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On Generalized Melvin's Solutions for Lie Algebras of Rank 2

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Abstract—We consider a class of solutions in multidimensional gravity which generalize Melvin's well-known cylindrically symmetric solution, originally describing the gravitational field of a magnetic flux tube. The solutions considered contain the metric, two Abelian 2-forms and two scalar fields, and are governed by two moduli functions $H_1(z)$ and $H_2(z)$ ($z = \rho^2$, ρ is a radial coordinate) which have a polynomial structure and obey two differential (Toda-like) master equations with certain boundary conditions. These equations are governed by a certain matrix A which is a Cartan matrix for some Lie algebra. The models for rank-2 Lie algebras A_2, C_2 and G_2 are considered. We study a number of physical and geometric properties of these models. In particular, duality identities are proved, which reveal a certain behavior of the solutions under the transformation $\rho \to 1/\rho$; asymptotic relations for the solutions at large distances are obtained; 2-form flux integrals over 2-dimensional regions and the corresponding Wilson loop factors are calculated, and their convergence is demonstrated. These properties make the solutions potentially applicable in the context of some dual holographic models. The duality identities can also be understood in terms of the Z_2 symmetry on vertices of the Dynkin diagram for the corresponding Lie algebra.

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1. INTRODUCTION

In this paper we deal with a certain generalization of Melvin's solution of GR [1]. Melvin's original solution in four dimensions describes the gravitational field of a magnetic flux tube. The multidimensional analog of such a flux tube, supported by a configuration of $(p+2)$-form fields, is referred to as a fluxbrane (a "thickened brane" of magnetic flux). The appearance of fluxbrane solutions was motivated by superstring/p-brane models and M-theory. A physical interest in such solutions is that they supply an appropriate background geometry for studying various processes involving p-branes, instantons, Kaluza–Klein (KK) monopoles, pair production of magnetically charged black holes and other configurations which can be studied via a special kind of KK reduction ("modding technique") of a certain multidimensional model in the presence of $U(1)$ isometry. It was shown, in particular, that Melvin's original solution (F1-fluxbrane) can be interpreted as a modding of flat space in one dimension higher. This "modding" technique is widely used in construction of new solutions in supergravity models and for various physical applications in superstring models and M-theory. For generalizations of Melvin's and fluxbrane solutions see [2–22] and references therein.

The solution we consider here was presented earlier in [23] and can be regarded as a special case of the generalized fluxbrane solutions investigated in [2]. Those solutions are governed by moduli functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$, where $z = \rho^2$, and ρ is a radial variable. These functions obey a set of n nonlinear differential master equations equivalent to Toda-like equations governed by a matrix $(A_{ss'})$ with the boundary conditions $H_s(+0) = 1$, $s = 1, ..., n$. In this paper we assume that $(A_{ss'})$ is a Cartan matrix of some simple finite-dimensional Lie algebra $\mathcal{G}$ of rank n ($A_{ss} = 2$ for all s). The appearance of Lie algebras is related to the integrability conditions of the set of differential equations considered.

Originally, the model from [23] contains n Abelian 2-forms and $l \geq n$ scalar fields. Here we consider special solutions with $n = l = 2$, governed by a 2×2 Cartan matrix (A_{ij}) for simple Lie algebras of rank 2: A_2, C_2, G_2. It is quite a simple choice to demonstrate some geometric properties of the model. On the other hand, classical Lie algebras correspond to

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certain physically interesting models [2] (for example, fluxbrane configurations for A_1 correspond to Melvin's solution, fluxbrane analogs of M2- and M5-branes in $D = 11$ supergravity model; A_2 is related to the KK dyonic flux tube). We note that G_2 is an exceptional algebra often appearing in the context of M-theory.

One of the goals of this paper is to study interesting geometric properties of the solution considered. In particular, we prove the so-called duality property which establishes a certain symmetry of the solutions under the inversion transformation $\rho \to 1/\rho$, which makes the model in tune with T-duality in string models, and also can be mathematically understood in terms of a representation of $\mathbb{Z}_2$ group acting on vertices of Dynkin diagrams for the corresponding Lie algebras. These duality identities may be used in deriving a $1/\rho$-expansion for solutions at large distances ρ. The corresponding asymptotic behavior of the solution is studied.

