentzian background, then one has the further notion of "stress-energy tensor" [4, 7–9] whose relation with the canonical energy-tensor is known as the "Belinfante-Rosenfeld formula" [2, 10, 11] In various concretely interesting cases the said two notions yield tensors which turn out to be different just by a numerical coefficient and, possibly, by a needed symmetrization.

In particular, the notion of energy tensor for the gravitational field has been variously debated in the literature [12–17]. Recent results [18] suggest that that role should be played by the Ricci tensor. On the other hand, precise covariant constructions [19] show that the Ricci tensor is to be seen as the canonical tensor of the gravitational field.

In this paper we are interested in applying the general formalism of Lagrangian field theory in the context where a gauge theory is coupled with tetrad-affine gravity "— indeed we regard that as the most natural and convenient setting. This also yields a canonical tensor for the gravitational field that, again, turns out to be essentially the Ricci tensor. It should be stressed, however, that we do not aim at a detailed discussion of the possible physical interpretations of the ensuing mathematical notions,[1] which are introduced just as natural extensions of the usual notions.

Tetrad gravity [20–30] has been introduced and studied mainly as a convenient "non-holonomic coordinate" formalism, but it is interesting to note that the tetrad θ acquires a neat geometric meaning if it is viewed as an isomorphism between the tangent bundle $\mathrm{T}M$ of the space-time manifold M and a further vector bundle H over M whose fibers are endowed with a Lorentz metric g—i.e., an $\mathrm{SO}(1,3)$-bundle. Moreover, such H is naturally generated by the spinor bundle needed for the description of Dirac fields, so that it does not actually constitute an *ad hoc* unphysical assumption; this result is especially well expressed in the context of 2-spinor geometry [31–37]. Now θ transforms g into a space-time metric; moreover, a metric connection Γ of H is transformed by θ into a metric space-time connection. Thus the couple (θ,Γ) can be regarded as representing the gravitational field, according to which we may call the "tetrad-affine representation". Note that the space-time structures, in this view, are derived, nonfundamental quantities. Though the space-time metric also determines the Levi-Civita (symmetric) connection, the space-time connection corresponding to Γ a has non-zero torsion, which turns out to interact with spin fields. The torsion is then an unavoidable, but not fundamental field since it can be essentially expressed

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[1] Indeed, a straightforward physical interpretation of the Ricci tensor in terms of energy is problematic as, for example, Schwarzschild space-time has a nonzero gravitational energy while the Ricci tensor vanishes.

as the covariant differential[2] of θ with respect to Γ. Furthermore, we observe that Γ can be essentially regarded as the spinor connection, as shown by Eq. (7).

In general, field theory topics can be most rigorously addressed in the context of a formulation exploiting jet bundle geometry [1, 3, 5, 6, 42–44]. In this presentation, however, we will skip some technicalities of that kind, limiting ourselves to plain coordinate expressions, even though a mathematically exigent reader might regard some statements as those not sufficiently justified.

2. TETRAD-AFFINE GRAVITY

If (e_λ) is an orthonormal frame of H, then the tetrad can be expressed as $\theta = \theta^\lambda_a \mathrm{dx}^a \otimes \mathrm{e}_\lambda$, where the components θ^λ_a have the physical dimension of length. We will use shorthands

$$|\theta| \equiv \det\theta = \tfrac{1}{4!}\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\theta^\lambda_a\theta^\mu_b\theta^\nu_c\theta^\rho_d,$$
$$\breve\theta^a_\lambda \equiv \partial|\theta|/\partial\theta^\lambda_a = \tfrac{1}{3!}\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\theta^\mu_b\theta^\nu_c\theta^\rho_d,$$
$$\breve\theta^{ab}_{\lambda\mu} \equiv \partial\breve\theta^a_\lambda/\partial\theta^\mu_b = \tfrac{1}{2}\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\theta^\nu_c\theta^\rho_d,$$
$$\breve\theta^{abc}_{\lambda\mu\nu} \equiv \partial\breve\theta^{ab}_{\lambda\mu}/\partial\theta^\nu_c = \varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\theta^\rho_d.$$

We observe that the above quantities are well-defined also if θ is degenerate; if θ is invertible, then $(\theta^{-1})^a_\lambda = \breve\theta^a_\lambda/|\theta|$.

