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On Exponential Solutions in the Einstein–Gauss–Bonnet Cosmology, Stability and Variation of G[1]

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Abstract—A D-dimensional gravitational model with Gauss–Bonnet and cosmological terms is considered. When an ansatz with a diagonal cosmological metric is adopted, we find new examples of solutions for $\Lambda \neq 0$ and $D = 8$ with an exponential dependence of the scale factors, which describe expansion of our 3D factor-space and contraction of 4D internal space. We also study the stability of the solutions with static Hubble-like parameters h^i and prove that two solutions with $\Lambda = 0$ in dimensions $D = 22, 28$, which were found earlier, are stable. For both solutions we find asymptotic relations for the effective gravitational constant.

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1. INTRODUCTION

In this paper we consider a D-dimensional gravitational model with Gauss–Bonnet and cosmological terms. The action reads

$$S = \int_M d^D z \sqrt{|g|} \{\alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g]\}, \quad (1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M, $\dim M = D$, $|g| = |\det(g_{MN})|$, Λ is the cosmological constant, and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2$$

is the standard Gauss–Bonnet term. Here α_1 and α_2 are nonzero constants.

The appearance of the Gauss–Bonnet term was motivated by string theory [1–5]. It is important to stress that not only the GB term can be motivated by string theory. Recently, it was shown in [6] that EGB theory (with or without a cosmological term) unavoidably leads to causality violation, unless the theory is completed by an infinite tower of higher-spin particles with fine-tuned couplings (perturbative string theory being an example). See also [7].

At present, the so-called Einstein–Gauss–Bonnet (EGB) gravitational model and its modifications—see [8] (for $D = 4$), [9–18] and references therein—are intensively used in cosmology, e.g., for explanation of the accelerated expansion of the Universe following from supernovae (type Ia) observational data [22–24]. Certain exact solutions in multidimesional EGB cosmology were obtained in [9–21] and some other papers.

Here we deal with the cosmological solutions with diagonal metrics (of Bianchi-I-like type) governed by n scale factors depending on one variable, where $n > 3$. We restrict ourselves to solutions with an exponential dependence of the scale factors. We present new examples of exact solutions in dimension $D = 8$ which describe an exponential expansion of 3-dimensional factor space and contraction of 4-dimensional internal space. We study the stability of the solutions with static Hubble-like parameters $h^i(t) = v^i$ and find asymptotic relations for variation of the effective gravitational constant for two stable solutions.

The paper is organized as follows. In Section 2, the equations of motion for the D-dimensional EGB model are considered. For diagonal cosmological-type metrics the equations of motion are equivalent to a set of Lagrange equations corresponding to a certain effective Lagrangian [19, 20] (see also [10]). In Section 3, some cosmological solutions are obtained

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with an exponential behavior of the scale factors, e.g., satisfying the observational restriction on the variation of the effective gravitational constant G for two isotropic factor spaces and a positive value of $\alpha = \alpha_2/\alpha_1$. Section 4 is devoted to an analysis of stability of the solutions with static Hubble-like parameters. We prove the stability of two solutions from [21] in dimensions $D = 22, 28$ and find an asymptotic relation for G in a linear approximation in the perturbations.

2. THE COSMOLOGICAL MODEL

We consider the manifold

$$M = (t_-, t_+) \times \mathbb{R}^n \quad (2)$$

with the metric

$$g = -dt \otimes dt + \sum_{i=1}^{n} e^{2\beta^i(t)} dy^i \otimes dy^i, \quad (3)$$

where $\beta^i(t)$ are smooth functions on (t_-, t_+), $i = 1, \ldots, n$.

We introduce "Hubble-like" variables $h^i = d\beta^i/dt$. The equations of motion for the action (1) read

$$\alpha_1(G_{ij}h^ih^j + 2\Lambda) - \alpha_2 G_{ijkl}h^ih^jh^kh^l = 0, \quad (4)$$

$$\left[2\alpha_1 G_{ij}h^j - \frac{4}{3}\alpha_2 G_{ijkl}h^jh^kh^l\right]\sum_{i=1}^{n} h^i + \frac{d}{dt}\left[2\alpha_1 G_{ij}h^j - \frac{4}{3}\alpha_2 G_{ijkl}h^jh^kh^l\right] - L = 0, \quad (5)$$

where $i = 1, \ldots, n$ and

$$L = \alpha_1(G_{ij}h^ih^j - 2\Lambda) - \frac{1}{3}\alpha_2 G_{ijkl}h^ih^jh^kh^l, \quad (6)$$

$$G_{ij} = \delta_{ij} - 1, \quad (7)$$

$$G_{ijkl} = G_{ij}G_{ik}G_{il}G_{jk}G_{jl}G_{kl}. \quad (8)$$

are the components of the 2-metric of pseudo-Euclidean signature and the Finslerian 4-metric on $\mathbb{R}^n$ [19, 20], respectively.

