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"General Theory of Particle Mechanics" Arising from a Fractal Surface

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Abstract—We trace the logical line of formulating a theory of mechanics founded on the basic relations of mathematics of hypercomplex numbers and associated geometric images. Namely, it is shown that the physical equations of quantum, classical and relativistic mechanics can be regarded as mathematical consequences of a single condition of stability of exceptional algebras of real, complex and quaternion numbers under transformations of primitive constituents of their units and elements. In the course of the study, the notion of a basic fractal surface underlying the physical three-dimensional space is introduced, and an original geometric treatment (admitting visualization) of some formerly considered abstract functions (mechanical action, space-time interval) are suggested.

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1. INTRODUCTION

In the previous paper [1], a logical line describing the growth of different branches of contemporary mechanics from mathematical roots was verbally exposed. In the present paper this logic is traced using purely math language finely demonstrating surprising results. The method used here implies a fine analysis of primitive elements constituting the basic structures of exceptional associative algebras (and other polynumber algebras). The simplest transformations of the elements induce involvement of a series of conditions (equations) providing the algebras' stability; these pure math equations are found to be equivalent to the laws of different branches of mechanics, and become them exactly when rewritten in physical units (instead of abstract magnitudes). This unique logical line leads consequently to the equations of quantum, classical and relativistic mechanics traditionally thought of as belonging to somewhat separate theories. Two types of main objects of the theory, a particle, indispensably emerge, a fractal one, made up from primitive elements, and a geometric one observed in physical space. By the way, some traditionally abstract functions (mechanical action, space-time interval) acquire an original geometric sense.

The paper is organized as follows. Section 2 contains a short review of the involved algebras, their units being represented by matrices. Section 3 outlines the principal stages and main points of the "general theory of mechanics." In particular, transformations of a fractal surface spoiling the algebras' basis are analyzed, and a condition of the algebras stability is introduced, then splitting into a series of fractal and geometric equations of mechanics (written in purely math and physical units). The respective particle model gives birth to a version of special relativity (with a helix-type Minkowski diagram) and to a general relativity-type geodesic equation. A brief discussion in Section 4 concludes the study.

2. ALGEBRAS INVOLVED

We start with *biquaternions*, hypercomplex numbers of the form [2]

$$b \equiv x + iy + (u_1 + iw_1)\mathbf{q}_1 + (u_2 + iw_2)\mathbf{q}_2 + (u_3 + iw_3)\mathbf{q}_3 \equiv x + iy + (u_k + iw_k)\mathbf{q}_k; \quad (1)$$

here x, y, u_k, w_k are real numbers, the scalar part $x + iy$ has the real unit factor 1 (traditionally not shown), $\mathbf{q}_k$ are three vector units, all four units satisfying the multiplication law

$$\mathbf{q}_k 1 = 1\mathbf{q}_k, \quad \mathbf{q}_k\mathbf{q}_n = -\delta_{kn} + \varepsilon_{knm}\mathbf{q}_m, \quad (2)$$

δ_{kn}, ε_{knm} are the 3D Kronecker and Levi-Civita symbols, summation in repeated (3D small Latin) indices is assumed. The numbers of the type (1) constitute the largest algebra in question here; according to the law (2), the algebra of biquaternions is noncommutative, but it is associative in multiplication. It is well

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known that the norm of a biquaternion is in general not defined as a positive real number, so this algebra has division defects, in particular, zero dividers.

If in Eq. (1) $x \neq 0$, $u_k \neq 0$ while all imaginary components vanish, $y = w_k = 0$, then one arrives to a *quaternion*, a hypercomplex number of the form $q \equiv x + u_k \mathbf{q}_k$, the units $(1, \mathbf{q}_k)$ obeying the law (2); so the quaternions can be regarded as a (4-unit) section of the set of bi-quaternions. The norm and the modulus of a quaternion are well defined:

$$||q||^2 \equiv q\bar{q} = (x + u_k\mathbf{q}_k)(x - u_k\mathbf{q}_k) = x^2 + u_k u_k,$$
$$|q| \equiv (q\bar{q})^{1/2} = \sqrt{x^2 + u_k u_k},$$

hence the inverse (left and right) number and division do exist. The Frobenius theorem proves that the algebra of quaternions is the last in dimension associative, though non-commutative, division algebra (a non-commutative ring).

If in Eq. (1), e.g., only $x \neq 0$, $w/2 \equiv w_1 = u_2 \neq 0$, while all other components vanish, then one obtains a "more narrow" (2-unit) section of bi-quaternions, the set of "exotic" *dual numbers* [3] of the form $d \equiv x + w(i\mathbf{q}_1 + \mathbf{q}_2)/2 \equiv x + w\boldsymbol{\varepsilon}$, its vector unit having a zero norm, $\boldsymbol{\varepsilon}^2 = 0$. The algebra of dual numbers is associative and commutative (according to Eq. (2), $1\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}1$), but it includes zero dividers since the norm of a dual number depends only on the scalar part: $||d||^2 \equiv d\bar{d} = (x + w\boldsymbol{\varepsilon})(x - w\boldsymbol{\varepsilon}) = x^2$, and a pure "imaginary" number has zero norm. It is evident that a dual number can be obtained from Eq. (1) in various ways, e.g., if only $x \neq 0$, $w/2 \equiv w_2 = u_3 \neq 0$, all other components vanishing or $x \neq 0$, $w/2 \equiv w_3 = u_1 \neq 0$, etc.

If in Eq. (1), e.g., only $x \neq 0$, $w \equiv w_1 \neq 0$, while the rest of the components vanish, then one meets another 2-unit section of biquaternions, the set of *double numbers* (*split complex numbers*) [4] of the form $h \equiv x + wi\mathbf{q}_1 \equiv x + w\mathbf{p}$, its units commuting, the square of the vector unit being equal to the real unit, $\mathbf{p}^2 = 1$. The norm of h is not well defined, $||h||^2 \equiv (x + w\mathbf{p})(x - w\mathbf{p}) = x^2 - w^2$, so the commutative and associative algebra of dual numbers also has zero dividers. It is evident that a double number can be obtained from Eq. (1) in various ways.

