ISSN 0202-2893, Gravitation and Cosmology, 2016, Vol. 22, No. 4, pp. 329–332. © Pleiades Publishing, Ltd., 2016.

On Stable Exponential Solutions in Einstein–Gauss–Bonnet Cosmology with Zero Variation of G

V. D. Ivashchuk*

Institute of Gravitation and Cosmology, RUDN University, Miklukho-Maklaya 6, Moscow, 117198 Russia,
Center for Gravitation and Fundamental Metrology, VNIIMS, Ozyornaya 46, Moscow, 119361 Russia

Received July 18, 2016

Abstract—A D-dimensional gravitational model with a Gauss–Bonnet term and the cosmological constant Λ is considered. Assuming diagonal cosmological metrics, we find, for certain $\Lambda > 0$, new examples of solutions with an exponential time dependence of two scale factors, governed by two Hubble-like parameters $H > 0$ and $h < 0$, corresponding to submanifolds of dimensions m and l, respectively, with $(m, l) = (4, 2), (5, 2), (5, 3), (6, 7), (7, 5), (7, 6)$ and $D = 1 + m + l$. Any of these solutions describes an exponential expansion of our 3-dimensional factor space with the Hubble parameter H and zero variation of the effective gravitational constant G. We also prove the stability of these solutions in the class of cosmological solutions with diagonal metrics.

DOI: 10.1134/S0202289316040095

1. INTRODUCTION

In this paper we consider a D-dimensional gravitational model with a Gauss–Bonnet term and a cosmological constant. We note that at present the so-called Einstein–Gauss–Bonnet (EGB) gravitational model and its modifications, see [1–11] and references therein, are intensively studied in cosmology, e.g., for a possible explanation of accelerated expansion of the Universe following from supernovae (type Ia) observational data [15–17].

Here we deal with cosmological solutions with diagonal metrics, governed by $n > 3$ scale factors depending on one variable, the synchronous time. We restrict ourselves to solutions with an exponential dependence of the scale factors and present six new examples of such solutions: five in dimensions $D = 7, 8, 9, 13$ and two for $D = 14$. Any of these solutions describes an exponential expansion of 3-dimensional factor-space with Hubble parameters $H > 0$ [18] and has a constant volume factor of the internal space, which implies zero variation of the effective gravitational constant G either in Jordan or Einstein frame [19, 20], see also [21, 22] and references therein. These solutions obey the most severe restrictions on variation of G [23].

We study the stability of these solutions in a class of cosmological solutions with diagonal metrics by using the results of [13, 14] and show that all solutions presented here are stable. It should be noted that two exponential solutions with two factor spaces (one of which is expanding and the other one contracting) and constant G were found for $D = 22, 28$ and $\Lambda = 0$ in [11]. In [22] it was proved that these solutions are stable.

2. THE COSMOLOGICAL MODEL

The action reads

$$S = \int_M d^D z \sqrt{|g|}\{\alpha_1(R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g]\}, \quad (1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M, $\dim M = D$, $|g| = |\det(g_{MN})|$, Λ is the cosmological constant,

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2$$

is the standard Gauss–Bonnet term, and α_1, α_2 are nonzero constants.

We consider the manifold

$$M = \mathbb{R} \times M_1 \times \ldots \times M_n \quad (2)$$

with the metric

$$g = -dt \otimes dt + \sum_{i=1}^{n} B_i e^{2v^i t} dy^i \otimes dy^i, \quad (3)$$

*E-mail: ivashchuk@mail.ru

where $B_i > 0$ are arbitrary constants, $i = 1, \dots, n$, and $M_1, ..., M_n$ are one-dimensional manifolds (either $\mathbb{R}$ or S^1) and $n > 3$.

The equations of motion for the action (1) give us the set of polynomial equations [13]

$$G_{ij}v^i v^j + 2\Lambda - \alpha G_{ijkl}v^i v^j v^k v^l = 0, \quad (4)$$

$$\left[2G_{ij}v^j - \frac{4}{3}\alpha G_{ijkl}v^j v^k v^l\right]\sum_{i=1}^{n} v^i - \frac{2}{3}G_{ij}v^i v^j + \frac{8}{3}\Lambda = 0, \quad (5)$$

$i = 1, \dots, n$, where $\alpha = \alpha_2/\alpha_1$. Here $G_{ij} = \delta_{ij} - 1$ and $G_{ijkl} = G_{ij}G_{ik}G_{il}G_{jk}G_{jl}G_{kl}$ are, respectively, the components of two metrics on $\mathbb{R}^n$ [9, 10]. The first one is a 2-metric, and the second one is a Finslerian 4-metric. For $n > 3$ we get a set of fourth-order polynomial equations.