We denote the components of the metric and of a connection of H by $g_{\lambda\mu}$, and $\Gamma_{a}{}^{\lambda}{}_{\mu}$, respectively, and the induced space-time quantities by

$$g_{ab} \equiv \theta^\lambda_a\theta^\mu_b g_{\lambda\mu},$$
$$\Gamma_a{}^c{}_b = \theta^c_\lambda(-\partial_a\theta^\lambda_b + \Gamma_a{}^\lambda{}_\mu\theta^\mu_b), \qquad (1)$$

where $(\theta^{-1})^a_\lambda = \theta^a_\lambda \equiv g^{ab}g_{\lambda\mu}\theta^\mu_b$. Then θ can be regarded as a "square root of the metric," and we also get $|\theta| \equiv \det\theta = \sqrt{|\det g|}$. The condition that the tetrad be covariantly constant characterizes a connection of the space-time manifold which turns out to be metric, but does not coincide with the standard space-time connection since it is not symmetric (remark: for the connection coefficients we use the sign convention yielding $\nabla_a \mathrm{dx}^c = \Gamma_a{}^c{}_b \mathrm{dx}^b$). The torsion is expressed as

$$T^c{}_{ab} = \Gamma_b{}^c{}_a - \Gamma_a{}^c{}_b = \theta^c_\lambda(\partial_{[a}\theta^\lambda_{b]} + \theta^\mu_{[a}\Gamma_{b]}{}^\lambda{}_\mu). \qquad (2)$$

[2] The notion of a covariant differential of a vector-valued forms, which has been variously present in the literature for several years, is strictly related to the *Frölicher-Nijenhuis bracket* [5, 38–41]. In this paper we will just write down the needed coordinate expressions.

Locally, we write the Lagrangian density of a field theory as $\ell\mathrm{d}^4\mathrm{x}$, where ℓ is a function of the fields and their first derivatives. For the gravitational field we set

$$\ell_{\mathrm{grav}} = \tfrac{1}{4G}R_{ab}{}^{\lambda\mu}\breve\theta^{ab}_{\lambda\mu} = -\tfrac{1}{2G}R_{ab}{}^{\lambda\mu}\theta^b_\lambda\theta^a_\mu|\theta|, \qquad (3)$$

where[3]

$$R_{ab}{}^{\lambda\mu} \equiv R_{ab}{}^\lambda{}_\nu g^{\nu\mu} = (-\partial_{[a}\Gamma_{b]}{}^\lambda{}_\nu + \Gamma_{[a}{}^\lambda{}_\rho\Gamma_{b]}{}^\rho{}_\nu)g^{\nu\mu}.$$

If θ is non-degenerate, then $R_{ab}{}^{\lambda\mu}\theta^b_\lambda\theta^a_\mu$ coincides with the scalar curvature of the space-time connection, but note that the above Lagrangian density is well-defined also in the degenerate case.

Independent variations of the fields θ^λ_a and $\Gamma_a{}^{\lambda\mu}$ then yield the Euler-Lagrange operator components

$$(\delta\ell_{\mathrm{grav}})^a_\lambda = \tfrac{1}{4G}\breve\theta^{abc}_{\lambda\mu\nu}R_{bc}{}^{\mu\nu} = \tfrac{1}{G}\breve\theta^b_\lambda E_b{}^a, \qquad (4)$$
$$(\delta\ell_{\mathrm{grav}})^a{}_{\lambda\mu} = -\tfrac{1}{4G}T^e{}_{bc}\theta^\nu_e\breve\theta^{abc}_{\lambda\mu\nu}, \qquad (5)$$

where $E_b{}^a$ is the Einstein tensor (not symmetric in this context).

