ISSN 0202-2893, Gravitation and Cosmology, 2016, Vol. 22, No. 3, pp. 230–233. © Pleiades Publishing, Ltd., 2016.

Relativistic Algebra of Space-Time and Algebrodynamics

V. V. Kassandrov[1]* and J. A. Rizcallah[2]**

[1]*Institute of Gravitation and Cosmology, Peoples' Friendship University of Russia, Moscow, Russia*

[2]*School of Education, Lebanese University, Beirut, Lebanon*

Received May 10, 2016

Abstract—We consider a manifestly Lorentz-invariant form $\mathbb{L}$ of the biquaternion algebra and its generalization to the case of a curved manifold. The conditions of $\mathbb{L}$-differentiability of $\mathbb{L}$-functions are formulated and considered as the primary equations for fundamental fields modeled with such functions. The exact form of the effective affine connection induced by $\mathbb{L}$-differentiability equations is obtained for flat and curved manifolds. In the flat case, the integrability conditions of the connection lead to self-duality of the corresponding curvature, thus ensuring that the source-free Maxwell and $SL(2,\mathbb{C})$ Yang-Mills equations hold on the solutions of the $\mathbb{L}$-differentiability equations.

DOI: 10.1134/S0202289316030087

1. LORENTZ-INVARIANT ALGEBRA OF A FLAT OR CURVED SPACE-TIME MANIFOLD

To construct a truly unified theory, one should rely on an exceptional geometric structure. On the other hand, the diversity of geometries of different dimensions, topologies or differential structures does not allow for a trustful choice of the candidate for (an extended) space-time geometry. If, however, an *algebraic* structure is laid in the foundation of the theory, the situation becomes much better, since there exist only a finite number of exceptional Lie groups or finite-dimensional linear algebras, the latter being exhausted by complex numbers, quaternions and (non-associative) octonions.

Since the times of Hamilton, it is well known that the Euclidean structure of 3D physical space can be regarded as a direct consequence of the existence of the exceptional quaternion algebra with its group of automorphism $SO(3)$. A lot of effort has been made to relate the structure of Minkowski space-time $\mathbf{M}$ to the properties of complex quaternions (*biquaternions*) $\mathbb{B}$, whose symmetry group $SO(3,\mathbb{C})$ is 2:1 isomorphic to the spinor Lorentz group $SL(2,\mathbb{C})$, see, e.g., the review [1]. However, the $4\mathbb{C}$ dimension of this algebra corresponds to the structure of a complexified space-time, in which $\mathbf{M}$ does not even constitute a subalgebra, but only a subspace.

In [2], an interesting 4D algebra $\mathbb{G}$ has been proposed with a manifestly Lorentz-covariant multiplication law:

$$(a\circ b)^\mu = a^\mu(b^\rho e_\rho) + b^\mu(a^\rho e_\rho) - e^\mu(a^\rho b_\rho) \pm \imath\epsilon^{\mu}_{.\nu\rho\lambda}a^\nu b^\rho e^\lambda, \quad (1)$$

where the ordinary notation using the Minkowski metric $\eta_{\mu\nu}$ is employed, and $\mu = 0,1,2,3$. For the distinguished element $e=\{e^\mu\}$ of $\mathbb{G}$, the constraint $e^\mu e_\mu = 1$ is imposed; then e plays the role of a *unit* element: $a\circ e = e\circ a = a, \forall a\in\mathbb{G}$. It is also straightforward to verify that $\mathbb{G}$ is an *associative* algebra.

Surprisingly, the fact that $\mathbb{G}$ is isomorphic to the algebra of biquaternions $\mathbb{B}$ was overlooked in [2]. Indeed, with the choice for the unit element $e^\mu = \{1,0,0,0\}$, the law (1) reproduces ordinary multiplication in $\mathbb{B}$. Precisely, for the basis vectors σ_μ, Eq. (1) yields

$$\sigma_\mu\circ\sigma_\nu = \sigma_\mu e_\nu + \sigma_\nu e_\mu - (e^\rho\sigma_\rho)\eta_{\mu\nu} \pm \imath\epsilon^{\rho}_{.\mu\nu\lambda}\sigma_\rho e^\lambda, \quad (2)$$

with $\sigma_0 = e$, and

$$\sigma_a\circ\sigma_b = \delta_{ab}e \pm \imath\epsilon_{abc}\sigma_c, \quad (3)$$

for the three space-like basis vectors σ_a, $a=1,2,3$, in full accord with the multiplication of the *Pauli matrices*. The different signs in (1) correspond to the left or right forms of the (bi)quaternion algebra. Under Lorentz boosts the unit element e transforms as a 4-vector while the defining law (1) preserves its form. Moreover, 3-rotations are canonical automorphisms of $\mathbb{G}$.