Due to (4) we have

$$L = \frac{2}{3}\alpha_1(G_{ij}h^ih^j - 4\Lambda). \quad (9)$$

In this paper we deal with the following solutions to equations (4) and (5):

$$h^i(t) = v^i, \quad (10)$$

with constant v^i, which correspond to the solutions $\beta^i = v^i t + \beta^i_0$, where β^i_0 are constants, $i = 1, \ldots, n$. In this case we obtain the metric (3) with the exponential dependence of scale factors

$$g = -dt \otimes dt + \sum_{i=1}^{n} B_i e^{2v^it} dy^i \otimes dy^i, \quad (11)$$

where $B_i > 0$ are arbitrary constants.

For a fixed point $v = (v^i)$ we have the set of polynomial equations

$$G_{ij}v^iv^j + 2\Lambda - \alpha G_{ijkl}v^iv^jv^kv^l = 0, \quad (12)$$

$$\left[2G_{ij}v^j - \frac{4}{3}\alpha G_{ijkl}v^jv^kv^l\right]\sum_{i=1}^{n} v^i - \frac{2}{3}G_{ij}v^iv^j + \frac{8}{3}\Lambda = 0, \quad (13)$$

where $i = 1, \ldots, n$, and $\alpha = \alpha_2/\alpha_1$. For $n > 3$ we get a set of forth-order polynomial equations.

With $\Lambda = 0$ and $n > 3$, the set of equations (12) and (13) has an isotropic solution $v^1 = ... = v^n = H$ only if $\alpha < 0$ [19, 20] $H = \pm 1/\sqrt{|\alpha|(n-2)(n-3)}$. This solution was generalized in [18] to the case $\Lambda \neq 0$.

It was shown in [19, 20] that there are no more than three different numbers among $v^1, ..., v^n$ when $\Lambda = 0$. This is valid also for $\Lambda \neq 0$.

3. EXAMPLES OF COSMOLOGICAL SOLUTIONS

In this section we consider some solutions to the set of equations (4), (5) of the following form:

$$v = (H, \ldots, H, h, \ldots, h), \quad (14)$$

where H a Hubble-like' parameter corresponding to the m-dimensional isotropic subspace with $m \geq 3$, and h is a Hubble-like parameter corresponding to the l-dimensional isotropic subspace, $l > 1$. We put $H > 0$ for describing an accelerated expansion of the 3D subspace (which may describe our Universe), and also put $h < 0$ for a possible description of a small enough variation of the effective gravitational constant (see below).

3.1. Polynomial Equations

According to the ansatz (14), the m-dimensional subspace is expanding with the Hubble parameter $H > 0$, while the l-dimensional subspace is contracting with the Hubble-like parameter $h < 0$.

We put in (12) and (13) $\alpha = \pm 1$ and denote $\Lambda = \lambda$, keeping in mind the general α-dependent form of the solutions

$$H(\alpha) = H|\alpha|^{-1/2}, \quad h(\alpha) = h|\alpha|^{-1/2}, \quad \Lambda = \lambda|\alpha|^{-1}. \quad (15)$$

3.2. Solutions with $\Lambda = 0$

Let $\Lambda = 0$ and $\alpha = 1$. It was shown in [21] that for $m = 9$ there exists an infinite series of cosmological solutions with $l = 3000,\ 3001, \ldots$, any of which describes an accelerated expansion of the 3D factor space with sufficiently small variation of the effective gravitational constant G obeying the observational restrictions [25]. This variation may be arbitrarily small for a large enough value of l.

We remind the reader that the effective gravitational constant G is proportional to the inverse volume scale factor of the internal space, see [26, 28–30] and references therein.

For $m = 11$ and $l = 16$, the following solution with

$$H = \frac{1}{\sqrt{15}}, \quad h = -\frac{1}{2\sqrt{15}} \qquad (16)$$

was found in [21], which describes a zero variation of the effective gravitational constant G.

Another solution of such a type with

$$H = \frac{1}{6}, \quad h = -\frac{1}{3} \qquad (17)$$

and a constant G appears for $m = 15$ and $l = 6$ [21].