3. GAUGE FIELD THEORIES IN TETRAD-AFFINE GRAVITY

A spin-zero "matter field" in a gauge theory is a section of some vector bundle whose fibers are not "soldered" to space-time. A field with nonzero spin can be seen as a section of a similar bundle tensorialized by a spin bundle; we denote its components by $\phi^{i\alpha}$, where α is the spin-related index (which may represent a sequence of ordinary spinor indices). The adjoint field $\bar\phi_{i\alpha}$ can be regarded as an independent section of the dual bundle.

The matter fields interact with a gauge field $A_a{}^i{}_j$ that is a connection of the 'unsoldered' bundle. Usually A is assumed to preserve some fiber structure and is accordingly valued into the appropriate Lie algebra, so one uses components A^I_a, but we do not need to deal with such a restriction explicitely—it is not difficult to see that the arguments presented here work seamlessly with respect to the needed restriction. The covariant derivative of a matter field has the expression

$$\nabla_a\phi^{i\alpha} = \partial_a\phi^{i\alpha} - A_a{}^i{}_j\phi^{j\alpha} - \omega_a{}^\alpha{}_\beta\phi^{i\beta},$$

where the "spinor connection" $\omega_a{}^\alpha{}_\beta$ is related to $\Gamma_a{}^\lambda{}_\mu$ by a linear relation of the type

$$\omega_a{}^\alpha{}_\beta = G^{\alpha|\mu}_{\beta|\lambda}\Gamma_a{}^\lambda{}_\mu.$$

[3] Here G is Newton's gravitational constant. We use natural units: $\hbar = c = 1$.

The coefficients $G^{\alpha|\mu}_{\beta|\lambda}$ can be expressed as combinations of Kronecker deltas in the case of integer spin, while Dirac matrices are involved for semi-integer spin. In particular, for spin one half we have

$$\omega^{\ \alpha}_{a\beta} = \tfrac{1}{4}\Gamma^{\ \lambda}_{a\mu}(\gamma_\lambda\gamma^\mu)^\alpha_{\ \beta}, \tag{6}$$

which can be inverted as

$$\Gamma^{\ \lambda}_{a\mu} = \tfrac{1}{2}\mathrm{Tr}(\gamma^\lambda\omega_a\gamma_\mu). \tag{7}$$

Thus our variable Γ could be regarded as the spinor connection, namely the gravitational field can be equivalently represented as the couple (θ, ω).

Remark: In this concise exposition, charges and other factors that usually appear in the literature are absorbed into the gauge field itself.

The Klein-Gordon Lagrangian, written in the form

$$\ell_\phi = \tfrac{1}{2|\theta|}g^{\lambda\mu}\theta^a_\lambda\theta^b_\mu\nabla_a\bar\phi_{i\alpha}\nabla_b\phi^{i\alpha} - \tfrac{1}{2}m^2\bar\phi_{i\alpha}\phi^{i\alpha}|\theta|, \tag{8}$$

yields the well-defined density $\ell_\phi\mathrm{d}^4\mathrm{x}$ for any matter field. For a spin-half field, one rather uses

$$\ell_\psi = \Big(\tfrac{\mathrm{i}}{2}(\bar\psi_{\alpha i}\nabla\!\!\!\!/\,\psi^{\alpha i} - \nabla\!\!\!\!/\,\bar\psi_{\alpha i}\psi^{\alpha i}) - m\bar\psi_{\alpha i}\psi^{\alpha i}\Big)|\theta|. \tag{9}$$

Note that the Dirac operator $\nabla\!\!\!\!/ \equiv \gamma^a\nabla_a$ depends on the tetrad that transforms the natural Clifford algebra structure of H (and its representation on the Dirac spinor bundle) into an object defined on $\mathrm{T}M$.

For matter fields of either integer or semi-integer spin greater than one half one may wish to consider an appropriate specialized setting, leading to possible generalizations of the Dirac equation [45–47]. However, issues about the Lagrangian treatment of such a setting suggest that we provisionally confine ourselves to the Lagrangian (8) for all matter fields of spin different from one half.

