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Some Bulk-Viscous Solutions in a First-Order Theory¶

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Abstract—We first motivate the study of viscosity in cosmology. Whilst most studies assume that the universe is filled with a perfect fluid, viscosity is expected to play a role, at least during some stages of the evolution of the Universe. There are several theories of viscosity. Eckart's first-order theory was found to permit superluminal signals, and equilibrium states were found to be unstable. To solve these problems, the Israel-Stewart second-order theory was proposed. More recently, a relatively new first-order theory has appeared, which is claimed to also solve these problems. We briefly review this first-order theory and present the basic field equations. Then we attempt to find homogeneous and isotropic solutions in the theory. It is noted that there do not exist stiff matter (pressure = energy density) solutions in the theory, in contrast to other theories. We then find power-law solutions without a cosmological term. Surprisingly, there do not exist simple exponential solutions, again in contrast to other theories. Finally, we present a solution with a cosmological term and make some concluding remarks.

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1. INTRODUCTION

Dissipative phenomena are expected to play a role at least during some stages of the evolution of the Universe. Misner [1] studied neutrino decoupling during the radiation era in homogeneous anisotropic models. In a large class of these models, he found that the initial anisotropy was damped out as the universe evolved to its presently observed isotropic state. Stewart [2] and Doroshkevich et al. [3, 4] showed that Misner had taken the initial anisotropy to be small. Collins and Stewart [5] showed that with arbitrary initial conditions, the anisotropy could be arbitrarily large today with shear viscosity alone. However, these studies were largely concerned with shear viscosity. We shall be concerned only with bulk viscosity, which is the only dissipative process that can arise in FLRW (Friedmann-Lemaıtre-Robertson-Walker) models.

The present-day features of the Universe are well described by the ΛCDM model filled with a perfect fluid. A perfect fluid is an idealization of an imperfect fluid, taking into account various dissipative processes, especially in the early universe. Dissipative processes could still become significant in the future. The Universe has currently a very high photon to baryon ratio, which is difficult to explain. Bulk viscosity can generate the right amount of entropy [6]. During the GUT phase transition ($T \sim 10^{15}$ GeV), gauge bosons acquire mass. The mixture of nonrelativistic and ultrarelativistic particles can give rise to bulk viscosity which can drive inflation.

In general relativity without viscosity, there exists a set of energy conditions that the fluid must obey. They lead to the initial big-bang singularity. Bulk viscosity violates the energy conditions, and hence the initial singularity can be avoided [7]. This is not a generic feature if anisotropy is introduced [8]. Bulk viscosity can lead to a better understanding of the initial singularity itself [9].

Bulk viscosity can provide a phenomenological description of particle creation in a strong gravitational field. Turok [10] showed that after the Planck time a rapid rhythm of string creation by strong quantum fields leads to an exponential expansion of the universe. The production of strings compensates its dilution by expansion. From the equation $\rho = 3H^2$ (in a flat FLRW model), it follows that $\rho =$ const

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implies $H = \text{const} \Rightarrow$ exponential expansion. Barrow [11] showed that this approach could be interpreted phenomenologically in terms of a bulk-viscous model.

Other reasons to study viscosity include the formation of galaxies [5], photon decoupling during the recombination era, interaction of dark energy with other matter, and as a candidate for dark energy itself. For philosophical and aesthetic reasons, a spatially closed FLRW model is appealing. Before the singularity theorems of Hawking and Penrose, an infinite series of oscillations was supposed to solve the problem of the creation/beginning of the universe. Bulk viscosity can give rise to an infinite series of growing oscillations. Such models involve quantum gravity, time asymmetry and thermodynamics. The actual transition from a big crunch to a big bang involves quantum processes. The universe is currently is undergoing acceleration. A natural explanation for this is the cosmological constant. However, this leads to the cosmological constant/fine tuning problem. There are several alternatives, amongst them bulk viscosity, a decaying cosmological parameter and exotic fluids, e.g., Chaplygin gas. Bulk viscosity can also model a decaying cosmological parameter as well as a Chaplygin gas.

There are several theories of viscosity. In the full nonlinear causal theory (NTIS), the pressure is modified by [12]:

$$\bar{p} = p + \Pi, \tag{1}$$

where $\bar{p}$ is the total pressure, p the equilibrium pressure, and Π the viscosity factor given by

$$\tau\dot{\Pi} = -3\zeta H - \Pi\left(1 + \Pi\frac{\tau_*}{\zeta}\right)^{-1} - \epsilon\frac{1}{2}\Pi\tau\left[3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T}\right]. \tag{2}$$

In this equation τ is the relaxation time for linear effects, ζ is the coefficient of bulk viscosity, T is th temperature, and τ_* is the characteristic time for nonlinear effects. The linear full causal Israel-Stewart theory (FIS) [13] is recovered for $\tau_* = 0$. If, in addition, $\epsilon = 0$, the truncated Israel-Stewart theory (TIS) [14] is obtained, and finally if, in addition, $\tau = 0$, then the Eckart theory [15] is obtained.