3.3. Solutions with $\Lambda \neq 0$

Here we present a few cosmological solutions for $\Lambda \neq 0$, $\alpha = 1$, $m = 3$ and $l = 4$:

$$H = \frac{1}{4}\sqrt{2}, \quad h = -\frac{1}{4}\sqrt{2} \qquad (18)$$

for $\lambda = 3/16$,

$$H = \frac{1}{4}\sqrt{6}, \quad h = -\frac{1}{12}\sqrt{6} \qquad (19)$$

for $\lambda = 13/48$ and

$$H = \frac{2}{29}\sqrt{29}, \quad h = -\frac{3}{58}\sqrt{29} \qquad (20)$$

for $\lambda = 21/116$.

These solutions describe and accelerated expansion of the 3D factor space and contraction of the 4D internal factor space for certain positive values of the reduced cosmological constant λ. There also exist examples of solutions with a negative cosmological constant, $\lambda = -21/80$:

$$H_{\pm} = \frac{1}{60320}(248 \pm 32\sqrt{30})R_{\mp} > 0, \qquad (21)$$

$$h_{\pm} = -\frac{1}{580}R_{\mp} < 0, \qquad (22)$$

where $R_{\mp} = \sqrt{68150 \mp 9280\sqrt{30}}$.

4. STABILITY ANALYSIS AND VARIATION OF G

4.1. Equations for Perturbations

Here we study the stability of static solutions $h^i(t) = v^i$ to Eqs. (4) and (5) under linear pertubations. We put

$$h^i(t) = v^i + \delta h^i(t), \qquad (23)$$

$i = 1, \ldots, n$. By substitution of (23) into Eqs. (4) and (5) we get in the linear approximation the following relations for the perturbations δh^i:

$$C_i(v)\delta h^i = 0, \qquad (24)$$

$$L_{ij}(v)\delta \dot{h}^j = B_{ij}(v)\delta h^j, \qquad (25)$$

where

$$C_i(v) = 2G_{ij}v^j - 4\alpha G_{ijks}v^j v^k v^s, \qquad (26)$$

$$L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks}v^k v^s, \qquad (27)$$

$$B_{ij}(v) = -L_{ij}(v)\sum_k v^k - L_i(v) + \frac{4}{3}G_{kj}v^k, \qquad (28)$$

and

$$L_i(v) = 2G_{ij}v^j - \frac{4}{3}\alpha G_{ijks}v^j v^k v^s, \qquad (29)$$

$i, j, k, s = 1, \ldots, n$.

We put the following restriction on the matrix $L = (L_{ij}(v))$:

$$\det L \neq 0. \qquad (30)$$

Thus the matrix L is considered to be invertible. Its inverse will be denoted $L^{-1} = (L^{ij}) = (L^{ij}(v))$. Then the relation (25) may be rewritten as follows:

$$\delta \dot{h}^i = A^i_j(v)\delta h^j, \qquad (31)$$

where the matrix $A = (A^i_j(v))$ is defined as $A = L^{-1}B$ with $B = (B_{ij}(v))$, or, explicitly,

$$A^i_j(v) = -\delta^i_j \sum_k v^k - \sum_s L^{is}L_s + \frac{4}{3}G_{kj}v^k \sum_s L^{is}. \qquad (32)$$

In what follows use the following agreement on the indices: $\mu, \nu = 1, \ldots, m$ and $\alpha, \beta = m+1, \ldots, m+n$. We also denote

$$S_{ij} = G_{ijks}v^k v^s, \quad A_i = S_{ij}v^j, \quad v_i = G_{ij}v^j. \qquad (33)$$

Note that $S_{ij} = S_{ji}$ and $S_{ii} = 0$.

4.2. Solution with $m = 11$, $l = 16$ and $\Lambda = 0$

Consider the solution (16) with $m = 11$, $l = 16$ and $\Lambda = 0$ ($\alpha = 1$). The calculations give

$$S_{\mu\nu} = \frac{4}{5}(\delta_{\mu\nu} - 1), \quad S_{\mu\alpha} = S_{\alpha\mu} = -\frac{1}{2},$$
$$S_{\alpha\beta} = \frac{1}{10}(1 - \delta_{\alpha\beta}). \qquad (34)$$

