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Virtual Ring and Quantum Elements of a Classical Particle

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Abstract—The fractal equations of mechanics (quantum and classical) are clearly demonstrated to be definitions of an arbitrary potential on a fractal complex number valued surface. The developed approach helps us to show that a translational motion of any rotating compact object (point-like particle) can be equivalently represented by a specific rotation of a virtual ring described in terms of a fractal "wave function", the model endowing the particle with a set of quantum characteristics including quantization of the ring's space translation.

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1. INTRODUCTION

Long but vain efforts of de Broglie [1, 2] to heuristically ascribe a physical meaning (or a geometric image) to his wave function attributed to a particle and later similar considerations of other eminent physicists have in fact resulted in the shaky Copenhagen probability conjecture though concerning only the function's amplitude. The exponential factor, initially responsible for the wave properties, dropped out from the interpreters' attention. It may be argued that the absence of such a factor in a series of the so-called stationary exact solutions of the Schrödinger equation does not cancel their ability to "perfectly describe" quantum models, among them the textbook solutions for a particle in a 1D-box, a quantum oscillator and Schrödinger's H-atom. The quality of description seems to be a matter of interpretation, while the absence of a momentum in a dynamic system nonetheless evokes reasonable questions.

A possible answer may come from recent studies revealing the unambiguous interdependence between mechanical motion of a particle and rotation of a quaternion triad "frozen" in its mass. This observation is made in the course of an unexpectedly precise and simple mathematical derivation of the Schrödinger equation and the related equations of classical and relativistic mechanics, all theories providing associative division algebras stability [3]. Despite its methodological value, demonstrating the logical unity of all branches of mechanics, the approach appears to endow the abstract mathematical (and physical) quantities with clear geometric images related though to a fractal surface in a way "underlying" the 3D physical world. In particular, it helps one to identify the phase of the wave function with a half angle of the related particle's proper rotation, and with the respective particle's action function of classical mechanics. This, in particular, establishes a link between the characteristics of a particle's motion and the math-originating phenomenon of its rotation.

Here we suggest a more detailed analysis of a 3D object's rotation, taking into account the difference between its real proper rotation (including an interior rotation of a pointlike particle) and a specific "propagational rotation" necessarily induced by the math description of motion at the fractal level, thus leading to an original geometric treatment of quantum characteristics and introduction of the notion of a virtual ring. In Section 2 we give a detailed version of the derivation of a Schrödinger-type equation with emphasis on the fractal geometry. In Section 3, the conjecture of a virtual ring replacing a moving rotating object is introduced, and the de Broglie function for any 3D mechanical object is rebuilt. A compact discussion in Section 4 concludes the study.

2. EQUATION OF MECHANICS AS A FRACTAL STABILITY CONDITION OF ASSOCIATIVE DIVISION ALGEBRAS

Distinguished by the Frobenius theorem, three associative (in multiplication) division algebras of real, complex and quaternion numbers can be in general shown [4] to be built on matrix units, the simplest set of which is given, e.g., by the 2×2 constant matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q}_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

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$$\mathbf{q}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{q}_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1)$$

obeying the multiplication law

$$\mathbf{q}_k\mathbf{q}_m = -\delta_{km} + \varepsilon_{kmj}\mathbf{q}_j, \quad I\mathbf{q}_k = \mathbf{q}_k I = \mathbf{q}_k \quad (2)$$

(small Latin indices are 3D, δ_{km} and ε_{kmj} are the Kronecker and Levi-Civita tensors, summing in repeated indices is assumed). The vector units $\mathbf{q}_k$ geometrically behave as a triad, an orthogonal frame initiating a 3D Cartesian system of coordinates. It has been found recently [5] that the scalar (real) unit I may be associated with a metric g of a locally flat 2D fractal space ("square root" from 3D space) endowed with a dyad ψ^+, ψ^-:

$$g\psi^\pm\psi^\pm \equiv \varphi^\pm\psi^\pm = 1, \quad g\psi^\pm\psi^\mp \equiv \varphi^\pm\psi^\mp = 0;$$

one readily shows that all units of the type (1) are built out of a single fractal dyad as different tensor products:

$$\begin{gathered} I = \psi^+\varphi^+ + \psi^-\varphi^-, \\ \mathbf{q}_1 = -\mathrm{i}(\psi^+\varphi^- + \psi^-\varphi^+), \\ \mathbf{q}_2 = \psi^+\varphi^- - \psi^-\varphi^+, \\ \mathbf{q}_3 = \mathrm{i}(\psi^+\varphi^+ - \psi^-\varphi^-). \end{gathered} \quad (3)$$