A convenient handling of gauge fields, analogous to the metric-affine gravity formalism, treats the gauge field A and the tensor field F as independent fields [33]. Indeed, consider the Lagrangian

$$\ell_{\text{gauge}} = -\tfrac{1}{2}\breve\theta^{ab}_{\lambda\mu}(\mathrm{d}[A]A)_{ab\ j}^{\ \ i}F^{\lambda\mu j}_{\ \ \ \ i} + \tfrac{1}{4}F^{\lambda\mu i}_{\ \ \ \ j}F_{\lambda\mu\ i}^{\ \ j}|\theta|, \tag{10}$$

where

$$(\mathrm{d}[A]A)^{\ \ i}_{abj} = \partial_{[a}A^i_{b]j} - A^{\ \ i}_{[ah}A^h_{b]j}$$

is the "covariant exterior differential" [39, 40]of A, coinciding with minus its curvature tensor. Since F is not present in other pieces of the total Lagrangian $\ell_{\text{tot}} \equiv \ell_{\text{grav}} + \ell_{\text{matter}} + \ell_{\text{gauge}}$, with ℓ_{matter} being either ℓ_ϕ or ℓ_ψ, the variation of ℓ_{gauge} with respect to F immediately yields

$$F_{abj}^{\ \ \ \ i} \equiv \theta^\lambda_a\theta^\mu_b F_{\lambda\mu\ j}^{\ \ i} = 2(\mathrm{d}[A]A)_{ab\ j}^{\ \ i}. \tag{11}$$

4. ENERGY TENSORS

In standard Einstein gravity, the general link between a field's Lagrangian and the related stress-energy tensor has nontrivial aspects [4, 48], mainly since one has to allow for the Lagrangian to depend on the derivatives of the metric. In the usual Lagrangians of matter fields this dependence comes from space-time connection coefficients in covariant derivatives, while the situation is somewhat different in the metric-affine approach. In the tetrad-affine approach, the total Lagrangians for all basic cases do not depend on the derivatives of the tetrad (later we will also consider such a possible dependence). Hence the role of the stress-energy tensor for each sector is played by $\mathcal{T}^a_\lambda \equiv \partial\ell/\partial\theta^\lambda_a = (\delta\ell)^a_\lambda$. We obtain

$$(\mathcal{T}_{\text{grav}})^a_\lambda = \tfrac{1}{4G}\breve\theta^{abc}_{\lambda\mu\nu}R_{bc}^{\ \ \mu\nu}, \tag{12}$$

$$(\mathcal{T}_{\text{gauge}})^a_\lambda = F_{\lambda\nu\ j}^{\ \ i}F^{\lambda\mu j}_{\ \ \ \ i}\breve\theta^c_\mu - \tfrac{1}{4}F_{\lambda\mu\ j}^{\ \ i}F^{\lambda\mu j}_{\ \ \ \ i}\breve\theta^c_\nu, \tag{13}$$

$$(\mathcal{T}_\phi)^c_\nu = \tfrac{1}{2|\theta|^2}g^{\lambda\mu}(\breve\theta^a_\lambda\breve\theta^b_\mu\breve\theta^c_\nu - \breve\theta^a_\lambda\breve\theta^b_\nu\breve\theta^c_\mu - \breve\theta^a_\nu\breve\theta^b_\mu\breve\theta^c_\lambda) \times \nabla_a\bar\phi_{\alpha i}\nabla_b\phi^{\alpha i} - \tfrac{1}{2}m^2\bar\phi_{\alpha i}\phi^{\alpha i}\breve\theta^c_\nu, \tag{14}$$

$$(\mathcal{T}_\psi)^c_\nu = \ell_\psi\theta^c_\nu - \tfrac{\mathrm{i}}{2|\theta|}g^{\lambda\mu}\breve\theta^a_\nu\breve\theta^c_\lambda \times (\bar\psi_{\alpha i}\gamma_{\mu\beta}^{\ \ \alpha}\nabla_a\psi^{\beta i} - \nabla_a\bar\psi_{\beta i}\gamma_{\mu\alpha}^{\ \ \beta}\psi^{\alpha i}). \tag{15}$$