The symmetric matrix $L = (L_{ij})$ has a block-diagonal form:

$$L_{\mu\nu} = \frac{6}{5}(1 - \delta_{\mu\nu}), \quad L_{\mu\alpha} = L_{\alpha\mu} = 0,$$
$$L_{\alpha\beta} = \frac{12}{5}(\delta_{\alpha\beta} - 1). \qquad (35)$$

The matrix L is invertible, and its inverse $L^{-1} = (L^{ij})$ reads

$$L^{\mu\nu} = \frac{5}{6}(\frac{1}{10} - \delta^{\mu\nu}), \quad L^{\mu\alpha} = L^{\alpha\mu} = 0,$$
$$L^{\alpha\beta} = \frac{5}{12}(\delta^{\alpha\beta} - \frac{1}{15}). \qquad (36)$$

Here we use the matrix identity $(\delta_{ab} - 1)^{-1} = (\delta_{ab} - 1/(N-1))$, $a, b = 1, \ldots, N$.

We also obtain $v_\mu = -2H$, $v_\alpha = -7H/2$,

$$A_\mu = -4H, \quad A_\alpha = -25H/4,$$
$$C_\mu = 12H, \quad C_\alpha = 18H, \qquad (37)$$

and $L_i = 4H/3$, where $H = 1/\sqrt{15}$. For the matrix (A^i_j) we get

$$A^\mu_\nu = -H(3\delta^\mu_\nu + \frac{1}{3}), \quad A^\mu_\beta = -\frac{H}{2},$$
$$A^\alpha_\nu = \frac{H}{9}, \quad A^\alpha_\beta = H(-3\delta^\alpha_\beta + \frac{1}{6}). \qquad (38)$$

Due to (37) and (38), the relations (24) and (31) read

$$2\sum_\mu \delta h^\mu + 3\sum_\alpha \delta h^\alpha = 0, \qquad (39)$$
$$\delta\dot{h}^i = -3H\delta h^i. \qquad (40)$$

We get the following solution for perturbations:

$$\delta h^i = a^i \exp(-3Ht), \qquad (41)$$
$$2\sum_{\mu=1}^{11} a^\mu + 3\sum_{\alpha=12}^{27} a^\alpha = 0, \qquad (42)$$

where $H = 1/\sqrt{15}$, $i = 1, \ldots, 27$. Thus the solution (16) is stable as $t \to +\infty$.

4.3. Solution with $m = 15$, $l = 6$ and $\Lambda = 0$

Now we consider the solution (17) with $m = 15$, $l = 6$ and $\Lambda = 0$ ($\alpha = 1$). We get

$$S_{\mu\nu} = \delta_{\mu\nu} - 1, \quad S_{\mu\alpha} = S_{\alpha\mu} = -1/2,$$
$$S_{\alpha\beta} = \frac{1}{2}(1 - \delta_{\alpha\beta}). \qquad (43)$$

The matrix $L = (L_{ij})$ is block-diagonal,

$$L_{\mu\nu} = 2(1 - \delta_{\mu\nu}), \quad L_{\mu\alpha} = L_{\alpha\mu} = 0,$$
$$L_{\alpha\beta} = 4(\delta_{\alpha\beta} - 1). \qquad (44)$$

The matrix L is invertible, and its inverse reads

$$L^{\mu\nu} = \frac{1}{2}\Big(\frac{1}{14} - \delta^{\mu\nu}\Big), \quad L^{\mu\alpha} = L^{\alpha\mu} = 0,$$
$$L^{\alpha\beta} = \frac{1}{4}(\delta^{\alpha\beta} - \frac{1}{5}). \qquad (45)$$

We also obtain $v_\mu = -1/3$, $v_\alpha = -5/6$,

$$A_\mu = -4/3, \quad A_\alpha = -25/12,$$
$$C_\mu = 14/3, \quad C_\alpha = 20/3, \quad L_i = 10/9. \qquad (46)$$

For the matrix (A^i_j) we have

$$A^\mu_\nu = -\frac{1}{2}\delta^\mu_\nu - \frac{1}{18}, \quad A^\mu_\beta = -\frac{5}{63},$$
$$A^\alpha_\nu = \frac{7}{90}, \quad A^\alpha_\beta = -\frac{1}{2}\delta^\alpha_\beta + \frac{1}{9}. \qquad (47)$$

Due to (46) and (47), the relations (24) and (31) read

$$7\sum_\mu \delta h^\mu + 10\sum_\alpha \delta h^\alpha = 0, \qquad (48)$$
$$\delta\dot{h}^i = -\frac{1}{2}\delta h^i. \qquad (49)$$

We get the following solution for perturbations :

$$\delta h^i = a^i \exp(-\frac{1}{2}t), \qquad (50)$$
$$\sum_{\mu=1}^{15} a^\mu + 10\sum_{\alpha=16}^{21} a^\alpha = 0, \qquad (51)$$

$i = 1, \ldots, 21$. Thus the solution (17) is stable as $t \to +\infty$.