Multiplication of the dyad by a complex number (the simplest transformation)

$$\begin{gathered} \psi'^\pm \equiv \sigma \mathrm{e}^{\pm \mathrm{i}\alpha}\psi^\pm \equiv \lambda\psi^\pm, \\ \varphi'^\pm \equiv \sigma \mathrm{e}^{\mp \mathrm{i}\alpha}\varphi^\pm \equiv \lambda^*\varphi^\pm, \end{gathered}$$

$\sigma \in \mathbb{R}$, $\sigma \neq 0$, rotates the triad by the angle 2α, but makes all four units differ from unity, e.g., $\psi'^+\varphi'^+ + \psi'^-\varphi'^- = \sigma^2 I$, $\psi'^+\varphi'^- - \psi'^-\varphi'^+ = \sigma^2\mathbf{q}_2$, thus violating the basic law (2). To save the algebra, we consider the transformation parameters, modulus σ and phase α, to be functions of a free variable θ and of the coordinate set ξ_A defined on an abstract M-dimensional space ($A = 1, ..., M$), and then we introduce the normalizing integral over a volume V_ξ of the space

$$f(\theta) = \int_{V_\xi} \lambda\lambda^* dV_\xi = \int_{V_\xi} \sigma^2 dV_\xi = 1. \quad (4)$$

Using Eq. (4) as a factor in the right-hand-side in Eqs. (3), e.g.,

$$\begin{gathered} f(\psi^+\varphi^+ + \psi^-\varphi^-) = I' = I, \\ f(\psi^+\varphi^- - \psi^-\varphi^+) = \mathbf{q}_{2'}, \end{gathered}$$

we restore all the units. If the normalizing integral (4) is constant, $\partial_\theta f = 0$, then all algebras in question remain valid for any value of θ; this leads to the continuity-type equation

$$\partial_\theta(\lambda\lambda^*) + \nabla_\xi(\lambda\lambda^*\mathbf{k}) = 0, \quad (5)$$

where $\mathbf{k}$ is an arbitrary vector. If it points to the direction of phase increase (i.e., of the angle of the triad rotation), $\mathbf{k} = \nabla_\xi\alpha$, then Eq. (5) acquires its fractal format. Here we give an expanded version of its derivation, never demonstrated in the literature before.

For simplicity (though not on account of generality), let the abstract space be 3D, so that Eq. (5) is rewritten with 3D indices:

$$\partial_\theta(\lambda\lambda^*) + \partial_n(\lambda\lambda^* k_n) = 0, \quad (6)$$

where

$$\lambda \equiv \sigma(\theta, \xi_m)e^{i\alpha(\theta,\xi_n)}, \quad (7)$$

so that $\alpha = (i/2)\ln(\lambda^*/\lambda)$. Since $k_n \equiv \partial_n\alpha$, we find

$$k_n = \frac{i}{2}\left(\frac{\partial_n\lambda^*}{\lambda^*} - \frac{\partial_n\lambda}{\lambda}\right) \quad (8)$$

and insert Eqs. (7) and (8) into Eq. (6) to get

$$\begin{gathered} e^{-i\alpha}[\partial_\theta(\sigma e^{i\alpha}) - \frac{i}{2}\partial_n\partial_n(\sigma e^{i\alpha})] \\ + e^{i\alpha}[\partial_\theta(\sigma e^{-i\alpha}) - \frac{i}{2}\partial_n\partial_n(\sigma e^{-i\alpha})] = 0. \end{gathered} \quad (9)$$