For $\Lambda = 0$ and $n > 3$ the set of equations (4) and (5) has an isotropic solution $v^1 = \ldots = v^n = H$ only if $\alpha < 0$ [9, 10]. This solution was generalized in [8] to the case $\Lambda \neq 0$.

It was shown in [9, 10] that there are no more than three different numbers among $v^1, \dots, v^n$ if $\Lambda = 0$. This is also valid for $\Lambda \neq 0$ if $\sum_{i=1}^{n} v^i \neq 0$ [14].

3. SOLUTIONS WITH CONSTANT G

In this section we present several solutions to the set of equations (4), (5) of the following form

$$v = (H, \dots, H, h, \dots, h). \quad (6)$$

where H is the Hubble-like parameter corresponding to an m-dimensional subspace with $m > 3$ and h is the Hubble-like parameter corresponding to an l-dimensional subspace, $l > 1$. We put $H > 0$ for the description of an accelerated expansion of a 3-dimensional subspace (which may describe our Universe) and also put

$$h = -(m-3)H/l < 0 \quad (7)$$

for the description of a zero variation of the effective gravitational constant G.

We remind the reader that the effective gravitational constant $G = G_{\text{eff}}$ in the Brans–Dicke–Jordan (or simply Jordan) frame [19] (see also [20]) is proportional to the inverse volume scale factor of the internal space, see [21, 22] and references therein.

According to the ansatz (6), the m-dimensional subspace is expanding with the Hubble parameter $H > 0$, while the l-dimensional subspace is contracting with the Hubble-like parameter $h < 0$.

For $\Lambda = 0$, $m = 11$, $l = 16$ and $\alpha = 1$ the solution with $H = 1/\sqrt{15}$, $h = -1/(2\sqrt{15})$, describing zero variation of G, was found in [11]. Another solution of such type with $\Lambda = 0$, $H = 1/6$, $h = -1/3$ and constant G appears for $m = 15$, $l = 6$ and $\alpha = 1$ [11]. It was proved in [13] that these two solutions are stable.

Here we present three solutions with constant G for $\alpha < 0$:

$$H = \frac{1}{\sqrt{6|\alpha|}}, \quad h = -\frac{1}{2\sqrt{6|\alpha|}} \quad (8)$$

for $\Lambda = 7/(8|\alpha|)$, $(m, l) = (4, 2)$;

$$H = \frac{1}{\sqrt{8|\alpha|}}, \quad h = -\frac{1}{\sqrt{8|\alpha|}} \quad (9)$$

for $\Lambda = 17/(16|\alpha|)$, $(m, l) = (5, 2)$, and

$$H = \frac{3}{2\sqrt{10|\alpha|}}, \quad h = -\frac{1}{\sqrt{10|\alpha|}} \quad (10)$$

for $\Lambda = 177/(80|\alpha|)$, $(m, l) = (5, 3)$.

We also present three solutions with constant G for $\alpha > 0$:

$$H = \frac{7}{2\sqrt{5\alpha}}, \quad h = -\frac{3}{2\sqrt{5\alpha}} \quad (11)$$

for $\Lambda = 928.2\alpha^{-1} = 4641/(5\alpha)$, $(m, l) = (6, 7)$;

$$H = \frac{5}{6\sqrt{\alpha}}, \quad h = -\frac{2}{3\sqrt{\alpha}}, \quad (12)$$

for $\Lambda = 169.72(2)\alpha^{-1} = 3055/(18\alpha)$, $(m, l) = (7, 5)$, and

$$H = \frac{3}{2\sqrt{5\alpha}}, \quad h = -\frac{1}{\sqrt{5\alpha}} \quad (13)$$

for $\Lambda = 84.9\alpha^{-1} = 849/(10\alpha)$, $(m, l) = (7, 6)$. All six solutions may be verified by MAPLE. The derivation of a more general class of solutions will be presented in a separate paper.