We also calculate the fluxes (i.e., integrals of 2-form fields over 2-dimensional disks) which have the meaning of Wilson loop factors, important objects in quantum field theory. They appear to be convergent (even over an infinite region) and have an interesting nontrivial property: each flux depends on only one integration constant, while the integrands depend on a mixed set of all such constants.

According to a conjecture suggested in [2], the moduli functions $H_s(z)$ with the above boundary conditions are polynomials:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \tag{1}$$

where $P_s^{(k)}$ are constants. Here $P_s^{(n_s)} \neq 0$, and

$$n_s = 2 \sum_{s'=1}^{n} A^{ss'}, \tag{2}$$

where we denote $(A^{ss'}) = (A_{ss'})^{-1}$. The integers n_s are components of the twice dual Weyl vector in the basis of simple (co-)roots [24].

The set of fluxbrane polynomials H_s defines a special solution to open Toda chain equations [25, 26] corresponding to a simple finite-dimensional Lie algebra $\mathcal{G}$ [27]. In [23, 28] a program (in Maple) for calculating these polynomials for classical series of Lie algebras (A-, B-, C- and D-series) was suggested. It was pointed out in [2] that the conjecture on a polynomial structure of $H_s(z)$ is valid for Lie algebras of A- and C- series.

In [29], the conjecture from [2] was verified for the Lie algebra E_6, and certain duality relations for six E_6 polynomials were found.

The paper is organized as follows. In Section 2 we present the generalized Melvin solution from [23] for the case of two scalar fields and two forms. In Section 3 we deal with solutions for the Lie algebras A_2, C_2, G_2. We find duality relations for polynomials and present asymptotic relations for the solutions. We also calculate 2-form flux integrals $\Phi^s(R) = \int_{M_R} F^s$ and the corresponding Wilson loop factors, where F^s are 2-forms, and M_R is a two-dimensional disc of radius R. The flux integrals have finite limits for $R = +\infty$.

2. THE SETUP AND THE GENERALIZED MELVIN SOLUTIONS

We consider a model governed by the action

$$S = \int d^D x \sqrt{|g|} \Big\{ R[g] - \delta_{ab} g^{MN} \partial_M \varphi^a \partial_N \varphi^b - \frac{1}{2} \sum_{s=1}^{2} \exp[2\lambda_{sa}\varphi^a)](F^s)^2 \Big\}, \tag{3}$$

where $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric, $\vec{\varphi} = (\varphi^a) \in \mathbb{R}^2$ is the vector of scalar fields, $F^s = dA^s = \frac{1}{2} F^s_{MN} dx^M \wedge dx^N$ is a 2-form, $\vec{\lambda}_s = (\lambda_{sa}) \in \mathbb{R}^2$ is a dilatonic coupling vector, $s = 1, 2$ $(a = 1, 2)$. We denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{M_1 M_2} F^s_{N_1 N_2} \times g^{M_1 N_1} g^{M_2 N_2}$, $s = 1, 2$.

Here we deal with a family of exact solutions to the field equations corresponding to the action (3) and depending on one variable ρ. The solutions are defined on the manifold

$$M = (0, +\infty) \times M_1 \times M_2, \tag{4}$$

where $M_1 = S^1$ and M_2 is a $(D-2)$-dimensional Ricci-flat manifold. The solution reads [23]

$$g = \Big(\prod_{s=1}^{2} H_s^{2h_s/(D-2)} \Big) \Big\{ d\rho \otimes d\rho + \Big(\prod_{s=1}^{2} H_s^{-2h_s} \Big) \rho^2 d\phi \otimes d\phi + g^2 \Big\}, \tag{5}$$

$$\exp(\varphi^a) = \prod_{s=1}^{2} H_s^{h_s \lambda_s^a}, \tag{6}$$

$$F^s = q_s \Big(\prod_{l=1}^{2} H_l^{-A_{sl}} \Big) \rho d\rho \wedge d\phi, \tag{7}$$

$s = 1, 2$, $a = 1, 2$, where $g^1 = d\phi \otimes d\phi$ is a metric on $M_1 = S^1$ and g^2 is a Ricci-flat metric of signature $(-, +, \ldots, +)$ on M_2. Here $q_s \neq 0$ are integration constants ($q_s = -Q_s$ in the notations of [23]), $s = 1, 2$.