If in Eq. (1), e.g., only $x \neq 0$, $u \equiv u_2 \neq 0$, while all other components vanish, then there emerges the last specific 2-unit section of biquaternions, that of *complex numbers* $z \equiv x + u\boldsymbol{q}_2 \equiv x + u\mathbf{i}$. It is evident that there are various ways to select a complex number from Eq. (1), in particular, $z \equiv x + iy$ (with the traditional scalar imaginary unit), all these representations being algebraically equivalent.

The simplest section of the biquaternion set is the 1-unit set of *real numbers* x, all other components in Eq. (1) being zero.

The algebras of real, complex and quaternion numbers are referred to as exceptional ones since only their elements (and 8-unit *octonion* numbers) satisfy the "square identities," the norm definition of a two-elements product; e.g., for two quaternions q_1, q_2 it is $||q_1q_2||^2 = ||q_1||^2||q_2||^2$. But the algebra of octonions is not associative, the property alien to known physical magnitudes; so octonions will not be considered here.

If the (bi-)quaternion algebra units are "canonically" represented by the 2×2 matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q}_{\tilde{1}} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbf{q}_{\tilde{2}} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{q}_{\tilde{3}} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (3)$$

then the multiplication law (2) is satisfied identically; the other units then can be chosen, e.g., as

$$\boldsymbol{\varepsilon} = (i\mathbf{q}_{\tilde{1}} + \mathbf{q}_{\tilde{2}})/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{p} \equiv i\mathbf{q}_{\tilde{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \boldsymbol{i} = \mathbf{q}_{\tilde{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But many other representations exist since the basic law (2) evidently holds for the transformed units

$$\mathbf{q}_k = S\mathbf{q}_{\tilde{k}}S^{-1}, \qquad (4)$$

the transformation matrices forming the spinor group[1] $S \in SL(2, \mathbb{C})$ [5]. We stress that after any such transformation the scalar unit remains a unit matrix, while the vector units may become a multi-component function of many parameters. Note also that representations of the units $(1, \mathbf{q}_k)$ by matrices of rank 2^N (N is a natural number) can be readily introduced.

3. GENERAL THEORY OF MECHANICS

3.1. Structures of Algebraic Units

(i) *All basic units of associative algebras (of real, complex, quaternion, double, dual and bi-quaternion numbers) can be regarded as matrices composed of a single dyad (a couple of 2D unit orthogonal vectors) on a fractal[2] surface.*

[1] The law (2) is as well invariant under $SO(3, \mathbb{C})$ transformations of the units, this group of rotations is twice covered by the reflection group $SL(2, \mathbb{C})$.

[2] "Fractal" means here that a line dimension on the surface is 1/2 of that of 3D geometric (physical) space.

Let two 2D vectors ψ^+, ψ^- (two-component column matrices) compose a local orthonormal basis (dyad) on a surface with a symmetric metric g, i.e., at a surface point $g\psi^+\psi^+ = 1$, $g\psi^-\psi^- = 1$, $g\psi^+\psi^- = g\psi^-\psi^+ = 0$; or with introduction of the covectors $\varphi^+ \equiv g\psi^+$, $\varphi^- \equiv g\psi^-$ (two-component row matrices), the dyad definition is

$$\varphi^\pm\psi^\pm = 1, \quad \varphi^\pm\psi^\mp = 0. \tag{5}$$

Due to (5), the following linear combinations of direct products (tensor squares) of the set $\psi^\pm$, $\varphi^\pm$ satisfy the law (2) for the units $(1, \mathbf{q}_k)$ [6]:

$$\psi^+\varphi^+ + \psi^-\varphi^- = 1, \quad -i(\psi^+\varphi^- + \psi^-\varphi^+) = \mathbf{q}_1,$$
$$\psi^+\varphi^- - \psi^-\varphi^+ = \mathbf{q}_2,$$
$$i(\psi^+\varphi^+ - \psi^-\varphi^-) = \mathbf{q}_3; \tag{6}$$

the other vector units can be chosen as (other combinations are possible)

$$\psi^-\varphi^+ = \boldsymbol{\varepsilon}, \quad \psi^+\varphi^- + \psi^-\varphi^+ = \mathbf{p},$$
$$\psi^+\varphi^- - \psi^-\varphi^+ = \mathbf{i}.$$

(ii) *The metric of the dyad's local vicinity (2D-cell) behaves as the real unit of all involved algebras, while the three vector units $\mathbf{q}_k$ behave as a Cartesian frame in 3D space.*

A domain of the fractal surface in the vicinity of the dyad's origin together with part of the tangent plane having the metric $\delta_{MN} = \delta^{MN} = \delta_M^N$ (the unit 2×2-matrix) will be called a "2D cell." So we can identify the scalar unit 1 with the metric of a 2D cell, the metric of the fractal surface structured by the dyad covectors $g = \varphi^+\varphi^+ + \varphi^-\varphi^-$. The quaternion vector units $\mathbf{q}_k$ from the times of Hamilton [7] are known to be geometrically identified with a frame initiating a Cartesian coordinate system in 3D space, often associated with the physical space.