4. STABILITY ANALYSIS

In [13, 14] we restricted ourselves to the exponential solutions (3) with a nonstatic volume factor, which is proportional to $\exp(\sum_{i=1}^{n} v^i t)$, i.e., we put

$$K = K(v) = \sum_{i=1}^{n} v^i \neq 0. \quad (14)$$

We consider the matrix

$$L = (L_{ij}(v)) = (2G_{ij} - 4\alpha G_{ijks}v^k v^s)$$

and put the restriction [13, 14]

$$\text{(R)} \quad \det(L_{ij}(v)) \neq 0. \quad (15)$$

For a general cosmological setup with the metric

$$g = -dt \otimes dt + \sum_{i=1}^{n} e^{2\beta^i(t)} dy^i \otimes dy^i,$$

we have obtained in [13, 14] the (mixed) set of algebraic and differential equations

$$f_0(h) = 0, \tag{16}$$

$$f_i(\dot{h}, h) = 0, \tag{17}$$

$i = 1, \ldots, n$, where $h = h(t) = (h^i(t)) = (\dot{\beta}_i(t))$ is the set of Hubble-like parameters; $f_0(h)$ and $f_i(\dot{h}, h)$ are fourth-order polynomials in h^i; $f_i(\dot{h}, h)$ are first-order polynomials in $\dot{h}^i$, see [13, 14]. The fixed-point solutions $h^i(t) = v^i$ $(i = 1, \ldots, n)$ to Eqs. (16), (17) correspond to the exponential solutions (3) which obey Eqs. (4), (5).

It has been proved in [14] that a fixed-point solution $(h^i(t)) = (v^i)$ $(i = 1, \ldots, n; n > 3)$ to Eqs. (16), (17) obeying the restrictions (14), (15) is stable under perturbations $h^i(t) = v^i + \delta h^i(t)$, $i = 1, \ldots, n$, (as $t \to +\infty$) if

$$K(v) = \sum_{k=1}^{n} v^k > 0, \tag{18}$$

and it is unstable (as $t \to +\infty$) if $K(v) = \sum_{k=1}^{n} v^k < 0$.

As was shown in [14] for the vector v from (6), obeying

$$mH + lh \neq 0, \quad H \neq h, \tag{19}$$

the matrix L has a block-diagonal form:

$$(L_{ij}) = \mathrm{diag}(L_{\mu\nu}, L_{\alpha\beta}), \tag{20}$$

where

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \tag{21}$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}), \tag{22}$$

and

$$S_{HH} = (m-2)(m-3)H^2 + 2(m-2)lHh + l(l-1)h^2, \tag{23}$$

$$S_{hh} = m(m-1)H^2 + 2m(l-2)Hh + (l-2)(l-3)h^2. \tag{24}$$

The matrix (20) is invertible if and only if $m > 1$, $l > 1$ and

$$S_{HH} \neq -\frac{1}{2\alpha}, \quad S_{hh} \neq -\frac{1}{2\alpha}. \tag{25}$$

Recall that the matrices $(G_{\mu\nu}) = (\delta_{\mu\nu} - 1)$ and $(G_{\alpha\beta}) = (\delta_{\alpha\beta} - 1)$ are invertible only if $m > 1$ and $l > 1$.

The first condition (18) is obeyed for the solutions under consideration since due to (7) we get $K(v) = 3H > 0$ [14].

Now, let us verify the second condition (25). Calculations give us

$$(-2\alpha)S_{HH} = -0.5,\ -1,\ -1.5,\ 21,\ 10,\ 6, \tag{26}$$

$$(-2\alpha)S_{hh} = 4,\ 5,\ 6,\ -39,\ -17,\ -9. \tag{27}$$

for solutions with $(m, l) = (4, 2), (5, 2), (5, 3), (6, 7), (7, 5), (7, 6)$, respectively. Thus the conditions (25) are satisfied for by our solutions. Hence all six solutions are stable in a class of cosmological solutions with diagonal metrics.

