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HYDROPHYSICAL PROCESSES

Groundwater Table Formation in the Coastal Zone over a Bed with Arbitrary Shape

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Abstract—The impact of tidal wave on a permeable shore consisting of porous material results in groundwater table lying above the average sea level. This effect, stemming from the nonlinearity of the process of seawater flow in the soil, is referred to as pumping effect.

Keywords: pumping effect, sea coastal zone, groundwater flow in porous medium in sea coastal zone

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INTRODUCTION

Studying pumping effect in the coastal zone of a sea is of great theoretical and practical importance. Thus, variations of groundwater table can cause various engineering, environmental, and, if freshwater level in soil is involved, even social problems. Many studies discuss the factors that govern the formation of groundwater table. For example, a vast review of publications on the effect of tidal waves on groundwater table in a coastal aquifer is given in [13].

Some studies (e.g., [2–4, 8]) provide a mathematical description of the process of water intrusion into the soil in coastal zone and various modifications of this process. For example, the model discussed in [11] incorporates capillary effect and that in [10] takes into account the effect of rains. Notwithstanding the wide range of publications, the major portion of them describes the process of water flow through soil in the case of a vertical or sloped shore, which corresponds to the common shore profile. However, in some cases, the researcher faces the problem of describing the pumping effect for situations more complicated than a flat sloped shore. In such cases, the problem is to describe the pumping effect for an arbitrary shore profile. This problem is the focus of this article.

MATHEMATICAL MODEL

A mathematical model of the process of groundwater table formation in the coastal zone with periodic variations of sea level is described in [1, 11, 12, 14], where an equation was derived to describe free groundwater surface as a function of horizontal distance from water edge to the point under consideration. The analysis of the equation of the free surface gives estimates of the scale of pumping effect; in particular, those studies provide some numerical estimates for the specific conditions of a research ground in Spitsbergen (Norway).

Below, we will use the results obtained in [1], and here we recall problem formulation and the main denotations.

We study a one-dimensional (along the horizontal coordinate x) groundwater flow, which can be described by a nonlinear Boussinesq equation [6, 7]:

$$m\frac{\partial h}{\partial t} = \mathbb{C}\frac{\partial}{\partial x}\left((H + h)\frac{\partial h}{\partial x}\right), \quad (1)$$

here m is aquifer porosity, dimensionless; $\mathbb{C}$ is hydraulic conductivity with the dimension of velocity; H is the depth to the impermeable bed, measured from the average sea level and assumed constant; and $h(x,t)$ is the equation of free groundwater table measured from the average water level in a coastal zone of the sea.

We consider an aquifer of infinite width; therefore, the boundary conditions are taken in the form of attenuation of groundwater flow velocity at the infinity at $x \to +\infty$

$$\frac{\partial h}{\partial x}_{x\to+\infty} = 0 \quad (2)$$

with a condition of equality of water levels in the sea $h_w(t)$ and in the soil $h(x,t)$ at the movable water edge

$$h(x(t),t) = h_w(t),$$

here $x = x(t)$ describes the motion of water edge in the horizontal direction.

Consider the simplest case, where the vertical motion of the free water surface in the sea follows the law:

$$h_w(t) = a\cos\omega t, \quad (3)$$

here a, ω are the amplitude and frequency of sea level variations.

We specify an arbitrary monotonous shape of bed profile as a dependence of x on y:

$$x = F(y). \quad (4)$$

Now, the motion of water edge in the vertical direction will follow (3), and its motion in horizontal direction will be described by the equation:

$$x(t) = F(h_w(t)). \quad (5)$$

Expanding (4) in powers of y yields the series:

$$x = m_1 y + m_2 y^2 + m_3 y^3 + \cdots. \quad (6)$$

The motion of water edge along the horizontal $x(t)$ can be derived from (6) by replacing y by (3) to yield

$$x = \sum_{n=1}^{\infty} m_n a^n \cos^n \omega t. \quad (7)$$

In particular, for a linear shore with a slope of α, we have from (7):

$$x(t) = a\cos\omega t \cot\alpha,$$

which corresponds to the reasoning in [1].

Now let us change to a movable coordinate system moving in the horizontal direction with water edge. With this in view, we change the coordinates:

$$\tilde{x} = x - x(t),$$

to obtain $h = h(\tilde{x}, t)$ in the new coordinate system.

In the movable coordinate system, equation (1) becomes:

$$\frac{\partial h}{\partial t} - \dot{x}(t)\frac{\partial h}{\partial \tilde{x}} = \varkappa \frac{\partial}{\partial \tilde{x}}\left((H + h)\frac{\partial h}{\partial \tilde{x}}\right), \quad (8)$$

where $\varkappa = \dfrac{\mathbb{C}}{m}$.