(iii) *The dimension of a line on a 2D cell (e.g. a dyad vector length) is a square root from the dimension of a line in 3D space (e.g. a vector unit length); from the 3D space viewpoint the dyad vectors are spinors.*

Equations (6) demonstrate that the dyad vectors (covectors) may be regarded as specific square roots of the units $(1, \mathbf{q}_k)$, but a single dyad is sufficient to build all units. Also note that the vectors (covectors) of this dyad are right and left eigenfunctions of the unit $\mathbf{q}_3$ with eigenvalues $\pm i$:

$$\mathbf{q}_3\psi^\pm = \pm i\psi^\pm, \quad \varphi^\pm\mathbf{q}_3 = \pm i\varphi^\pm, \tag{7}$$

(that is why the parity indicators $\pm$ arise). Eigenfunctions of the simplest operator $\mathbf{q}_{\tilde{3}}$ from Eqs. (3a) are

$$\tilde{\psi}^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{\psi}^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\tilde{\varphi}^+ = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \tilde{\varphi}^- = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{8}$$

We emphasize that the transformations of the dyad constituents by matrices $S \in SL(2,\mathbb{C})$

$$\psi^\pm = S\tilde{\psi}^\pm, \quad \varphi^\pm = \tilde{\varphi}^\pm S^{-1} \tag{9}$$

should be considered the prior one since they induce the transformations (4) of 3D units leaving the multiplication law (2) intact; so the dyad vectors (covectors) $\psi^\pm$, $\varphi^\pm$ are spinors.

(iv) *If 3D space and objects in it are traditionally attributed to "geometry," then the fractal surface and objects on it (2D-cell, dyad vectors) may be related to "pregeometry."*

We dare remind the reader that the notion of pregeometry was introduced by Wheeler in an attempt to find a plausible image of a space where functions and operators of quantum mechanics act [8].

3.2. Transformations of a 2D Cell

(v) *The 2D cell's area can be "pumped over" from the real sector to the imaginary one with a certain phase, this "flickering" does not change the metric, but the respective 3D frame rotates by an angle equal to the doubled phase. The 2D flickering (and the respective 3D rotation) produces no damage to the involved algebras.*

The simplest transformation of the type (9) for the dyad (8) is a "rotation" about the vector $\mathbf{q}_{\tilde{3}}$ at an angle α, the rotation matrix is $S = \cos\alpha + \mathbf{q}_3 \sin\alpha$; the results of the transformation are

$$\psi^\pm = (\cos\alpha \pm i\sin\alpha)\tilde{\psi}^\pm = e^{\pm i\alpha}\tilde{\psi}^\pm,$$
$$\varphi^\pm = e^{\mp i\alpha}\tilde{\varphi}^\pm. \tag{10}$$

Equations (10) state that the lengths of real and imaginary constituents of the dyad vectors $\psi^\pm$ harmonically change with α, so that an area of the fractal space formed by the vectors is "pumped over" (flickers) from real sector to imaginary sector of the 2D cell (the same with the covectors $\varphi^\pm$). Using Eqs. (6), we compute the results of the respective transformations of the algebraic units:

$$1 = \tilde{1}, \quad \mathbf{q}_1 = \mathbf{q}_{\tilde{1}}\cos 2\alpha + \mathbf{q}_{\tilde{2}}\sin 2\alpha,$$
$$\mathbf{q}_2 = \mathbf{q}_{\tilde{1}}\cos 2\alpha - \mathbf{q}_{\tilde{2}}\sin 2\alpha, \quad \mathbf{q}_3 = \mathbf{q}_{\tilde{3}}, \tag{11}$$

i.e., the scalar unit does not change while the 3D frame is rotated about the vector $\mathbf{q}_3 = \mathbf{q}_{\tilde{3}}$ by 2α (twice the spinor vectors' "rotation"); the units (11) well fit all the involved algebras, this may be verified by direct computation.

(vi) *The flickering 2D cell can be stretched ("loaded with a fractal density"); this transformation causes a 2D metric defect and changes* *the lengths of the rotated 3D frame vectors thus damaging the algebras.*

Let the flickering dyad (10) be subject to an extra transformation, conformal stretching

$$\psi'^{\pm} = \sigma e^{\pm i\alpha}\tilde{\psi}^{\pm} \equiv \lambda\tilde{\psi}^{\pm},$$
$$\varphi'^{\pm} = \sigma e^{\mp i\alpha}\tilde{\varphi}^{\pm} \equiv \lambda^*\tilde{\varphi}^{\pm}, \qquad (12)$$

$\sigma \in \mathbb{R}$. The mapping (12) injects the 2D cell's metric defect

$$1' = \psi'^{+}\varphi'^{+} + \psi'^{-}\varphi'^{-} = \sigma^2, \qquad (13)$$

all vector units acquiring the same square factor, e.g., $\mathbf{q}_{3'} = \sigma^2\mathbf{q}_3$; this evidently damages the multiplication law (2), so all involved algebras are violated. In 3D space σ^2 may be thought of as a density, so the factor σ will be called "fractal density" loading a formerly "empty" 2D cell.

3.3. Abstract (Exterior) Space and the Algebras' Stability Condition

(vii) *A normalizing integral (functional) of a dyad vector square length over a volume of an abstract M-dimensional space (in particular, an abstract 3D space) smooths down the 2D metric defect and returns the unit lengths of 3D frame vectors thus restoring the algebras.*

The metric defect (13) is smoothed down if the factor

$$\lambda = \sigma e^{i\alpha} \qquad (14)$$

is a compact function $\lambda(\xi_\Lambda, \theta)$ (of the coordinates ξ_Λ, $\Lambda = 1, 2, ..., M$ of M-dimensional abstract space and a free parameter θ, all magnitudes measured in no units) in a volume V_Λ and

$$f \equiv \int_{V_\Lambda} \varphi'^{\pm}\psi'^{\pm} dV_\Lambda = \int_{V_\Lambda} \lambda\lambda^* dV_\Lambda$$
$$= \int_{V_\Lambda} \sigma^2 dV_\Lambda = 1. \qquad (15)$$