The representation (1) is very useful for generalizations to curved manifolds [3, 4]. To this end, we consider the *tetrad field* $h^\alpha_\mu(x)$, $\alpha,\beta,\ldots = 0,1,2,3,$

*E-mail: vkassan@sci.pfu.edu.ru

**E-mail: joeriz68@gmail.com

and the *local algebra*[1] $\mathbb{L}$ defined by the basis vectors $\Sigma_\mu(x) := h^\alpha_\mu(x)\sigma_\alpha$ depending on the point of the manifold. The multiplication table (2) for $\mathbb{L}$ takes then the obvious form

$$\Sigma_\mu \circ \Sigma_\nu = \Sigma_\mu E_\nu + \Sigma_\nu E_\mu - (E^\rho \Sigma_\rho) g_{\mu\nu} \pm \imath\sqrt{-g}\epsilon^{\rho}_{.\mu\nu\lambda}\Sigma_\rho E^\lambda, \quad (4)$$

in which a unit field $E_\mu(x) := h^\alpha_\mu e_\alpha$, satisfying $g^{\mu\nu}E_\mu E_\nu = 1$, and a metric tensor of the manifold $g_{\mu\nu} := h^\alpha_\mu h^\beta_\nu \eta_{\alpha\beta}$ ($g := |g_{\mu\nu}|$) naturally arise.

The existence of the local algebra (4) on a (complexified) 4D manifold requires, apart from the metric tensor, an algebro-geometric structure—the unit time-like 4-vector field (*U-field*) $E^\mu(x)$[2]. Its physical interpretation may be related to a matter flow, etc. Below, we shall see that the properties of $\mathbb{L}$-valued functions differentiable over $\mathbb{L}$ can impose strong restrictions on the metric and the U-field, and, in a sense, determine the geometry of the manifold itself. First and foremost, however, they define a set of relativistic fields and guarantee the fulfillment of the corresponding field equations.

2. ANALYSIS AND INDUCED FIELD DYNAMICS ON THE RELATIVISTIC ALGEBRA OF SPACE-TIME

In the algebrodynamic program (see, e.g., [5, 7, 8] and references therein) one considers some algebraic structure (“space-time algebra”) $\mathbb{A}$ which predetermines both the physical geometry and the equations of fundamental fields. Specifically, one should formulate the *differentiability conditions* for $\mathbb{A}$-valued functions, in close analogy to the Cauchy-Riemann conditions for holomorphic functions of a complex variable. For quaternion-like, non-commutative yet associative algebras, appropriate conditions have been proposed in [5, 7] in the following Pfaffian form:

$$dF = \Phi \circ dZ \circ \Psi, \quad (5)$$

in which $F(Z)$, $\Phi(Z)$, $\Psi(Z)$ are the principal $\mathbb{A}$-function of $\mathbb{A}$-variable Z, and two auxiliary $\mathbb{A}$-functions (the so-called left and right *semi-derivatives*), respectively. dF denotes the linear part of the increment (differential) of $F(Z)$, while ($\circ$) is the operation of multiplication in $\mathbb{A}$.

[1] The concept of a local algebra has been introduced in [5, ch. 2] and elaborated further in [3, 4]; the analogous concept of the so-called $\mathbb{Q}$-basis was considered in [6].

[2] The situation resembles that in Weyl geometry which, together with the metric tensor, is defined by the nonmetricity 1-form field (identified by H. Weyl with the electromagnetic potentials).

Thus, an $\mathbb{A}$-function $F(Z)$ is called differentiable over $\mathbb{A}$ if its increment can be represented in an invariant form (5), i.e., only through the operation of multiplication in $\mathbb{A}$.

For a commutative complex algebra, the above condition reduces to $dF = (\Phi \circ \Psi) \circ dZ$, and, being written out in components, is equivalent to the Cauchy-Riemann equations. For real quaternions $\mathbb{Q}$, (5) proves to be a condition of *conformity* of the corresponding mapping $Z \mapsto F(Z)$ in $\mathbf{E}^4$, in full analogy with the complex case. However, the class of such mappings is known to be very restricted, defined by 15 parameters only (the *Liouville theorem*). Fortunately, upon complexification of $\mathbb{Q}$, that is, transition to the algebra of biquaternions $\mathbb{B}$, the class of solutions to (5) substantially widens, on the account of elements $\Phi(Z)$ and/or $\Psi(Z)$ with *null* norm.