The functions $H_s(z) > 0$, $z = \rho^2$, obey the master equations

$$\frac{d}{dz}\left(\frac{z}{H_s}\frac{d}{dz}H_s\right) = P_s \prod_{l=1}^{2} H_l^{-A_{sl}}, \qquad (8)$$

with the boundary conditions

$$H_s(+0) = 1, \qquad (9)$$

where

$$P_s = \frac{1}{4}K_s q_s^2, \quad s = 1, 2. \qquad (10)$$

The boundary condition (9) guarantees the absence of a conic singularity in the metric (5) at $\rho = +0$.

The parameters h_s satisfy the relations

$$h_s = K_s^{-1}, \quad K_s = B_{ss} > 0, \qquad (11)$$

where

$$B_{sl} \equiv 1 + \frac{1}{2-D} + \vec{\lambda}_s \vec{\lambda}_l, \quad s, l = 1, 2. \qquad (12)$$

In the above relations we denote $\lambda_s^a = \lambda_{sa}$ and

$$(A_{sl}) = (2B_{sl}/B_{ll}). \qquad (13)$$

The latter is the so-called quasi-Cartan matrix.

It can be shown that if (A_{sl}) is a Cartan matrix for a simple Lie algebra $\mathcal{G}$ of rank 2, there exists a set of vectors $\vec{\lambda}_1, \vec{\lambda}_2$ obeying (13), see the Remark in the next section.

The solution under consideration is as a special case of the fluxbrane solution from [2, 20].

Thus we deal with a multidimensional generalization of Melvin's solution for the case of two scalar fields and two 2-forms [1]. Melvin's solution without a scalar field corresponds to $D = 4$, one 2-form, $M_1 = S^1$ $(0 < \phi < 2\pi)$, $M_2 = \mathbb{R}^2$ and $g^2 = -dt \otimes dt + dx \otimes dx$.

3. SOLUTIONS RELATED TO SIMPLE LIE ALGEBRAS OF RANK 2

We are dealing with solutions which corresponds to a simple Lie algebras $\mathcal{G}$ of rank 2, i.e., the matrix $A = (A_{sl})$ coincides with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -k & 2 \end{pmatrix}, \qquad (14)$$

where $k = 1, 2, 3$ for $\mathcal{G} = A_2, C_2, G_2$, respectively. This matrix is described graphically by the Dynkin diagrams shown in Fig. 1 for these three Lie algebras.

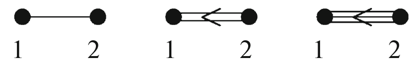

Fig. 1. Dynkin diagrams for the Lie algebras A_2, C_2, G_2, respectively.

Due to (11)–(13) we get

$$K_s = \frac{D-3}{D-2} + \vec{\lambda}_s^2, \quad h_s = K_s^{-1}, \qquad (15)$$

and

$$\vec{\lambda}_s \vec{\lambda}_l = \frac{1}{2}K_l A_{sl} - \frac{D-3}{D-2} \equiv G_{sl}, \qquad (16)$$

$s, l = 1, 2$; (15) is a special case of (16).

It follows from (11)–(13) that

$$\frac{h_1}{h_2} = \frac{K_2}{K_1} = \frac{A_{21}}{A_{12}} = k, \qquad (17)$$

where $k = 1, 2, 3$ for $\mathcal{G} = A_2, C_2, G_2$, respectively.

Remark. For large enough K_1 in (17), there exist vectors $\vec{\lambda}_s$ obeying (16) (and hence (15)). Indeed, the matrix $G = (G_{sl})$ is positive definite if $K_1 > K_*$, where K_* is a positive number. Hence there exists a matrix Λ such that $\Lambda^T \Lambda = G$. We put $(\Lambda_{as}) = (\lambda_s^a)$ and get the set of vectors obeying (16).