Moreover, we consider the canonical energy tensor that for a generic field ϕ^i has the expression

$$\mathcal{U}^a_b = \ell\delta^a_b - \nabla_b\phi^i P^a_i, \quad P^a_i \equiv \partial\ell/\phi^i_{,a}. \tag{16}$$

Note the covariant derivative $\nabla_b\phi^i$ above, in contrast to the ordinary partial derivative $\phi^i_{,b}$ appearing most commonly in the literature. This modification, which is necessary for $\mathcal{U}$ to be geometrically well defined in general, was introduced by Hermann [49]; see also Hehl et al. [24], Eq. (3.10). A precise geometric construction and a discussion of the meaning of this object can be found in the previous work [19, 50].

Briefly, $\mathcal{U}$ relates infinitesimal transformations of the space-time manifold M, represented by vector fields X on M, to currents of the field theory under consideration, that are expressed as $J^a = \mathcal{U}^a_b X^b$. To do that, one needs a way to "lift" a vector field so that it acts on the theory's "configuration bundle;" if the latter is not trivial, then the required construction can be performed by means of a connection. In terms of the coordinate expression of $\mathcal{U}$ this eventually amounts to replacing $\phi^i_{,b}$ with $\nabla_b\phi^i$ in the basic expression.

It is well known that the two notions of energy tensor turn out to be strictly related, though in general they do not coincide [2]. In our present context, we can try a generic comparison between $\mathcal{T}$ and $\mathcal{U}$ by observing that writing $\ell = \tilde{\ell}|\theta|$, and assuming that ℓ is independent of the derivatives of θ, we get

$$\mathcal{T}^a_\lambda = \ell\theta^a_\lambda + \frac{\partial\tilde{\ell}}{\partial\theta^\lambda_a}|\theta|,$$
$$\mathcal{U}^a_\lambda = \theta^b_\lambda \mathcal{U}^a_b = \ell\theta^a_\lambda - \theta^b_\lambda \nabla_b \phi^i P^a_i.$$

Then the two tensors coincide if

$$\frac{\partial\tilde{\ell}}{\partial\theta^\lambda_a} = -\frac{1}{|\theta|}\nabla_b\phi^i P^a_i \theta^b_\lambda.$$

Straightforward computations then show that this situation actually occurs in the basic cases presently under consideration, including the Dirac spinor case. Interestingly, this also holds true for the energy tensors of the gauge and gravitational fields, provided that we use the right notion of "covariant derivative" of such fields. Various arguments [19, 41] clearly indicate that the role of the covariant derivative of a connection is to be taken up by the exterior covariant differential of the connection with respect to itself, that is, minus its curvature tensor. Namely, we insert $\nabla_b A^{\ i}_{cj} \equiv (\mathrm{d}[A]A)_{bc\ j}^{\ \ i}$ into

$$(\mathcal{U}_{\text{gauge}})^a_\lambda = \ell_{\text{gauge}}\theta^a_\lambda - \theta^b_\lambda \nabla_b A^{\ i}_{cj}\frac{\partial \ell_{\text{gauge}}}{\partial(\partial_a A^{\ i}_{cj})},$$

and obtain the stated identity. As for the gravitational field (θ, Γ), since ℓ_{grav} is independent of the derivatives of θ, we get

$$\begin{aligned}(\mathcal{U}_{\text{grav}})^a_\lambda &= \tilde{\ell}_{\text{grav}}\breve{\theta}^a_\lambda - \nabla_b\Gamma_c^{\ \mu\nu}\frac{\partial\tilde{\ell}_{\text{grav}}}{\partial(\partial_a\Gamma_c^{\ \mu\nu})}\breve{\theta}^b_\lambda|\theta|^{-1} \\ &= -\tfrac{1}{2G}R\breve{\theta}^a_\lambda - \tfrac{1}{2G|\theta|}\breve{\theta}^{ac}_{\mu\nu}R_{bc}^{\ \ \mu\nu}\breve{\theta}^b_\lambda \\ &= \tfrac{1}{G}(R_b^{\ \mu}\theta^a_\mu\breve{\theta}^b_\lambda - \tfrac{1}{2}R\breve{\theta}^a_\lambda) = (\mathcal{T}_{\text{grav}})^a_\lambda.\end{aligned}$$