Therefore, the $\mathbb{B}$-algebra (or, equivalently, the isomorphic $\mathbb{G}$-algebra) does constitute a basis of an algebrodynamic theory. $\mathbb{B}$-differentiable functions of the $\mathbb{B}$-variable should then be considered as the primary physical fields (together with the corresponding semi-derivatives) while the field equations are represented by the differentiability conditions (5) or secondary constraints following from the latter (e.g., via successive differentiations, etc.). In this approach, *particles* are modeled as singularities of the $\mathbb{B}$-fields.

To avoid problems with the complex extension of space-time (related to the complex 4$\mathbb{C}$ structure of the vector space of $\mathbb{B}$-algebra), in our previous papers we restricted the coordinate space to the subspace with Minkowski metric. In the matrix representation of $\mathbb{B}$ this corresponds to Hermitian matrices,

$$Z \mapsto X = X^+ = x^\mu \sigma_\mu, \ \{x_\mu\} \in \mathbb{R}, \quad (6)$$

while the components of the fundamental fields $F(X)$, $\Phi(X)$ and $\Psi(X)$ are generally assumed to be complex-valued. Finally, we regard the $\mathbb{B}$-differentiability conditions on the Minkowski coordinate subspace

$$dF = \Phi(X) \circ dX \circ \Psi(X) \quad (7)$$

as the only constraints to determine the corresponding functions-fields and their singular locus, which is identified with particle-like formations.

Using the $SL(2,\mathbb{C})$ matrix representation of the $\mathbb{B}$-algebra, it was proved [5, 7] that any matrix component $\varphi = F^B_A(x)$ of the $\mathbb{B}$-field satisfies, in view of (7), the complex eikonal equation

$$\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 0, \quad (8)$$

instead of the linear Laplace equation for complex functions. In general, the set of PDE corresponding to (7) is Lorentz-invariant and *nonlinear* (the latter is a direct consequence of the noncommutativity of $\mathbb{B}$-algebra).

For the most important [9] case where $\Psi(x) \equiv F(X)$ (equivalently, one can take $\Phi(X) \equiv F(X)$), after spinor splitting of (7), we obtain

$$d\xi = \Phi dX \xi, \tag{9}$$

for the 2-spinor $\xi = \{\xi_A(x)\}$ and the complex 4-vector $\Phi = \{\Phi_{AB'(x)}\}$ fields $(A, A', ... = 0, 1)$. Both $\xi_A(x)$ and $\Phi_{AB'}(x)$ can be found from the overdetermined structure of the set of equations (9).

In particular, the integrability conditions of (9) read

$$dd\xi = R\xi = 0, \quad R := (d\Phi - \Phi dX \Phi) \wedge dX, \tag{10}$$

where the matrix-valued 2-form R can be regarded as the *curvature 2-form* of the matrix-valued *connection 1-form* $\Omega := \Phi dX$ entering into the initial equations (9). The latter can thus be interpreted as the conditions for the 2-spinor field $\xi(X)$ to be *covariantly constant* w.r.t. the complex affine connection Ω,

$$d\xi - \Omega\xi = 0. \tag{11}$$

Let us now return to the integrability conditions (10). Since the spinor $\xi(x)$ is not arbitrary, the curvature R does not vanish identically. Thus, the $\mathbb{B}$-differentiability conditions define *dynamically*, on the flat Minkowski background, a non-trivial geometric structure, the *complex curved 4D space* with the affine connection Ω. Moreover, it has been demonstrated in [7, 10] that the spinor components can be eliminated from (10), and the curvature R turns out to be *self-dual* on the solutions of (9). Specifically, one obtains from (10):

$$(\vec{\mathbf{R}})_a := R_{oa} + \frac{\imath}{2}\epsilon_{abc}R_{bc} = 0, \tag{12}$$

with the following structure of the self-dual part of the curvature:

$$\vec{\mathbf{R}} = \vec{P} + D\vec{\sigma} - \imath\vec{P} \times \vec{\sigma}, \tag{13}$$

where the quantities $\vec{P} := \vec{E} + \imath\vec{H}$ and D are defined through the components of the 4-vector field $\Phi = A^\mu(x)\sigma_\mu$ as follows:

$$\vec{E} := -\partial_o\vec{A} - \nabla A_o, \;\; \vec{H} := \nabla \times \vec{A},$$
$$D := \partial_\mu A^\mu + 2A_\mu A^\mu, \tag{14}$$

and represent, consequently, the components of 4-potentials and field strengths of an effective complex electromagnetic field. Now, from the full self-duality condition (12) and the curvature structure (13), the self-duality of the electromagnetic field follows immediately,

$$\vec{P} = \vec{E} + \imath\vec{H} = 0, \tag{15}$$

together with the "inhomogeneous Lorentz condition" $D = 0$.