Polynomials. The moduli functions $H_1(z)$, $H_2(z)$, obeying Eqs. (8) and (9) with the matrix $A = (A_{sl})$ from (14) are polynomials with powers $(n_1, n_2) = (2, 2), (3, 4), (6, 10)$ for $\mathcal{G} = A_2, C_2, G_2$, respectively.

In what follows we list these polynomials. Here, as in [27], we use the rescaled variables

$$p_s = P_s/n_s, \quad s = 1, 2. \qquad (18)$$

A_2**-case.** For the Lie algebra $A_2 = sl(3)$ we have [2, 20, 27]

$$H_1 = 1 + 2p_1 z + p_1 p_2 z^2, \qquad (19)$$

$$H_2 = 1 + 2p_2 z + p_1 p_2 z^2. \qquad (20)$$

C_2**-case.** For the Lie algebra $C_2 = so(5)$ we obtain the following polynomials [20, 27]:

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1^2 p_2 z^3, \qquad (21)$$

$$H_2 = 1 + 4p_2 z + 6p_1 p_2 z^2 + 4p_1^2 p_2 z^3 + p_1^2 p_2^2 z^4. \qquad (22)$$

G_2**-case.** For the Lie algebra G_2 the fluxbrane polynomials read [20, 27]

$$H_1 = 1 + 6p_1 z + 15p_1 p_2 z^2 + 20p_1^2 p_2 z^3 + 15p_1^3 p_2 z^4 + 6p_1^3 p_2^2 z^5 + p_1^4 p_2^2 z^6, \qquad (23)$$

$$H_2 = 1 + 10p_2 z + 45p_1 p_2 z^2 + 120p_1^2 p_2 z^3 + p_1^2 p_2(135p_1 + 75p_2)z^4 + 252p_1^3 p_2^2 z^5 + p_1^3 p_2^2(75p_1 + 135p_2)z^6 + 120p_1^4 p_2^3 z^7 + 45p_1^5 p_2^3 z^8 + 10p_1^6 p_2^3 z^9 + p_1^6 p_2^4 z^{10}. \qquad (24)$$

Let us denote

$$H_s = H_s(z) = H_s(z, (p_i)), \qquad (25)$$

where $s = 1, 2$, $(p_i) = (p_1, p_2)$.

Due to the relations for polynomials, we have the following asymptotic behavior as $z \to \infty$:

$$H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{2} (p_l)^{\nu^{sl}}\right) z^{n_s} \equiv H_s^{as}(z, (p_i)), \quad s = 1, 2. \qquad (26)$$

Here $\nu = (\nu^{sl})$ is an integer valued matrix

$$\nu = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix} \qquad (27)$$

for the Lie algebras A_2, C_2, G_2, respectively.

In the last two cases (C_2 and G_2) we have $\nu = 2A^{-1}$, where A^{-1} is an inverse Cartan matrix. For A_2 the matrix ν is related to the inverse Cartan matrix as follows:

$$\nu = A^{-1}(I + P), \qquad (28)$$

where I is a 2×2 identity matrix, and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (29)$$

is the permutation matrix. This matrix corresponds to the permutation $\sigma \in S_2$ (S_2 is the symmetric group)

$$\sigma : (1, 2) \mapsto (2, 1), \qquad (30)$$

by the relation $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here σ is a generator of the group $S_2 = \{\sigma, \mathrm{id}\}$, which is the symmetry group of the Dynkin diagram (for A_2); S_2 is isomorphic to the group $\mathbb{Z}_2$.

We note that in all cases we have

$$\sum_{l=1}^{2} \nu^{sl} = n_s, \quad s = 1, 2. \qquad (31)$$

Let us denote $\hat{p}_i = p_{\sigma(i)}$ for A_2, and $\hat{p}_i = p_i$ for C_2 and G_2 cases, $i = 1, 2$. We call the ordered set $(\hat{p}_i)$ the dual one to the ordered set (p_i). Using the relations for polynomials, we obtain the following identity which can be easily verified "by hand".