More generally, one may wish to consider a Lagrangian that also depends on the derivatives of the tetrad. Then the question arises if one can generalize the construction of the canonical energy tensor to this case. Without being involved in technical details, we state that two constructions turn out to be legitimate, the difference between them being the way in which the action of a vector field on M is properly lifted. Essentially, both ways eventually lead to an expression of the type $\mathcal{U}^a_b = \ell^a_{\ b} - \mathrm{D}_b\theta^\lambda_c P_{a,c}^{\ \ \lambda}$, where D_b is a suitable differential operator. One construction yields just $\mathrm{D}_b\theta^\lambda_c = \nabla_b\theta^\lambda_c = 0$. More interestingly, the other construction determines $\mathrm{D}_b\theta^\lambda_c$ to be—somewhat similarly to the connection—the covariant differential of θ, that is essentially the torsion. Namely, one gets

$$\mathcal{U}^a_b = \ell\delta^a_{\ c} - P^{a,c}_{\ \ \ \lambda}\theta^\lambda_e T^e_{\ cb}, \quad P^{a,c}_{\ \ \ \lambda} \equiv \partial\ell/\theta^\lambda_{c,a}.$$

For example, one may consider the standard "ghost Lagrangian" that in terms of the tetrad can be written as

$$\ell_{\text{ghost}} \equiv g^{\lambda\mu}\theta^a_\lambda\theta^b_\mu\bar{\chi}_{I,a}\nabla_b\chi^I|\theta| - \tfrac{1}{2\xi}f_I f^I|\theta|,$$
$$f^I \equiv |\theta|^{-1}g^{\lambda\mu}\partial_a(\theta^a_\lambda\theta^b_\mu|\theta|A^I_b).$$

Here χ^I and $\bar{\chi}_I$ are the ghost and anti-ghost fields, ξ is a constant, and the index I denotes components in the appropriate Lie algebra. Then the "gauge-fixing Lagrangian" $\ell_{\text{fix}} \equiv -f_I f^I|\theta|/(2\xi)$ introduces into the total canonical energy tensor, constructed in the above described way, a term which is linear in the torsion. Similarly, the stress-energy tensor gets a term that can be expressed as the "variational derivative" of ℓ_{fix} with respect to θ; in turn, this can be expressed in terms of the torsion, through somewhat intricate computations.

5. FIELD EQUATIONS

Besides Eq. (11), the field equations obtained from variations of ℓ_{tot} with respect to θ, Γ, A, ϕ, and $\bar{\phi}$ yield, respectively, the gravitational equation, the torsion equation, the non-Abelian generalization of the second Maxwell equation, and a generalization of either the Klein-Gordon or Dirac equation.

The gravitational equation is

$$0 = \mathcal{T}_{\text{tot}} \equiv \mathcal{T}_{\text{grav}} + \mathcal{T}_{\text{gauge}} + \mathcal{T}_{\text{matter}}, \tag{17}$$

where $\mathcal{T}_{\text{matter}}$ is either $\mathcal{T}_\phi$ or $\mathcal{T}_\psi$.

The other field equations—in a somewhat concise form—can be written in the KG case as

$$0 = -\tfrac{1}{G}\breve{\theta}^{abc}_{\lambda\mu\nu}T^e_{\ bc}\breve{\theta}^\nu_e + 2g^{ab}G^\alpha_{\beta|\lambda\mu} \times (\bar{\phi}_{\alpha i}\nabla_b\phi^{\beta i} - \nabla_b\bar{\phi}_{\alpha i}\phi^{\beta i}), \tag{18}$$

$$0 = (\mathrm{d}[A]{*F})_{aj}^{\ \ i} + \tfrac{1}{2}|\theta|g^{ab} \times (\bar{\phi}_{\alpha i}\nabla_b\phi^{\alpha j} - \nabla_b\bar{\phi}_{\alpha i}\phi^{\alpha j}), \tag{19}$$

$$0 = (\mathrm{d}[\Gamma\otimes A]{*\nabla\bar{\phi}})_{\alpha i} + m^2\bar{\phi}_{\alpha i}|\theta|, \tag{20}$$

$$0 = (\mathrm{d}[\Gamma\otimes A]{*\nabla\phi})^{\alpha i} + m^2\phi^{\alpha i}|\theta|. \tag{21}$$