Then the objects built from the dyad (12) as in Eq. (6), but "seen" from the space, do not differ from those of Eq. (11) and can serve as good algebra units, e.g.,

$$1' = f(\psi'^{+}\varphi'^{+} + \psi'^{-}\varphi'^{-}) = \tilde{1},$$
$$\mathbf{q}_{3'} = if(\psi'^{+}\varphi'^{+} - \psi'^{-}\varphi'^{-}) = \mathbf{q}_{\tilde{3}},$$
$$\mathbf{q}_{1'} = -if(\psi'^{+}\varphi'^{-} + \psi'^{-}\varphi'^{+})$$
$$= (\cos 2\alpha)\mathbf{q}_{\tilde{1}} + (\sin 2\alpha)\mathbf{q}_{\tilde{2}} = \mathbf{q}_1. \qquad (16)$$

(viii) *The algebras are saved "forever" in the sense of the free parameter θ if the normalizing functional is constant with respect to the parameter change; this condition of the algebras' stability entails a continuity-type equation for the squared fractal density.*

The normalization (15) "lasts forever" in the sense of θ (thus providing the algebras' stability) if the function $\lambda\lambda^*$ satisfies the continuity-type equation

$$\partial_\theta(\lambda\lambda^*) + \partial_\Lambda(\lambda\lambda^* k_\Lambda) = 0, \qquad (17)$$

where $\partial_\theta \equiv \partial/\partial\theta$, $\partial_\Lambda \equiv \partial/\partial\xi_\Lambda$, summing in the index Λ is implied, k_Λ is a vector of 2D cell "propagation" in abstract space.

3.4. Basic Fractal Equations, Consequences of the Stability Condition

(ix) <u>Schrödinger-type equation</u>. *If the propagation vector of the 2D cell (in the abstract space) is just a gradient of the flickering phase, then the continuity-type equation splits into a couple of mutually conjugate equations, each mathematically equivalent to the Schrödinger equation of quantum mechanics.*

Let the propagation of a 2D cell be determined by phase increase. The phase is expressed from Eq. (14) as $\alpha = (i/2)\ln(\lambda^*/\lambda)$, then the propagation vector $k_\Lambda = \partial_\Lambda\alpha = (i/2)(\partial_\Lambda\lambda^*/\lambda^* - \partial_\Lambda\lambda/\lambda)$, when inserted in Eq. (17), brings it to the form

$$\lambda^*\left(\partial_\theta - \frac{i}{2}\partial_\Lambda\partial_\Lambda\right)\lambda + \lambda\left(\partial_\theta + \frac{i}{2}\partial_\Lambda\partial_\Lambda\right)\lambda^* = 0,$$

If each of two imaginary parts (zero in sum) of this equation is defined as $W(\xi, \theta)$ (an arbitrary real function) then the equation is split into mutually conjugate vanish parts; that for λ

$$\left[\partial_\theta - \frac{i}{2}\left(\partial_\Lambda\partial_\Lambda - 2W\right)\right]\lambda = 0 \qquad (18)$$

is an exact math analogue of the Schrödinger equation of quantum mechanics.

(x) <u>Pauli-type equation</u>. *If the propagation vector of a 2D cell, apart from the phase gradient, includes an exterior vector field, then the continuity-type equation splits into mutually conjugate math equivalents of the Pauli equation of quantum mechanics.*

Consider a more general case of the 2D cell propagation vector $k_n = \partial_n\alpha + A_n$, where $A_k(x_n, t)$ is some exterior vector field (for simplicity the abstract space is chosen here to be three-dimensional with coordinates x_n). The presence of a vector field induces returning to full spinor functions in the normalizing integral $\int_{V_n} \varphi'\psi' dV_n = 1$ (here φ', ψ' and other spinors are chosen, e.g., of positive parity), and the representation of the 3D space metric in the Clifford algebra format $\delta_{kn} \equiv (1/2)\mathbf{p}_k\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_k)$, where $\mathbf{p}_k \equiv i\mathbf{q}_k$. The respective continuity-type equation is written as

$$\partial_\theta(\tilde{\varphi}\lambda^*\lambda\tilde{\psi}) + \frac{1}{2}(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)$$

$$\times \partial_m[\tilde{\varphi}\lambda^*\lambda\tilde{\psi}(\partial_n\alpha + A_n)] = 0,$$

and it decays into mutually Hermitian conjugate parts; the equation for the function ψ' has the form

$$\left[i\partial_\theta - \frac{1}{2}(-i\partial_k + A_k)(-i\partial_k + A_k) - \frac{1}{2}\mathbf{p}_k B_k - W\right]\psi' = 0, \tag{19}$$

where $B_k \equiv \varepsilon_{kmn}\partial_m A_n$. Equation (19) is a math analogue of the Pauli equation of quantum mechanics describing the electron motion in an exterior magnetic field. The details of derivation of Eq. (19) are found in [9].

(xi) Klein-Gordon-type equation. *If the exterior space is a Minkowski-type space-time with θ the timelike coordinate, and the propagation vector of a 2D cell is a 4D gradient of the phase, then the continuity-type equation splits into mutually conjugate math equivalents of the Klein-Gordon-type equation.*

Let the exterior space be formally represented as a "space-time" having the indefinite metric $\delta^{\mu\nu} = \mathrm{diag}(1, -1, -1, ..., -1)$ and the coordinates ξ^μ $(\mu, \nu, ... = 0, 1, 2, ..., M)$ with $\xi^0 \equiv \theta$; let also the complex factor $\lambda = \sigma e^{i\alpha}$ be "re-gauged" (shown as a product of two complex numbers) $\lambda(\xi^\mu) = \gamma\bar{\lambda}$, where $\gamma\gamma^* \equiv \bar{k}_0$, and $\bar{\lambda} \equiv \bar{\sigma}e^{i\Phi}$. Then, defining the $(M+1)$-dimensional propagation vector $\bar{k}^\mu = \{\bar{k}^0, \bar{k}^\Lambda\}$, where $\bar{k}^\Lambda \equiv \bar{k}_0 k_\Lambda$, k_Λ being the propagation vector from Eq. (17), we can identically rewrite this continuity-type equation in the form