Performing all needed derivations, we arrive at the equation

$$\begin{gathered} \partial_\theta\sigma + i\sigma\partial_\theta\alpha - \frac{i}{2}\partial_n\partial_n\sigma + \partial_n\sigma\partial_n\alpha \\ + \frac{1}{2}\sigma\partial_n\partial_n\alpha + \frac{i}{2}\sigma\partial_n\alpha\partial_n\alpha \\ + \partial_\theta\sigma - i\sigma\partial_\theta\alpha + \frac{i}{2}\partial_n\partial_n\sigma + \partial_n\sigma\partial_n\alpha \\ + \frac{1}{2}\sigma\partial_n\partial_n\alpha - \frac{i}{2}\sigma\partial_n\alpha\partial_n\alpha = 0, \end{gathered} \quad (10)$$

comprising two couples of identical parts, a real one [it doubles in Eq. (10) and is equaled to zero],

$$\partial_\theta\sigma + \partial_n\sigma\partial_n\alpha + \frac{1}{2}\sigma\partial_n\partial_n\alpha = 0, \quad (11)$$

and an imaginary one [denoted by $W\sigma$; in Eq. (10) it vanishes in sum]

$$i(\sigma\partial_\theta\alpha - \frac{1}{2}\partial_n\partial_n\sigma + \frac{1}{2}\sigma\partial_n\alpha\partial_n\alpha) \equiv -iW\sigma. \quad (12)$$

Altogether there remain one *equation* (11) and one *definition* (12), respectively, rewritten below as

$$\partial_\theta\sigma + \partial_n\sigma\partial_n\alpha + \frac{1}{2}\sigma\partial_n\partial_n\alpha = 0, \quad (13)$$

$$\sigma\partial_\theta\alpha - \frac{1}{2}\partial_n\partial_n\sigma + \frac{1}{2}\sigma\partial_n\alpha\partial_n\alpha + W\sigma = 0, \quad (14)$$

with $W(\theta, \xi_n)$ being an arbitrary function. The last set of equations can be recombined into one complex-number equation as [Eq. (13) $+i$ Eq. (14)] $e^{i\alpha}$, the result having the form

$$\left[\partial_\theta - \frac{i}{2}\partial_n\partial_n + iW\right](\sigma e^{i\alpha}) = 0. \qquad (15)$$

We recall here that all the above equations are pure mathematical, hence abstract, no physical units are involved. But we can introduce a physical length (space coordinate) $x_k \equiv \varepsilon\xi_k$ and time $t \equiv \tau\theta$ using certain standards $\varepsilon = \hbar/(mV)$, $\tau = \varepsilon/V = \hbar/(mV^2)$, where $\hbar$ is some angular momentum (e.g., Planck's constant), m is a particle's mass, V is a characteristic velocity. In variables x_k, t Eq. (15) becomes precisely the Schrödinger equation with the potential $U \equiv (mV^2)W$, while Eqs. (13, 14) take the form of the Bohm equations [6], i.e., a decomposition of the Schrödinger equations into real and imaginary parts.

So the algebras' stability condition (5), (6), formulated in abstract M-dimensional (or 3D) space, has its fractal (square-root) analog (13), (14) or (15), describing (as it surprisingly turns out) in physical units a quantum particle characterized by the wave function $\psi'^{+} = \sigma e^{i\alpha}\binom{0}{1}$ and affected by the field of an arbitrary potential. The wave-function absolute value is treated here as a fractal relative mass density $\sigma \equiv \sqrt{\rho(x,t)/\rho_{\text{mean}}}$, so that the integral (4) has the meaning of the particle's mass,

$$\int\limits_V \rho dV = \rho_{\text{mean}}\varepsilon^3 = m.$$

Then Eq. (13) of the fractal set in physical units (the first Bohm equation)

$$\partial_t\sigma + \frac{\hbar}{m}\partial_n\sigma\partial_n\alpha + \frac{\hbar}{2m}\sigma\partial_n\partial_n\alpha = 0 \qquad (16)$$

is a square root from the fundamental mass conservation law $\partial_t\rho + \partial_n(\rho u_n) = 0$, while Eq. (14) in physical units (the second Bohm equation)