In turn, the complex self-duality condition (15) guarantees the source-free Maxwell equations, separately for the (mutually dual) real and imaginary parts of the electromagnetic fields (14). Moreover, for two independent components $\psi(x)$ of the potential matrix $\Phi(X)$ (the other two can always be nullified by a gauge transformation) *the 2-spinor Weyl equation holds for any solution of (9)* (for details, see [11]). Remarkably, the $SL(2, \mathbb{C})$ Yang-Mills fields can be also defined through the same matrix field $\Phi(X)$, and, on the solutions of (9), satisfy the correspondent source-free equations (for proofs and details we refer the reader to [7, 10]).

From a 4-vector perspective, the principal equation (9) (complemented from the 2-spinor $\xi(X)$ to corresponding *null* 4-vector $F = F^\mu\sigma_\mu$) reads

$$\partial_\nu F = \Phi \circ \sigma_\nu \circ F, \tag{16}$$

or, in components,

$$\partial_\nu F^\rho\sigma_\rho = A^\mu F^\rho\sigma_\mu \circ \sigma_\nu \circ \sigma_\rho. \tag{17}$$

Using then (2) to evaluate the product in the r.h.s., we find

$$\partial_\nu F^\rho = \Gamma^\rho_{\nu\mu}F^\mu, \tag{18}$$

with the connection $\Gamma^\rho_{\mu\nu}$ being

$$\Gamma^\rho_{\mu\nu} = \delta^\rho_\nu A^\alpha(2e_\mu e_\alpha - \eta_{\mu\alpha}) - A_\nu\eta^{\beta\rho}(2e_\mu e_\beta - \eta_{\mu\beta})$$
$$\pm\, \imath\{\epsilon^\rho_{.\alpha\nu\gamma}e_\mu + \epsilon^\rho_{.\alpha\mu\gamma}e_\nu - \epsilon^\rho_{.\nu\mu\gamma}e_\alpha + \epsilon_{\alpha\nu\mu\gamma}e^\rho\}e^\gamma A^\alpha$$
$$+ A^\rho(2e_\nu e_\mu - \eta_{\nu\mu}). \tag{19}$$

In this form, Eqs. (18) and (19) define a covariantly constant 4-vector field and can be readily generalized to a curved metric-affine space. Specifically, instead of (5), we now can deal with the $\mathbb{L}$-differentiability conditions of the form (in the principal case when $\Psi(X) \equiv F(X)$)

$$DF = \Phi(X) \circ dX \circ F(X), \tag{20}$$

with DF being the *covariant* differential w.r.t. the metric $g_{\mu\nu}$ (that is, w.r.t. the Levi-Civita connection γ)[3]. Again, (20) can be interpreted as the condition for a 4-vector $F(X)$ to be covariantly constant w.r.t the connection

$$\Gamma = \gamma + G, \tag{21}$$

where $G := \{G^\rho_{\mu\nu}\}$ is the connection of the form (19) generalized, in a natural way, to the case of the local $\mathbb{L}$-algebra (4):

$$G^\rho_{\mu\nu} = \delta^\rho_\nu A^\alpha(2E_\mu E_\alpha - g_{\mu\alpha})$$

[3] A more sophisticated and mathematically substantiated approach to the generalization of the differentiability conditions (5) has been elaborated in [4].

$$- A_\nu g^{\beta\rho}(2E_\mu E_\beta - g_{\mu\beta})$$
$$\pm \imath\sqrt{-g}\{\epsilon^{\rho}_{.\alpha\nu\gamma}E_\mu + \epsilon^{\rho}_{.\alpha\mu\gamma}E_\nu - \epsilon^{\rho}_{.\nu\mu\gamma}E_\alpha$$
$$+ \epsilon_{\alpha\nu\mu\gamma}E^\rho\}E^\gamma A^\alpha + A^\rho(2E_\nu E_\mu - g_{\nu\mu}). \quad (22)$$

It follows from (22) that the generalized $\mathbb{L}$-connection is not symmetric in low indexes and thus possesses *torsion* of rather a specific form. On the other hand, calculating the covariant derivative of the metric tensor w.r.t. the full connection (21), one has:

$$\nabla_\rho g_{\mu\nu} = -2g_{\mu\nu}\tilde{A}_\rho, \quad (23)$$

with $\tilde{A}_\rho := (2E_\rho E_\lambda - g_{\rho\lambda})A^\lambda$. Thus, the connection (22) under consideration possesses a *nonmetricity* of the Weyl type, and we are now in a position to identify the effective space, dynamically induced by the primary algebraic structure, as a *Weyl-Cartan manifold* [7].