Here $*$ stands for the "Hodge isomorphism" of exterior forms,[4] namely,

$$*F^{abj}_{\ \ \ \ i} = g^{ac}g^{bd}|\theta|F_{cd\ j}^{\ \ \ i}, \quad *\nabla\phi^{a\alpha i} = g^{ab}|\theta|\nabla_b\phi^{\alpha i},$$

and $\mathrm{d}[A]$ and $\mathrm{d}[\Gamma\otimes A]$ are the exterior covariant differentials with respect to the connections indicated between brackets. A generalized version of the so-called "replacement principle" states that these differ

[4] Exterior form components ξ^a, ξ^{ab} with higher indices are to be intended relative to frames $i(\partial\mathrm{x}_a)\mathrm{d}^4\mathrm{x}$, $i(\partial\mathrm{x}_a\wedge\partial\mathrm{x}_b)\mathrm{d}^4\mathrm{x}$ etc.

from the usual "covariant divergences" by torsion terms [19]. In fact, we have the identities

$$\nabla_a \xi^{ai} = (\mathrm{d}[K]\xi)^i - T^b{}_{ab}\xi^{ai},$$
$$2\nabla_a \xi^{bai} = (\mathrm{d}[K]\xi)^{bi} - \tfrac{1}{2}\xi^{aci}T^b{}_{ac} - \xi^{bai}T^c{}_{ac},$$

where ξ is a $(4-r)$-form $(r = 1, 2)$ valued in a vector bundle, and K is a connection of that same bundle.

Equation (18) is the torsion equation; equation (19) is the "second Maxwell equation;" Eqs. (20) and (21) are the "Klein-Gordon equations" for $\bar{\phi}$ and ϕ.

In the Dirac case we find the field equations

$$\begin{aligned} 0 &= -\tfrac{1}{G}\breve{\theta}^{abc}_{\lambda\mu\nu}T^e{}_{bc}\theta^\nu_e \\ &\quad + \tfrac{\mathrm{i}}{4}\breve{\theta}^a_\nu\bar{\psi}_{\alpha i}(\gamma_\lambda \wedge \gamma_\mu \wedge \gamma^\nu)^\alpha{}_\beta\psi^{\beta i}, \end{aligned} \tag{22}$$

$$0 = (\mathrm{d}[A]{*}F)^{aj}_{\ i} - \mathrm{i}\breve{\theta}^a_\lambda\bar{\psi}_{\alpha i}\gamma^{\lambda\alpha}{}_\beta\psi^{\beta j}, \tag{23}$$

$$0 = -(\mathrm{i}\nabla\!\!\!\!/\,\bar{\psi}_{\beta j} + m\bar{\psi}_{\beta j} + \tfrac{\mathrm{i}}{2}\bar{\psi}_{\alpha j}\tau_\lambda\gamma^{\lambda\alpha}{}_\beta)|\theta|, \tag{24}$$

$$0 = (\mathrm{i}\nabla\!\!\!\!/\,\psi^{\beta j} - m\psi^{\beta j} + \tfrac{\mathrm{i}}{2}\tau_\lambda\gamma^{\lambda\beta}{}_\alpha\psi^{\alpha j})|\theta|, \tag{25}$$

where $\tau_\lambda \equiv \theta^a_\lambda T_b{}^{ab}$. Equations (24) and (25) are the Dirac equations with torsion.