$$\hbar\partial_t\alpha + \frac{\hbar^2}{2m}(\partial_n\alpha)(\partial_n\alpha) + U - \frac{\hbar^2}{2m}\partial_n\partial_n\sigma/\sigma = 0 \qquad (17)$$

is in fact a definition of the potential U. There are two ways to treat Eq. (17). Either it is a conventional quantum-mechanical equation for a particle in a field $U(t, x_n)$, or, if $\sigma = \text{const}$ (or if $\Delta\sigma - (2m/\hbar^2)U_{\text{int}}\sigma = 0$) it is a Hamilton-Jacoby equation of a classical particle

$$\partial_t S + \frac{(\nabla S)^2}{2m} + U_{\text{ext}} = 0$$

with

$$S = \hbar\alpha, \qquad (18)$$

and $U = U_{\text{int}} + U_{\text{ext}}$ the potential parts responsible for an interior and exterior effect on the particle.

There are three essential points worth emphasizing at the end of this section. (i) The Schrödinger and Hamilton-Jacoby equation are in fact fractal equations of mechanics, formulated in the quantum case for a full fractal function (a square-root of a spoiled algebra unity) ψ', or, in the classical case, for its phase part only, known in classical mechanics as the action. (ii) The same particle admits two different models, the first one is a 2D fractal function ψ' (or a scalarλ) having discrete (σ) and wave (α) properties, the second one is a pointlike 3D particle with frozen-in triad rotation by the angle equal to the doubled phase. (iii) A specific feature of the particle in question is its necessary intrinsic rotation, i.e., possession of a proper angular momentum. The latter property evokes the concept of a virtual ring and compels to form an alternative view at quantum characteristics of particle motion.

3. A PARTICLE'S VIRTUAL RING MODEL

Consider a pointlike particle (a compact 3D object) of mass m inertially moving in physical space with a velocity $V = \text{const}$. The crucial point of further considerations is that the object must be simply rotating (about one axis) with the angular frequency $\Omega = \text{const}$, the rotation being described by a frozen-in triad, so that, e.g., $\mathbf{q}_3 = \text{const}$. We would like to stress here that the particle can either be really rotating (e.g., as a bullet or a planet on their trajectories), or otherwise the interior rotation frequency Ω can be artificially ascribed to it for purposes of a specific description (or measurement) of the motion. For the rotational period $T = 2\pi/\Omega$ the particle covers the path

$$l = VT = 2\pi V/\Omega \qquad (19)$$

that can be associated with the particle's characteristic "wavelength." These conditions lead to a specific model of the particle's motion and endow it with a set of "quantum" characteristics. Indeed, Eq. (19) can be rewritten in the form $l = 2\pi r$, the velocity-frequency link $V = \Omega r$ is defined by

$$r \equiv l/(2\pi) = V/\Omega, \qquad (20)$$

taken for the radius of a virtual circumference of length l; then the respective particle's "wave number" is

$$k \equiv 2\pi/l = 1/r. \qquad (21)$$

Provided the mass m is uniformly (for simplicity) distributed along the circumference, we obtain a massive virtual ring, rotating with the frequency Ω, each its point moving with linear velocity V, the ring's proper angular momentum being

$$L = mr^2\Omega = mrV = \text{const.} \tag{22}$$

Taking L, k, Ω for basic constants, we can express through them the particle's main dynamic characteristics, the values of its momentum

$$p = mV = L/r = kL \tag{23}$$

and energy

$$E = mV^2/2 = L\Omega/2 = \omega L, \tag{24}$$

where, according to the results of Section 2, $\omega = \Omega/2$ is the phase frequency of the wave $\lambda \sim e^{i\alpha}$, $\alpha = \omega t$ describing, on the 2D fractal surface, the 3D rotation of the ring.

Thus both 3D physical and 2D fractal characteristic of particle motion are expressed through the parameters of the virtual ring, precisely as they were formulated in de Broglie's postulates, provided $L \to \hbar$.