2. DISCONZI THEORY

A new first-order formulation of bulk viscosity was recently given by Disconzi et al. [16, 17]. It is claimed that it is causal and that the equilibrium states are stable.

The field equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}, \tag{3}$$

where R_{ab} is the Ricci tensor, R the Ricci scalar, g_{ab} the metric tensor, Λ the cosmological term (which need not necessarily be constant), and the energy-momentum tensor T_{ab} is given by

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab} - \zeta(g_{ab} + u_a u_b)\nabla_d C^d, \tag{4}$$

where ρ is the energy density, p the pressure, u_a the 4-velocity of the fluid and ζ the coefficient of bulk viscosity. The quantity C^a is the dynamic velocity of the fluid defined by

$$C_a = Fu_a, \tag{5}$$

and F is the specific enthalpy of the fluid given by

$$F = (\rho + p)/\mu, \tag{6}$$

where μ is the rest mass density of the fluid, conserved along the fluid flow lines:

$$\nabla_a(\mu u^a) = 0. \tag{7}$$

3. FIELD EQUATIONS

The FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{8}$$

where a is the scale factor, and $k = 0, +1, -1$ corresponding to flat, closed, and open models, respectively. Now we get

$$\nabla_a u^a = 3\dot{a}/a, \tag{9}$$

from which we find that

$$\nabla_a C^a = \dot{F} + 3F\dot{a}/a. \tag{10}$$

From the previous equations, we get the following Raychaudhuri-type equation:

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p - 3\zeta\dot{F} - 9\zeta F\frac{\dot{a}}{a} - 2\Lambda\right), \tag{11}$$

where $H = \dot{a}/a$ is the Hubble parameter. In addition, we also get the modified energy conservation equation

$$\dot{\rho} + 3(\rho + p)H - 3\zeta(\dot{F} + 3FH)H + \dot{\Lambda} = 0. \tag{12}$$

Note that we have allowed for the possibility of a variable cosmological parameter. For comparison with other theories, we note that the modified energy conservation equation (12) can also be written, for $\Lambda = 0$, as

$$\dot{\rho} + 3(\rho + \bar{p})H = 0, \tag{13}$$

where the total pressure $\bar{p}$ is given by Eq. (1), and the viscous pressure Π is:

$$\Pi = -3\zeta(\dot{F} + 3FH)H. \tag{14}$$

From Eqs. (11) and (12) we can derive a Friedmann-type equation

$$3H^2 + \frac{3k}{a^2} = \rho + \Lambda. \tag{15}$$

We shall mostly assume a linear equation of state,

$$p = \omega\rho, \tag{16}$$

where ω is not necessarily constant. Equations (11) and (12) then become, respectively,

$$\dot{H} + H^2 = \ddot{a}/a$$
$$= -\frac{1}{6}\left((3\omega + 1)\rho - 3\zeta\dot{F} - 9\zeta FH - 2\Lambda\right), \tag{17}$$
$$\dot{\rho} + 3(\omega{+}1)\rho H - 3\zeta(\dot{F} + 3FH)H + \dot{\Lambda} = 0. \tag{18}$$

Equations (15), (17), (18) are the basic equations that we will use for our analysis. The function F is given by

$$F = \frac{(1+\omega)\rho a^3}{\mu_0}. \tag{19}$$

4. SOLUTIONS

Firstly, we note that Disconzi et al. [16] pointed out that it is impossible to have any stiff matter solutions ($p = \rho$) in this theory. Lichnerowicz [18] has shown that the condition $p = \rho$ leads to $\nabla_a C^a = 0$. From Eq. (10) we then see that $\dot{F} + 3F\dot{a}/a = 0$. Hence, from Eq. (14), we see that the bulk viscous pressure $\Pi = 0$. Hence, it is impossible to have any stiff matter solution with bulk viscosity in this theory, in stark contrast to the Eckart, TIS, FIS, and NTIS theories.

We now present some simple solutions. It is possible to find some more general solutions than those presented here using Mathematica or Maple, but these solutions are very complicated and not very useful to analyze. As is the usual practice, we take the viscosity coefficient to have the form $\zeta = \zeta_o\rho^\alpha$, where ζ_o and α are constants.