4.4. Variation of the Effective Gravitational Constant

For both solutions under consideration we get in the linear approximation in δh^i

$$\dot{\beta}^i = v^i + \delta h^i = v^i + a^i \exp(-Kt), \qquad (52)$$

where $K = 3H > 0$ and $\sum_{i=1}^n C_i a^i = 0$. This implies the following asymptotic relation as $t \to +\infty$:

$$\beta^i(t) = v^i t + \beta^i_0 - K^{-1} a^i \exp(-Kt), \qquad (53)$$

$i = 1, \ldots, n$.

For the effective gravitational constant $G(t) = \text{const} \cdot \exp(-\sum_{i=4}^{n} \beta^i(t))$ we get the following asymptotic relation:

$$G(t) = G_0 \exp\left(+K^{-1} e^{-Kt} \sum_{i=4}^{n} a^i\right), \qquad (54)$$

for the two solutions under consideration (with $\sum_{i=4}^{n} v^i = 0$), where $K = 3H$ and G_0 is the asymptotic value of $G(t)$ for $t \to +\infty$.

Hence we obtain the asymptotic relation for the variation of G

$$\frac{\dot{G}}{G} = -e^{-Kt} \sum_{i=4}^{n} a^i. \qquad (55)$$

In the general case $\alpha > 0$ we have $K = 3H(\alpha)$, where

$$H(\alpha) = \frac{\alpha^{-1/2}}{6}, \quad \frac{\alpha^{-1/2}}{\sqrt{15}} \qquad (56)$$

for $D = 22, 28$, respectively.

Let us consider the most stringent observational restriction on $\dot{G}$ obtained from the set of ephemerides [25]

$$\dot{G}/G = (0.16 \pm 0.6) \times 10^{-13} \text{ year}^{-1}, \qquad (57)$$

allowed at a 95% confidence (2-σ) level. It follows from (55) that in both cases under consideration the restriction (57) is satisfied for either a large enough value of t for fixed a^i or a small enough value of $\sum_{i=4}^{n} a^i$ for fixed t.

Here we present for completeness the value of the Hubble parameter [31]

$$H_0 = (67.80 \pm 1.54) \text{ km/s Mpc}^{-1} = (6.929 \pm 0.157) \times 10^{-11} \text{ year}^{-1}, \qquad (58)$$

with a 95% confidence level. The relation $H(\alpha) = H_0$ gives the value of α in any of the two cases.

5. CONCLUSIONS

We have considered the D-dimensional Einstein–Gauss–Bonnet (EGB) model with Λ term. By using the ansatz of a diagonal cosmological type metric, we have found new solutions with an exponential dependence of the scale factors with respect to the synchronous time variable t in dimensions $D = 1 + 3 + 4$. Any of these solutions describes an exponential expansion of our 3D factor space with a Hubble parameter $H > 0$ and exponential contraction of the 4D internal space.

We have also studied two cosmological solutoins from [21] in the EGB model with $\Lambda = 0$ for $D = 22, 28$. These solutions describe an exponential expansion of our 3D factor space with the Hubble parameter $H > 0$ and zero temporal variation of the effective gravitational constant G. We have proved that these solutions are stable as $t \to +\infty$. We have found solutions for perturbations in both cases: $\delta h^i(t) = a^i \exp(-3H(\alpha)t)$, with certain linear constraints on the amplitudes a^i, where

$$H(\alpha) = \frac{\alpha^{-1/2}}{6}, \ \frac{\alpha^{-1/2}}{\sqrt{15}} \quad (\alpha > 0)$$

for $D = 22, 28$, respectively. In the linear approximation for perturbations, we have found a relation for variation of G: $\dot{G}/G$ is exponentially damped and tends to zero as $t \to +\infty$. Thus we have shown that, in the framework of the EGB model, there exists a variety of solutions describing an accelerated expansion of the 3D factor space with a sufficiently small (or even zero) value of variation of the effective gravitational constant G.

An open question here is to compare our scheme of the stability analysis of the exponential solutions with that of [32]. This may be a subject of a separate publication.

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