$$\partial_\mu(\bar{\lambda}\bar{\lambda}^*\bar{k}^\mu) = 0. \tag{20}$$

Next, we demand that the propagation vector be a gradient of the phase $\bar{k}^\mu = \delta^{\mu\nu}\partial_\nu\Phi = (i/2)\delta^{\mu\nu}\partial_\nu \times \ln(\bar{\lambda}^*/\bar{\lambda})$, then Eq. (20), quadratic in $\bar{\lambda}$, splits into a conjugate couple of Klein-Gordon-type equations, linear in the factor, which for $\bar{\lambda}$ is

$$(\delta^{\mu\nu}\partial_\mu\partial_\nu - \bar{W})\bar{\lambda} = 0, \tag{21}$$

where $\bar{W}(\xi^\nu)$ is an arbitrary function. Under simple conditions Eq. (21) is reduced to the Schrödinger-type equation (18). Indeed, put the function $\bar{\lambda}$ in the form

$$\bar{\lambda} = \gamma^{-1}\lambda \equiv \varsigma e^{i\eta}\lambda, \tag{22}$$

with $\varsigma = 1 + o_1$, $\eta = \theta + o_2$, where o_1, o_2 are small functions as well as all their derivatives. Insert the function (22) into Eq. (21); straightforward computations yield the sought-for result

$$\left[i\partial_0 - \frac{1}{2}\partial_\Lambda\partial_\Lambda + \frac{1}{2}(-\bar{W} - 1)\right]\lambda = 0,$$

i.e. Eq. (18) with $\bar{W} + 1 \equiv -2W$.

3.5. Transition to Physical Space and Introduction of Scales

(xii) *All the above math equations contain quantities measured in no physical units; a transition from abstract to physical space compels to introduce space-time standards. Short-scale standards are chosen.*

Instead of M-dimensional abstract space (where ξ_Λ and θ are dimensionless), the math equations can be regarded over 3D physical space and time necessarily scaled: $\xi_\Lambda \to x_k/\varepsilon$, $\theta \to t/\tau$, where ε and τ are spatial length and time-interval standards. The characteristic spatial length is chosen to be equal to the Compton wavelength,

$$\varepsilon \equiv \hbar/(mc), \tag{23a}$$

where $\hbar$ is the Planck constant, c is the velocity of light in vacuum, and m is the electron rest mass (in these constants the spatial scale is assessed as $\varepsilon \cong 10^{-11}$ cm). The respective time standard is the time interval needed for light to travel (in vacuum) along the characteristic length,

$$\tau \equiv \varepsilon/c = \hbar/(mc^2) \tag{23b}$$

(the time scale is assessed as $\tau \cong 10^{-21}$ s).

(xiii) *In these units the 2D cell describes a pregeometric protoparticle, its fractal density function acquiring the sense of a relative fractal mass density (the density function per mean density), so that the normalizing functional (15) is converted into a definition of mass of a 3D particle (based on a 2D protoparticle).*

The function σ remains dimensionless (measured in no units), so it may have a meaning of a "relative fractal mass density" $\sigma \equiv \sqrt{\rho(x,t)/\rho_{\text{mean}}}$, where ρ_{mean} is the mean mass density of the particle (electron) in a 3D volume it is supposed to occupy. Now a model of a "protoparticle" emerges. As a fractal object, it is conceived as a σ-weighted 2D cell $\{\sigma e^{i\alpha}\tilde{\psi}^{\pm}\}$; with a phase change, its area (hence, weight) is flickering between real and imaginary sectors. In fact, it is a "visual image" of the particle's state (wave) function of quantum mechanics. The normalizing integral (15) is converted to a definition of the particle mass:

$$\frac{1}{\varepsilon^3}\int_V \sigma^2(x,t)\,dV = 1,$$

$$\Rightarrow \quad \int_V \rho(x,t)\,dV = \varepsilon^3\rho_{\text{mean}} = m. \tag{24}$$

3.6. In Physical Space, Math Equations Become Physical Laws

(xiv) *In the physical units chosen, Eq. (18) and Eq. (19) become the exact Schrödinger and Pauli equations, respectively, Eq. (21) becomes the extended Klein-Gordon equation.*

One can easily verify that in the coordinates and time scaled as in Eqs. (23), Eq. (18) takes the form of the Schrödinger equation

$$\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_k\partial_k - U\right)\lambda(x,t) = 0, \quad (25)$$

where $U \equiv mc^2W$ is a scalar potential; Eq. (19) takes the form of the Pauli equation

$$\left[i\hbar\partial_t - \frac{1}{2m}(-i\hbar\partial_k + \frac{q}{c}\tilde{A}_k)(-i\hbar\partial_k + \frac{q}{c}\tilde{A}_k) - \frac{q\hbar}{2mc}\mathbf{p}_k\tilde{B}_k - U\right]\Psi(x,t) = 0, \quad (26)$$

where q is the electric charge, $\tilde{A}_k \equiv \frac{mc^2}{q}A_k$ and $\tilde{B}_k \equiv \frac{mc^2}{q}B_k$ are the potential and the intensity of the magnetic field, respectively, and $U \equiv mc^2W$ is a scalar potential. Eq. (21) in the chosen physical units becomes the extended Klein-Gordon equation

$$\left[\hbar^2\left(\frac{1}{c^2}\partial_t\partial_t - \partial_n\partial_n\right) - m^2c^2(1+W')\right]\lambda(x,t) = 0, \quad (27)$$

the free function being presented in the form $1 + W' \equiv \bar{W}$; Eq. (27) obviously admits fractalization, e.g., the Dirac square root format.