Duality Relations

Proposition. *The fluxbrane polynomials corresponding to Lie algebras A_2, C_2, and G_2 obey, for all $p_i > 0$ and $z > 0$, the identities*

$$H_s(z, (p_i)) = H_s^{as}(z, (p_i)) H_s(z^{-1}, (\hat{p}_i^{-1})), \qquad (32)$$

$s = 1, 2$. We call the relation (32) the duality relation.

Fluxes. Now let us consider the oriented 2-dimensional manifold $M_R = (0, R) \times S^1$, $R > 0$, and the flux integrals

$$\Phi^s(R) = \int_{M_R} F^s = 2\pi \int_0^R d\rho \rho \mathcal{B}^s, \qquad (33)$$

where

$$\mathcal{B}^s = q_s \prod_{l=1}^{2} H_l^{-A_{sl}}, \quad s = 1, 2. \qquad (34)$$

The total flux integrals $\Phi^s = \Phi^s(+\infty)$ are convergent. Indeed, due to (26) we have

$$H_s \sim C_s \rho^{2n_s}, \quad C_s = \prod_{l=1}^{2} (p_l)^{\nu^{sl}}, \qquad (35)$$

as $\rho \to +\infty$. From (34), (35) and the equality $\sum_1^2 A_{sl} n_l = 2$, following from (2) ($n = 2$), we get

$$\mathcal{B}^s \sim q_s C^s \rho^{-4}, \quad C^s = \prod_{l=1}^{2} C_l^{-A_{sl}}, \qquad (36)$$

hence the integral (33) converges for any $s = 1, 2$.

Due to (28), we get, for A_2, $A\nu = I + P$ and

$$C^s = \prod_{l=1}^{2} p_l^{-(I+P)_{sl}} = \prod_{l=1}^{2} p_l^{-\delta^l_s - \delta^l_{\sigma(s)}} = p_s^{-1} p_{\sigma(s)}^{-1}, \qquad (37)$$

or

$$C^s = p_1^{-1} p_2^{-1}, \quad s = 1, 2, \qquad (38)$$

while for C_2 and G_2 we obtain

$$C^s = p_s^{-2}, \quad s = 1, 2. \qquad (39)$$

Now we calculate $\Phi^s(R)$. Using the master equations (8), we obtain

$$\int_0^R d\rho \rho \mathcal{B}^s = q_s P_s^{-1} \frac{1}{2} \int_0^{R^2} dz \frac{d}{dz} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = \frac{1}{2} q_s P_s^{-1} \frac{R^2 H_s'(R^2)}{H_s(R^2)}, \qquad (40)$$

where $H'_s = dH_s/dz$. With (33) we obtain

$$\Phi^s(R) = 4\pi q_s^{-1} h_s \frac{R^2 H'_s(R^2)}{H_s(R^2)}, \quad s = 1, 2. \qquad (41)$$

Using the Stokes theorem, we get

$$\Phi^s(R) = \int_{M_R} F^s = \int_{M_R} dA^s = \int_{C_R} A^s, \qquad (42)$$

where C_R is a circle of radius R with proper orientation (the boundary of M_R). Using the definition of an Abelian Wilson loop (factor), we get

$$W^s(C_R) = \exp\left(i \int_{C_R} A^s\right) = e^{i\Phi^s(R)},$$
$$s = 1, 2. \qquad (43)$$

The relations (1), (42) imply (see (10))

$$\Phi^s = \Phi^s(+\infty) = 4\pi n_s q_s^{-1} h_s, \quad s = 1, 2. \qquad (44)$$

Any (total) flux Φ^s depends on one integration constant $q_s \neq 0$, while the integrand form F^s depends on both constants: q_1, q_2.

We get in the A_2 case

$$(\Phi^1, \Phi^2) = 8\pi h(q_1^{-1}, q_2^{-1}), \qquad (45)$$

where $h_1 = h_2 = h$.

In the C_2 case we find

$$(\Phi^1, \Phi^2) = 4\pi(3h_1 q_1^{-1}, 4h_2 q_2^{-1}), \qquad (46)$$

where $h_1 = 2h_2$.