4.1. Power-Law Solution

- $k = 0$, $\Lambda = 0$ (we shall consider $\Lambda \neq 0$ later),
- $\omega = \text{const}$.

The equations admit the following solution:

$$a = a_o t^{2/[3(\omega+1)]}, \tag{20}$$
$$\rho = \rho_o/t^2, \tag{21}$$
$$F = F_o t^{2/(\omega+1)}, \tag{22}$$
$$\Pi = \Pi_o t^{-2\omega/(\omega+1)}, \tag{23}$$
$$\omega_{\text{eff}} \equiv \frac{\bar{p}}{\rho} = \omega + \frac{\Pi_o}{\rho_o} + t^{2/(\omega+1)}, \tag{24}$$
$$q = (3\omega + 1)/2, \tag{25}$$

where $\omega_{\text{eff}} \equiv \bar{p}/\rho$ gives the effective equation of state.

4.2. Exponential Solution

In the Eckart theory, it is well known that a simple exponential solution of the type [11]

$$a = a_o e^{H_o t}, \tag{26}$$

where $a_o, H_o = \text{const}$, exists for all values of $\omega \neq -1$. In fact, such a simple solution exists in the TIS [20], FIS [6], and NFIS [12] as well.

Let us find out if such a solution exists in this formulation. We consider:

- $k = 0$, $\Lambda = 0$ (we consider $\Lambda \neq 0$ later),
- $\alpha = 0 \implies \zeta = \text{const} = \zeta_o$,
- $\omega = \text{const}$.

From Eq. (15), the density ρ will also be constant. From the modified energy conservation law (12), we get the following equation:

$$\mu_o(\omega + 1) - 6(\omega + 1)\zeta_o H_o a^3 = 0, \tag{27}$$

from which we conclude that $a = \text{const}$ unless $\omega = -1$. However, if $\omega = -1$, then we see from Eq. (19) for F that there is no viscosity.

We conclude that a simple exponential solution of the type $a = a_o e^{H_o t}$ does not exist in this theory, in contrast to Eckart, TIS, LIS, and NIS theories.

4.3. Variable Λ Solution

Consider:

- $\omega = \text{const} \neq \pm 1$,
- $\Lambda = H^2$,
- $\zeta = \zeta_o\rho^\alpha$, $\alpha \geq 0$.

Then we find the following solution:

$$a(t) = (-2kt^2 + k_1 t + k_2)^{1/2}, \tag{28}$$

where $k, k_1, k_2 = \text{const}$. We can easily find the other parameters from our equations. We list them for the case $k = 0$ (k_3 is an integration constant):

$$\rho^{\alpha} = \frac{2\alpha(\omega - 1)\mu_0}{(6\alpha - 1)k_1(1+\omega)\zeta_o(k_1 t + k_2)^{1/2}} + \frac{k_3}{(k_1 t + k_2)^{3\alpha}}, \quad (29)$$

$$\zeta = \zeta_o \frac{2\alpha(\omega - 1)\mu_0}{(6\alpha - 1)k_1(1+\omega)\zeta_o(k_1 t + k_2)^{1/2}} + \zeta_o \frac{k_3}{(k_1 t + k_2)^{3\alpha}}, \quad (30)$$

$$\Lambda = \frac{k_1^2}{4(k_1 t + k_2)^2}, \quad (31)$$

$$\omega_{\text{eff}} = \omega - \frac{3(1+\omega)\zeta_o k_1 (k_1 t + k_2)^{1/2}}{2\mu_o} x, \quad (32)$$

where x is given by

$$x = \frac{(1-\omega)\mu_o}{(6\alpha-1)(1+\omega)\zeta_o(k_1 t + k_2)^{3/2}} - \frac{3k_1 k_3}{(k_1 t + k_2)^{3\alpha+2}} + \frac{6k_1\alpha(\omega - 1)\mu_o}{(6\alpha - 1)(1+\omega)k_1\zeta_o(k_1 t + k_2)^{3/2}}, \quad (33)$$

$$q = -1/2. \quad (34)$$

We note that this solution is accelerating since q is negative.

5. CONCLUSION

In this paper, we have studied the first-order theory of Disconzi et al. [16, 17, 19], which is claimed to be causal and stable. We have found several simple solutions. A further study is required, especially the evolution of the temperature and a more detailed comparison with other viscosity theories. We are presently carrying out a dynamic system analysis and hope to report on this elsewhere.

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