3.7. The Hamilton-Jacobi Equation

(xv) *The Schrödinger-type math equation has a complex-number structure; it can be separated into real and imaginary parts; the real part is a math equivalent of a conservation law for the fractal density function.*

Returning to the Schrödinger-type equation (18) and using $\lambda = \sigma e^{i\alpha}$, we decompose it into real and imaginary parts, thus obtaining the set of Bohm-type equations [10]

$$\partial_\theta\sigma + \partial_\Lambda\sigma\partial_\Lambda\alpha + \frac{1}{2}\sigma\partial_\Lambda\partial_\Lambda\alpha = 0, \quad (28)$$

$$\partial_\theta\alpha + \frac{1}{2}(\partial_\Lambda\alpha)(\partial_\Lambda\alpha) + W - \frac{1}{2}\partial_\Lambda\partial_\Lambda\sigma/\sigma = 0. \quad (29)$$

The real component (28) multiplied by σ is converted into the density conservation-type equation

$$\partial_\theta\sigma^2 + \partial_\Lambda(\sigma^2\partial_\Lambda\alpha) = 0. \quad (30)$$

(xvi) *The imaginary part is a math equivalent of the Hamilton-Jacobi equation of classical mechanics, the 2D cell's flickering phase (or angle of a 3D frame rotation) playing the role of the action function.*

The imaginary component (29) of the Bohm-type system has the form

$$\partial_\theta\alpha + \frac{1}{2}(\partial_\Lambda\alpha)(\partial_\Lambda\alpha) + W - \frac{1}{2}\partial_\Lambda\partial_\Lambda\sigma/\sigma = 0. \quad (31)$$

If all terms in Eq. (31) are rapidly changing functions ("inside" the 2D cell), then the set of equations (30), (31) is just an equivalent of Eq. (18). But it may happen that only the last term in Eq. (31) (depending on σ) is a rapidly changing (short-scale) one while the terms depending on α are functions slowly changing "outside" the 2D cell (long-scale ones). In this case we have to consider the free function $W = W_{\text{in}} + W_{\text{ex}}$ as split into "interior" and "exterior" parts; so Eq. (30) splits into the respective equations

$$\partial_\Lambda\partial_\Lambda\sigma - 2W_{\text{in}}\sigma = 0, \quad (32)$$

$$\partial_\theta\alpha + \frac{1}{2}(\partial_\Lambda\alpha)(\partial_\Lambda\alpha) + W_{\text{ex}} = 0. \quad (33)$$

The static equation (32) determines the fractal density distribution under the influence of some interior reason W_{in}. But Eq. (33) is the familiar math analogue of the Hamilton-Jacobi equation of classical mechanics, the phase α of a 2D cell flickering (or half angle of the frame $\mathbf{q}_k$ rotation about $\mathbf{q}_3$) playing the role of the action function, the term W_{ex} behaving as an exterior potential.

(xvii) *In physical units (on the laboratory scale) the phase of a 2D cell's flickering is a mechanical action function measured in units of the Planck constant. Then the dynamic math equations following from the Bohm-type equations become the mass conservation equation and Hamilton-Jacobi equation. The static equation determines the fractal mass density distribution.*

Let the phase be a slowly changing (laboratory scale) function in the physical units chosen above. Then replacing $\partial_\Lambda \to (\hbar/mc)\partial_n$, $\partial_\theta \to (\hbar/mc^2)\partial_t$ and taking into account that $\sigma^2 \sim \rho$, Eq. (30) is rewritten as

$$\partial_t\rho + \partial_n(\rho u_n) = 0, \quad (34)$$

where $u_k \equiv \partial_k S/m$ is the 3D velocity, and S is the classical mechanical action function, the phase measured in the Planck constant units:

$$S(x,t) \equiv \hbar\alpha(x,t). \quad (35)$$

Equation (31) in the physical units becomes precisely the Hamilton-Jacobi equation

$$\partial_t S + \frac{1}{2m}(\partial_m S)(\partial_m S) + U_{\text{ex}} = 0, \qquad (36)$$

where $U_{\text{ex}} \equiv mc^2 W_{\text{ex}}$. We emphasize that Eq. (36), deduced in fact as a square root from the 3D continuity-type equation, should be referred to as a spinor (fractal) equation of classical mechanics.

Finally, Eq. (32) in physical units has the form

$$\partial_m \partial_m \sigma - R_{\text{in}} \sigma = 0, \qquad (37)$$

where $R_{\text{in}} \equiv 2W_{\text{in}}/\varepsilon^2$ is some interior "potential" measured (as, e.g., curvature) in cm^{-2}; we leave it for future explorations.

3.8. Geometric (Physical) Equations

(xviii) *The minimum value of the flickering phase on the free parameter segment entails a math equivalent of the Euler-Lagrange equation (Newton's dynamic law) of classical mechanics.*

Returning to the math equations (33), let us replace in it the partial derivative with the full one, $\partial_\theta \alpha = d_\theta \alpha - d_\theta \xi_\Lambda \cdot \partial_\Lambda \alpha$, to obtain the phase value integral on the segment $[\theta_1, \theta_2]$

$$\alpha = \int_{\theta_1}^{\theta_2} \left(d_\theta \xi_\Lambda \partial_\Lambda \alpha - \frac{1}{2} \partial_\Lambda \alpha \partial_\Lambda \alpha - W_{\text{ex}} \right) d\theta. \quad (38)$$

The "minimum phase" requirement selects the extreme lines $\xi_\Lambda(\theta)$ with "observables" $d_\theta \xi_\Lambda$ (M-dimensional velocity) and $\partial_\Lambda \alpha$ (momentum) obeying the equation

$$\partial_\theta \left[\partial_{\text{K}} \alpha + \frac{\partial(\partial_\Lambda \alpha)}{\partial(d_\theta \xi_{\text{K}})} (d_\theta \xi_\Lambda - \partial_\Lambda \alpha) \right] + \partial_{\text{K}} W_{\text{ex}} = 0. \qquad (39)$$

We recognize in the integrand in Eq. (38) a math analogue of the Lagrangian function of classical mechanics, Eq. (39) is a math analogue of the dynamic equation of Newtonian dynamics. In physical units, the derivatives are $d_\theta \xi_\Lambda \to \tau d_t (x_n/\varepsilon) = u_n/c$, $\partial_\Lambda \alpha \to \varepsilon \partial_n (S/\hbar) = u_n/c$; their insertion into Eqs. (38), (39) converts them into the classical action functional

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2} m u_n u_n - U_{\text{ex}} \right) dt, \qquad (40)$$

where $U_{\text{ex}} \equiv mc^2 W_{\text{ex}}$, and the Newtonian dynamic equation

$$\partial_t (m u_n) + \partial_n U_{\text{ex}} = 0. \qquad (41)$$

Strangely enough, the basic physical law (41) discovered in experiment appears to be just a very special case of a simple purely mathematical model.