5. CONCLUSIONS

We have considered a D-dimensional Einstein–Gauss–Bonnet (EGB) model with a cosmological constant. By using the ansatz with diagonal cosmological metrics, we have found, for certain $\Lambda > 0$ and $\alpha = \alpha_2/\alpha_1$, six new solutions with exponential dependence of two scale factors on synchronous time t in dimensions $D = 1 + m + n$, where $(m, l) = (4, 2)$, $(5, 2)$, $(5, 3)$, $(6, 7)$, $(7, 5)$, $(7, 6)$. Here $m > 3$ is the dimension of the expanding subspace, and $l > 1$ is the dimension of the contracting subspace. The first three solutions correspond to $\alpha < 0$, the other three to $\alpha > 0$.

Any of these solutions describes an exponential expansion of our 3-dimensional factor space with the Hubble parameter $H > 0$ and an anisotropic behavior of the $(m - 3 + l)$-dimensional internal space: expansion in $(m - 3)$ dimensions with the Hubble-like parameter H and contraction in l dimensions, with the Hubble-like parameter $h < 0$. Each solution has a constant volume factor of internal space, and hence it describes zero variation of the effective gravitational constant G. By using the results of [14] we have proved that all these solutions are stable as $t \to +\infty$.

ACKNOWLEDGMENTS

This paper was funded by the Ministry of Education and Science of the Russian Federation in the Program to increase the competitiveness of Peoples' Friendship University (RUDN University) among the world's leading research and education centers in 2016-2020 and by the Russian Foundation for Basic Research, grant No. 16-02-00602.

REFERENCES

1. H. Ishihara, Phys. Lett. B **179**, 217 (1986).
2. N. Deruelle, Nucl. Phys. B **327**, 253 (1989).
3. E. Elizalde, A. N. Makarenko, V. V. Obukhov, K. E. Osetrin, and A. E. Filippov, Phys. Lett. B **644**, 1 (2007).
4. A. Toporensky and P. Tretyakov, Grav. Cosmol. **13**, 207 (2007).
5. I. V. Kirnos, A. N. Makarenko, S. A. Pavluchenko, and A. V. Toporensky, Gen. Rel. Grav. **42**, 2633 (2010).
6. S. A. Pavluchenko and A. V. Toporensky, Mod. Phys. Lett. A **24**, 513–521 (2009).
7. S. A. Pavluchenko, Phys. Rev. D **80**, 107501 (2009).
8. D. Chirkov, S. Pavluchenko, and A. Toporensky, Mod. Phys. Lett. A **29**, 1450093 (2014).
9. V. D. Ivashchuk, Grav. Cosmol. **16**, 118 (2010).
10. V. D. Ivashchuk, Int. J. Geom. Meth. Mod. Phys. **7**, 797 (2010).
11. V. D. Ivashchuk and A. A. Kobtsev, Eur. Phys. J. C **75**, 177 (2015).
12. S. A. Pavluchenko, Phys. Rev. D **92**, 104017 (2015).
13. K. K. Ernazarov, V. D. Ivashchuk, and A. A. Kobtsev, Grav. Cosmol. **22**, 245 (2016).
14. V. D. Ivashchuk, "On stability of exponential cosmological solutions with non-static volume factor in the Einstein–Gauss–Bonnet model," arXiv: 1607.01244; Eur. Phys. J. C 76, 431 (2016).
15. A. G. Riess et al., Astron. J. **116**, 1009 (1998).
16. S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).
17. M. Kowalski, D. Rubin et al., "Improved cosmological constraints from new, old and combined supernova datasets," arXiv: 0804.4142.
18. P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. **571**, A1 (2014).
19. M. Rainer and A. Zhuk, Gen. Rel. Grav. **32**, 79–104 (2000); gr-qc/9808073.
20. V. D. Ivashchuk and V. N. Melnikov, Grav. Cosmol. **2**(3), 211–220 (1996); hep-th/9612054.
21. K. A. Bronnikov, V. D. Ivashchuk, and V. N. Melnikov, Nuovo Cimento B **102**, 209 (1998).
22. V. D. Ivashchuk and V. N. Melnikov, Grav. Cosmol. **20**, 26 (2014).
23. E. V. Pitjeva, Astron. Vestnik **47**, 419 (2013).