Moreover, Eqs. (20)–(24) in a way explain the mechanical Lagrangian formula as a difference between the kinetic and potential energies; indeed, for a particle with constant momentum and energy, the phase of its fractal functions has the De Broglie format

$$\alpha = kx - \omega t = \frac{1}{L}(px - Et) =$$
$$= \frac{1}{L}(2E_{kin} - E)t = \frac{1}{L}(E_{kin} - U)t = \frac{S}{L}, \tag{25}$$

so the Lagrangian is $\Lambda \equiv E_{kin} - U$, as postulated in manuals. As well, Eq. (25) states that, due to the wave-type form of the phase, the virtual ring is in general supposed to rotate in two opposite directions simultaneously, counterclockwise ("positively") with the frequency $2px/L = 2\Omega$, and clockwise ("negatively") with the frequency $-2\omega = -\Omega$, the sum being Ω; in particular, this means that the physical particle propagates with a double phase velocity computed for the fractal function, $V = \Omega r = 2\omega/k = 2V_\varphi$.

We also note that the energy of the ring's rotation given by Eq. (24) is equal to the kinetic energy of particle translation, no particle's proper rotational energy taken into account. This fact additionally emphasizes that the virtual ring model is just a different way to mathematically describe the same physical phenomenon, translational motion of an object (though with necessary intrinsic rotation prescribing the model's frequency). But the description of space motion in this approach appears to be quite specific because this rotating ring cannot be considered somehow moving along the trajectory as a real physical object, e.g., rolling with the velocity V, since in this case its energy would be twice as great as that of the particle. So one has to admit that the ring (only imagined with a particle) must be regarded as propagating along the trajectory discretely; i.e., during one rotation period T the ring (occupying the length $2r$ on the orbit) only rotates without translation in space, and straight after completing one full revolution it is instantly replaced at the wavelength (19) to start the next rotation. Meanwhile the particle's momentum and energy are linked to the ring's proper angular momentum by the de Broglie ratios $p = kL$, $E = \omega L$ with $k \equiv 1/r$, r being the radius of the virtual ring. One straightforwardly finds that the function $\lambda = \sigma e^{i\alpha}$ with the phase (25) is a solution to the Schrödinger-type equation

$$\left(iL\partial_t + \frac{L^2}{2m}\nabla^2 - U\right)\lambda = 0$$
$$\Rightarrow E - \frac{p^2}{2m} = 0 \tag{26}$$

for a free and structureless particle $U = 0$, $\sigma = \text{const.}$

4. DISCUSSION

Of course the virtual ring concept is just a math model that represents a classical object allegedly possessing quantum properties. This model could hardly emerge without the fractal surface math theory and the possibility to describe the same object as a 2D fractal and 3D physical entity. This remarkably distinguishes the ring model from heuristic though genius suggestions of the quantum mechanics creators.

But there are a couple of aspects of the theory that seem to be worth discussing.

First, the virtual ring approach may be looked at as a method of measurement of the physical systems' characteristics by specific units L, k, ω. Apart from momentum and energy given by Eqs. (23), (24), these units produce derived standards for the length $l = 2\pi/k$ ("wavelength") and time $T = 4\pi/\omega$ (period of the ring rotation). Since this method is based on counting discreet ring positions on a trajectory, the measurement of orbital lengths of finite motion (e.g., that of circular orbits) immanently implies an integer number of such positions to provide coincidence of the last ring position with the initial one. Precisely this is done in the quantum model of an H-atom postulated by Bohr, but later shown to be a solution of the Schrödinger equation [7]; the Planck's constant was there taken for an angular momentum unit, while the characteristics of the first energy level were chosen for units of frequency and length. The respective classical problem for a star or a planetary system considered from the virtual ring model viewpoint seems promising.

The second aspect distinguishing the ring model approach is its clear formulation for the case of inertial motion; this feature together with a discrete translation of the virtual ring hints at a description of an arbitrary motion as a series of inertial segments with discreetly changing dynamical characteristics. As a challenging example, a virtual ring model for the classical problem of a linear harmonic oscillator will be analyzed in the forthcoming paper.

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