3.9. The Helix Model and a Relativistic Particle

(xix) *The model of a particle in 3D space is a point-like mass (distributed in a very small volume of the characteristic length size) with a frozen-in 3D frame able to rotate. If it is permanently rotating, then at a point of the particle's ultimate radius (half of the scale unit) the rotation velocity equals that of light; if this particle moves, then this point depicts in space a cylindrical helix line. The velocity of a moving particle's border point remains maximum, i.e., that of light.*

Geometrically, the particle is conceived as a mass m distributed in a small 3D volume of the size ε with a triad $\mathbf{q}_k$ "frozen" at its center and rotating (with the mass) about one of its vectors by an angle equal to the doubled 2D cell's flickering phase; the angle's gradient shows the direction of the particle motion. A free particle moving in 3D space along the coordinate z with the velocity $dz/dt \equiv u = \text{const}$, ($t$ is the observer's time) satisfies two conditions:

(1) the particle's triad rotates permanently about $\mathbf{q}_3$ with the frequency $d(2\alpha)/dt \equiv 2\omega = \text{const}$, the angle between $\mathbf{q}_3$ and the velocity vector is $\beta = \text{const}$;

(2) a point at the particle's ultimate radius $\varepsilon/2$ depicts a helix-type line, the point's linear velocity is always maximum, i.e., that of light.

(xx) *The difference between the squares of the free particle's helix small length and the path has the form of the space-time interval of special relativity; computed in physical units, this interval gives the action function of a relativistic particle.*

The helix line, given by the coordinate functions

$$x = (\varepsilon/2) \cos 2\alpha \cos \beta,$$
$$y = (\varepsilon/2) \sin 2\alpha,$$
$$z = ut - (\varepsilon/2) \cos 2\alpha \sin \beta, \qquad (42)$$

has the line element $dl^2 = \varepsilon^2 d\alpha^2 + 2\varepsilon \sin 2\alpha \times \sin \beta d\alpha u dt + u^2 dt^2$. Condition (2) means $dl = c dt$, then from the line element we obtain

$$c^2 = \varepsilon^2 \omega^2 + 2u\varepsilon\omega \sin(2\omega t) \sin \beta + u^2. \qquad (43)$$

For a moving free particle ($u = \text{const}$) Eq. (43) holds if $\beta = 0, \pi$, i.e., the regular helix lies on the circular cylinder $dl^2 = c^2 dt^2 = \varepsilon^2 d\alpha^2 + u^2 dt^2$; let us find the 2D cell's phase on the segment $[t_1, t_2]$:

$$\alpha = \pm \frac{c}{\varepsilon} \int_{t_1}^{t_2} \sqrt{1 - \frac{u^2}{c^2}}\, dt, \qquad (44)$$

the signs indicating the right or left helicity. Let us insert $\varepsilon = \hbar/(mc)$ and choose the minus sign, then, taking into account Eq. (35), Eq. (44) yields the action of a free relativistic particle

$$S = \alpha \hbar = -mc^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{u^2}{c^2}}\, dt. \qquad (45)$$

So the line element of special relativity $\varepsilon^2 d\alpha^2 \equiv ds^2 = c^2dt^2 - dz^2$ acquires a specific geometric meaning. The "space-time interval" $ds = \varepsilon d\alpha$ has the meaning of an arc length of the particle's "ultimate circumference," the immobile particle's border in the rotation plane. For a free particle in its own frame this arc length is a definite unchanging number (an invariant of special relativity).

(xxi) *Reduction to the non-relativistic case automatically establishes relations between classical and quantum quantities, thus determining the free particle 2D model (protoparticle) as a De Broglie wave, the particle's rest energy linked to the permanent flickering of the 2D cell.*

We rewrite Eq. (45) in a differential form and reduce it to the non-relativistic case:

$$\hbar d\alpha \cong -Edt + p_n dx_n$$
$$\equiv -(mc^2 + mu^2/2)dt + mu_n dx_n \cong dS, \qquad (46)$$

the free particle is considered to be a quantum one, $\hbar d\alpha = \hbar\partial_t \alpha dt + \hbar \partial_n \alpha dx_n$. Then from Eq. (46) we have $\hbar\omega dt + \hbar k_n dx_n \cong -Edt + p_n dx_n$, thus automatically obtaining the de Broglie energy-frequency and momentum-wave vector ratios

$$E = (mc^2 + mu^2/2) = |\omega|\hbar, \quad p_n = k_n \hbar; \qquad (47)$$

the state function of the particle (e.g., with positive parity)

$$\psi'^{+} = \sigma e^{i\alpha}\tilde{\psi}^{+} = \sigma e^{i(p_n x_n - Et)/\hbar}\tilde{\psi}^{+}$$
$$= \sigma e^{i(k_n x_n - \omega t)}\tilde{\psi}^{+} \qquad (48)$$

has precise the form of a de Broglie wave. From Eqs. (47) one notes, in particular, that even for a particle immobile in space ($u = 0$) the 2D cell must be permanently "pumped over" ($\omega \neq 0$), the flickering frequency $|\dot{\alpha}| \equiv \omega_0$ determining the particle's rest energy $E_{\rm rest} \equiv mc^2 = \omega_0 \hbar$.