For G_2 we obtain the relation

$$(\Phi^1, \Phi^2) = 4\pi(6h_1 q_1^{-1}, 10h_2 q_2^{-1}), \qquad (47)$$

where $h_1 = 3h_2$.

We note that for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, q_s coincides with the value of the x-component of the sth magnetic field on the axis of symmetry.

Due to (17) we get fixed numbers of the ratios

$$\frac{q_1 \Phi^1}{q_2 \Phi^2} = \frac{n_1 h_1}{n_2 h_2} = \frac{n_1}{n_2} k = 1,\ \frac{3}{2},\ \frac{9}{5} \qquad (48)$$

for $k = 1, 2, 3$, or $\mathcal{G} = A_2, C_2, G_2$, respectively.

Asymptotic relations. For the solution under consideration, at the asymptotic $\rho \to +\infty$ we obtain

$$g_{as} = \left(\prod_{l=1}^{2} p_l^{a_l}\right)^{2/(D-2)} \rho^{2A} \Bigg\{ d\rho \otimes d\rho$$
$$+ \left(\prod_{l=1}^{2} p_l^{a_l}\right)^{-2} \rho^{2-2A(D-2)} d\phi \otimes d\phi + g^2 \Bigg\}, \qquad (49)$$

$$\varphi^a_{as} = \sum_{s=1}^{2} h_s \lambda^a_s \left(\sum_{l=1}^{2} \nu^{sl} \ln p_l + 2n_s \ln \rho\right), \qquad (50)$$

$$F^s_{as} = q_s p_s^{-1} p_{\theta(s)}^{-1} \rho^{-3} d\rho \wedge d\phi, \qquad (51)$$

where $a, s = 1, 2$, and

$$a_l = \sum_{s=1}^{2} h_s \nu^{sl}, \quad A = 2(D-2)^{-1} \sum_{s=1}^{2} n_s h_s. \qquad (52)$$

In (51) we have put $\theta = \sigma$ for $\mathcal{G} = A_2$, and $\theta = \mathrm{id}$ for $\mathcal{G} = C_2, G_2$. In the derivation of the asymptotic relations, Eqs. (31) and (35)–(39) were used. We note that for $\mathcal{G} = C_2, G_2$ the asymptotic value of the form F^s_{as} depends on q_s ($s = 1, 2$), while in the A_2 case any F^s_{as} depends on both q_1 and q_2.

4. CONCLUSIONS

We have considered generalizations of Melvin's solution corresponding to simple finite-dimensional Lie algebras of rank 2: $\mathcal{G} = A_2, C_2, G_2$. Any solution is governed by a set of 2 fluxbrane polynomials $H_s(z)$, $s = 1, 2$. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $\mathcal{G}$.

The polynomials $H_s(z)$ also depend on the parameters q_s, which coincide for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

We have found duality identities for polynomials, which may be used in deriving $1/\rho$ expansion for solutions at large distances ρ, e.g., for asymptotic relations which are presented in the paper.

The power-law asymptotic relations for the polynomials $H_s(z)$ at large ρ are governed by an integer-valued matrix ν which coincides with twice the inverse Cartan matrix $2A^{-1}$ for Lie algebras C_2 and G_2, while in the A_2 case $\nu = A^{-1}(I + P)$, where I is the identity matrix and P is the permutation matrix, corresponding to a generator of the Z_2 symmetry group of the Dynkin diagram.

We have calculated 2D flux integrals $\Phi^s(R) = \int_{M_R} F^s$ ($s = 1, 2$) on a disc M_R of radius R and the corresponding Wilson loop factors $W^s(C_R)$ over a circle C_R of radius R. Any total flux $\Phi^s(\infty)$ depends on only one parameter q_s, while the integrand F^s depends on both parameters q_1, q_2. The calculation of "partial" fluxes $\Phi^s(R)$ and the Wilson loop factors $W^s(C_R)$ may have an application in certain holographic dual model [30].

An open problem is to study the convergence of flux integrals for non-polynomial solutions for moduli functions corresponding to non-Cartan matrices $(A_{ss'})$ (e.g. for a model with two 2-forms from [31]).

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