6. DIVERGENCES

In the standard, torsion-free formulation of general relativity, the stress-energy tensor in the right-hand side of the Einstein equation is divergence-free on-shell (that is, when the field equations are taken into account). This well-known result [4, 48] is a consequence of the naturality of the Lagrangian, and holds, in particular, for gauge theories provided that the stress-energy tensor contains the contributions of the matter field *and* the gauge field [19]. This property is interpreted as local energy conservation.[5]

In the presence of torsion, the situation is more intricate. The gravitational equation $\mathcal{T}_{\mathrm{tot}} = 0$ implies $\nabla_a(\mathcal{T}_{\mathrm{tot}})^a_\lambda = 0$, but the single contributions have a nonvanishing divergence. In particular, we remark that the "Einstein tensor" appearing in Eqs. (4) and (12) is not divergence-free; actually,

$$\nabla_a(\mathcal{T}_{\mathrm{grav}})^a_\lambda = \tfrac{1}{G}\breve{\theta}^c_\lambda(T^b{}_{ca}R^a_b - \tfrac{1}{2}T^b{}_{ad}R_{bc}{}^{ad}).$$

Hence we expect that the on-shell divergence of $\mathcal{T}_{\mathrm{gauge}} + \mathcal{T}_{\mathrm{matter}}$ depends on the torsion linearly. Indeed, this can be checked, by not-so-short computations. The vanishing of $\nabla_a(\mathcal{T}_{\mathrm{tot}})^a_\lambda$ expressed in terms of the torsion can be regarded as an "integrability condition" for the gravitational equation. In the KG case we obtain

$$\begin{aligned} \nabla_a(\mathcal{T}_{\mathrm{tot}})^a_\lambda &= \tfrac{1}{G}\breve{\theta}^c_\lambda R_{\{a}{}^b{}T^a{}_{c\}b} + \breve{\theta}^c_\lambda F_{ca}{}^i_j T^{\{e}{}_{be}F^{a\}bj}{}_i \\ &\quad + \tfrac{1}{2}g^{ae}\breve{\theta}^{\{c}_\lambda T^{b\}}{}_{ca}(\nabla_e\bar{\phi}_{\alpha i}\nabla_b\phi^{\alpha i} + \nabla_b\bar{\phi}_{\alpha i}\nabla_e\phi^{\alpha i}). \end{aligned}$$

In the Dirac case,

$$\begin{aligned} \nabla_a(\mathcal{T}_{\mathrm{tot}})^a_\lambda &= \tfrac{1}{G}\breve{\theta}^c_\lambda(T^a{}_{cb}R^a_b - \tfrac{1}{2}T^a{}_{cb}R_{ac}{}^{bd}) \\ &\quad + \breve{\theta}^c_\lambda F_{ca}{}^i_j(T^a{}_{eb}F^{ebj}{}_i - F^{abj}{}_i\tau_b) \\ &\quad + \tfrac{\mathrm{i}}{2}\breve{\theta}^c_\lambda\big[\tau_\lambda\gamma^{\lambda\alpha}{}_\beta(\bar{\psi}_{\alpha i}\nabla_c\psi^{\beta i} - \nabla_c\bar{\psi}_{\alpha i}\psi^{\beta i}) \\ &\quad + \bar{\psi}_{\alpha i}(\gamma^b R_{ab} + R_{ab}\gamma^b)^\alpha{}_\beta\psi^{\beta i}\big]. \end{aligned}$$

The last term in the above equation can be further elaborated. By Clifford algebra we get

$$\gamma^b R_{ab} + R_{ab}\gamma^b = -\tfrac{1}{3}R_{a[bcd]}\gamma^b\gamma^c\gamma^d,$$

and $R_{a[bcd]}$, vanishing in the torsion-free situation, can be expressed in terms of the exterior covariant differential $\mathrm{d}[\Gamma]T$, which is essentially the right-hand side of the first Bianchi equation with torsion.

7. CONCLUSIONS

The offered results support the view that the tetrad-affine representation of gravity is natural and convenient in various respects. In a gauge field theory coupled to gravity there is essentially one energy tensor for each sector. The total energy-tensor is divergence-free, while the single contributions are not—on account of the torsion. The torsion itself is unavoidable in this setting, but it should be regarded as a "by-product" rather than a fundamental, independent field.

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[5] As previously observed, in this paper we are not involved with detailed discussions about physical interpretations of the presented mathematical notions.

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