(xxii) *In the classical case the helix model expectedly yields the Hamilton-Jacobi equation.*

Let the free particle be classical with $dS = \partial_t S + u_n \partial_n S$. Then from Eq. (46) we have $-mc^2 + mu^2/2 = \partial_t S + u_n \partial_n S$, or the Hamilton-Jacobi equation $\partial_t S + \partial_n S \partial_n S/(2m) + E_{\rm rest} = 0$, here appearing as a consequence of the helix model (no exterior potentials for a free particle).

3.10. An Irregular Helix Model and "More General" Relativity

(xxiii) *Making the 3D particle's helix line irregularly curved and compressed (as if the moving particle is subject to a variable external force) leads to appearance of variable metric components and general relativity-type "space-time" line element, the space components though being negligibly small on a laboratory scale. A metric function measuring the helix compression emerges in the respective Hamilton-Jacobi equation as an external potential.*

A point of a particle's triad vector (e.g., of $\mathbf{q}_1$) in a force field must move along a distorted helix line, its length element being determined as follows. Let $\mathbf{q}_n(t, x)$ be the particle's Frenet-type triad with $\mathbf{q}_3$ tangent to the particle trajectory $x_k(t)$, its local curvature being $R(x)$. Define a quaternion radius vector $\mathbf{l} \equiv \varepsilon \mathbf{q}_1 + r\mathbf{q}_3$, where ε is the helix diameter (constant), r is a length along the trajectory; the differential of $\mathbf{l}$ is

$$d\mathbf{l} \equiv r\omega_{331} dr \mathbf{q}_1 + (\varepsilon\omega_{312} + r\omega_{332}) dr \mathbf{q}_2$$
$$+ (1 - \varepsilon\omega_{331}) dr \mathbf{q}_3,$$

since $d\mathbf{q}_n = \omega_{jnk} dx_j \mathbf{q}_k$ where ω_{nkj} are connection components, among them $\omega_{312} = d\alpha/dr$ is "torsion" (rotation of $\mathbf{q}_1$ about $\mathbf{q}_3$), $\omega_{331} \equiv R$ is the trajectory's first curvature, and ω_{332} (neglected for simplicity) is the second curvature [5]. So the "curved" helix line element has the form

$$dl^2 = d\mathbf{l} d\bar{\mathbf{l}} = \varepsilon^2 d\alpha^2 + [(1 - \varepsilon R)^2 + r^2 R^2] dr^2$$
$$\equiv \varepsilon^2 d\alpha^2 + e^{2G(x)} dr^2.$$

If the helix is additionally "compressed" with the measure $e^{-2W(x)}$, then the line element becomes $dl^2 = e^{-2W(x)}[\varepsilon^2 d\alpha^2 + e^{2G(x)} dr^2]$, and the helix "space-time interval" acquires the features of general relativity:

$$\varepsilon^2 d\alpha^2 = ds^2 = e^{2W(x)} c^2 dt^2 - e^{2G(x)} dr^2. \qquad (49)$$

For small spatial curvatures $e^{2G} \approx 1$, and for a nonrelativistic classical particle Eq. (49) gives the action differential ($dS = \hbar d\alpha$) as

$$dS = -mc^2 dt \sqrt{e^{2W} - (u/c)^2}$$
$$\approx -[mc^2(1 + W) - mu^2/2]\, dt,$$

or equivalently, the Hamilton-Jacobi equation $\partial_t S + \partial_n S \partial_n S/(2m) + U = 0$ with the potential $U \equiv mc^2 W + E_{\rm rest}$.

(xxiv) *The 3D space Euler-Lagrange equation of a "squeezed helix particle" exactly coincides with the 4D space-time geodesic equation, thus demonstrating a convergence of general relativity and the helix model theory.*

Variation of the "space-time" interval $\delta \int_a^b \sqrt{g_{\mu\nu} u^\mu u^\nu} ds = 0$, $u^\mu \equiv dx^\mu/ds$ of the compressed helix $ds^2 = g_{00}(dx^0)^2 - \delta_{kn} dx^k dx^n = e^{2W(x)} c^2 dt^2 - dr^2$ yields the equation of an extremal (geodesic) line $d_s(g_{\mu\lambda} u^\lambda) = g_{\alpha\beta} u^\alpha u^\beta/2 \to m\partial_t u_k =$

$-\partial_k U(e^{2W}\delta_{kn} - 2u_k u_n/c^2)$, the same as the Euler-Lagrange equation following from the Lagrangian $L(x, \dot{x}) = -mc^2\sqrt{e^{2W(x)} - \dot{x}_k\dot{x}_k/c^2}$. So the "irregular relativistic helix" model, a consequence of the 2D cell conjecture, partly explains the heuristic geometrization of interactions.

4. CONCLUSION

Here we summarize the logic and structure of the suggested theory. There are two parallel but drastically different realms in the theory, one is an "unobservable" area comprising primitive math relations and pregeometric images on the fractal surface, the other is an "observable" area containing equalities composed of the primitive ones, and usual geometric objects in physical space. We start with deformations of a small pregeometric domain, at the same time trying not to wreck the properties of geometric objects thus saving a set of algebras; the price is a series of fractal equations. Written in physical units, these pure math equalities lead precisely to equations of quantum (Schrödinger, Pauli) and classical mechanics (the Hamilton-Jacobi equation, the Newtonian dynamic equation). Simultaneously a fractal protoparticle model arises, its phase having the meaning of mechanical action; the respective geometric analog is a rotating massive pointlike particle. This model leads to an original "helix-line" formulation of mechanics of a free relativistic particle (if the helix is a regular cylindrical "spring"); for an irregular "spring" the space-time metric becomes point-dependent, and the relativistic mechanics is described by the geodesic equation, so that the theory acquires features of general relativity. The respective nonrelativistic fractal equation is again that of Hamilton-Jacobi.

There are some challenging problems waiting for solution; among them are the analysis of the static equation for a particle's fractal density (as well as physical density) distribution, construction of pregeometric models for massless particles, and possibly for the electric charge.

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