It is worth noting that the structure of $\mathbb{L}$-connection (22) and the covariant derivative of the metric (23) essentially involve the effective metric tensor

$$\tilde{g}_{\rho\lambda} := 2E_\rho E_\lambda - g_{\rho\lambda}, \quad (24)$$

which, rather surprisingly, in the "flat" limit of $\mathbb{B}$-algebra and connection (19) reduces to the metric of 4D *Euclidean* (!) space and preserved its Euclidean signature upon generalization to (22).

Moreover, a third metric $g^*_{\mu\nu}(X)$ defined algebraically via the *structure functions* $C^\rho_{\mu\nu}(X)$ of the local $\mathbb{L}$-algebra,

$$C^\rho_{\mu\nu} = E_\mu\delta^\rho_\nu + E_\nu\delta^\rho_\mu - E^\rho g_{\mu\nu}$$
$$\pm \imath\sqrt{-g}\epsilon^{\rho}_{.\mu\nu\lambda}E^\lambda, \quad (25)$$

and thus invariant under the automorphisms of $\mathbb{L}$ *turns out to coincide with (24)* (at least in the case $g = |g_{\mu\nu}| = |\eta_{\mu\nu}| = -1$):

$$g^*_{\mu\nu} := \frac{1}{4}C^\beta_{\mu\alpha}C^\alpha_{\nu\beta} \equiv \tilde{g}_{\mu\nu}. \quad (26)$$

The complete meaning of this effective metric and consequences of the above obtained remarkable coincidence certainly deserve further investigation.

In the light of all the above-said, it is not unreasonable to assume that the integrability conditions of $\mathbb{L}$-differentiability equations (20) might impose restrictions not only on the vector field A_μ, but also on the metric $g_{\mu\nu}$ as well as the unit vector field E_μ. However, the task of analyzing these conditions and the equations they yield for the metric and fields is left for future work.

3. CONCLUSION

We study a manifestly covariant form of the biquaternion algebra $\mathbb{B}$, which coincides with the relativistic ring extension proposed in [2]. We consider the $\mathbb{B}$-differentiability conditions of $\mathbb{B}$-valued functions and suggest to regard them as the fundamental generating system for the set of fundamental physical fields. In the most important case, these conditions admit a geometrical interpretation as those defining a covariantly constant vector field w.r.t. the affine connection of a very specific form. Their integrability conditions lead then to self-duality of the corresponding curvature, which in turn yields Maxwell, Yang-Mills and Weyl equations for the associated fields. It is hoped that $\mathbb{L}$-generalization of the $\mathbb{B}$-algebra and $\mathbb{B}$-differentiability equations to Riemannian or general metric-affine space-times will enable us, in addition, to obtain sufficient constraints on the connection, metric and unit U-field and thus, in a purely algebraic way, to determine physical geometry.

REFERENCES

1. A. Gsponer and J.-P. Hurni, math-ph/0510059; math-ph/0511092
2. E. Grgin, Phys. Lett. B **431**, 15 (1998)
3. V. N. Trishin, Vestnik RUDN University **10**, 13 (2002); V. V. Kassandrov and V. N. Trishin, in *Proc. Int. Conf. "Theoretical and experimental problems of GR and gravitation"* (Tomsk, 2002), p. 61.
4. V. N. Trishin, *Candidate's Dissertation in Mathematics and Physics* (RUDN University, Moscow, 2004)
5. V. V. Kassandrov, *Algebraic Structure of Space-Time and Algebrodynamics* (RUDN University, Moscow, 1992) [in Russian].
6. A. P. Yefremov, Grav. Cosmol. **2** (1), 77 (1996); A. P. Yefremov, Grav. Cosmol. **2** (4), 335 (1996); A. P. Yefremov, *Quaternionic Spaces, Reference Frames and Fields* (RUDN University, Moscow, 2005) [in Russian].
7. V. V. Kassandrov, Grav. Cosmol. **3**, 216 (1995); arXiv:gr-qc/0007027.
8. V. V. Kassandrov, Phys. Atom. Nuclei **72**, 813 (2009); arXiv:0907.5425.
9. J. A. Rizcallah, *Candidate's Dissertation in Mathematics and Physics* (RUDN University, Moscow, 1999).
10. V. V. Kassandrov and J. A. Rizcallah, gr-qc/0012109.
11. V. V. Kassandrov, in *Has the Last Word been Said on Classical Electrodynamics?* Eds. A. Chubykalo, V. Onoochin, A. Espinoza, and V. Smirnov-Rueda. (Rinton Press, 2003), p. 42; physics/0308045.