The boundary conditions for this case are

$$h(0,t) = \alpha\cos\omega t, \quad \frac{\partial h}{\partial \tilde{x}}_{\tilde{x}\to\infty} = 0. \quad (9)$$

Now, the first equation in (9) corresponds to (5); and the second, to (2).

Let us consider in more detail the description of coefficient $\varkappa$. This coefficient is calculated as the ratio of the hydraulic conductivity to soil porosity; it can vary within wide limits, depending on soil permeability. According to [6], the porosity ratio of sand is 0.3–0.45, and its hydraulic conductivity is 10^{-3}–10^{-5} $\frac{\text{m}}{\text{s}}$. For clay, the porosity ratio is 0.40–0.55, and its hydraulic conductivity is 5×10^{-7}–5×10^{-9} $\frac{\text{m}}{\text{s}}$. As shown below, such "sensitivity" of $\varkappa$ to the permeability of material can have a strong effect on groundwater table in the aquifer.

To make equation (8) dimensionless, we introduce the following dimensionless variables:

$$X = \frac{\tilde{x}}{L}, \quad \tau = \frac{t}{T} = \omega t, \quad \eta(X, \tau) = \frac{h}{a},$$

here L is the horizontal penetration distance of wave perturbation into the coastal aquifer. In [1], a characteristic value

$$k = \sqrt{\frac{\omega}{2\varkappa H}}, \quad T = \frac{1}{\omega} \quad (10)$$

was used for L.

A method for evaluating L will be given below.

$$\frac{\partial \eta}{\partial \tau} - \varepsilon\, \dot{x}(\tau)\frac{T}{L}\frac{\partial \eta}{\partial X} = \frac{\varkappa H T}{L^2}\frac{\partial}{\partial X}\left((1 + \varepsilon\eta)\frac{\partial \eta}{\partial X}\right), \quad (11)$$

where $\varepsilon = \dfrac{a}{H}$ is a dimensionless parameter.

Expression (11) for the case of a flat sloped shore takes the form:

$$\frac{\partial \eta}{\partial \tau} - \varepsilon\beta\sin\tau\frac{T}{L}\frac{\partial \eta}{\partial X} = \frac{\varkappa H T}{L^2}\frac{\partial}{\partial X}\left((1 + \varepsilon\eta)\frac{\partial \eta}{\partial X}\right), \quad (12)$$

here we introduce a new dimensionless parameter:

$$\beta = \frac{H}{L}\cot\alpha, \quad (13)$$

α is the slope of the linear part in (5).

Considering expressions for L and T, we have from (10):

$$\frac{\partial \eta}{\partial \tau} = \varepsilon\beta\sin\tau\frac{T}{L}\frac{\partial \eta}{\partial X} + \frac{1}{2}\frac{\partial}{\partial X}\left((1 + \varepsilon\eta)\frac{\partial \eta}{\partial X}\right). \quad (14)$$

Expression (12) contains two independent dimensionless parameters: ε and $\delta = \varepsilon\beta$. In the general case, the values of these parameters can be arbitrary. For the conditions of the experimental ground on Spitsbergen, $\varepsilon \sim 0.166$, and $\beta \sim 0.94$; therefore, both parameters ε, δ can be assumed to be of the same order of smallness. Therefore, both summands in the right-hand part of (12) are also small variables of the same order.

Equation (12) in dimensional form, with the preserved smallness of appropriate summands, can be written with the use of formally "small parameters," which will be taken equal to unit after the completion of calculations. In the general form, expression (11) can be written as

$$\frac{\partial h}{\partial t} - \delta \dot{x}(t)\frac{\partial h}{\partial \tilde{x}} = \varkappa \frac{\partial}{\partial \tilde{x}}\left((H + \varepsilon h)\frac{\partial h}{\partial \tilde{x}}\right) \quad (15)$$

with boundary conditions

$$h(0,t) = a\cos\omega t, \quad \left.\frac{\partial h}{\partial \tilde{x}}\right|_{\tilde{x}\to\infty} = 0. \quad (16)$$

The summand $\delta \dot{x}(t)\frac{\partial h}{\partial \tilde{x}}$ in (15) accounts for bed slope; and $\varkappa \frac{\partial}{\partial \tilde{x}}\left(\varepsilon h \frac{\partial h}{\partial \tilde{x}}\right)$, for the nonlinearity of the flow. Considering the smallness of summands, expression (15) can be used to search for the solution of the equation through expanding the function sought for in series in a single small parameter ε, with ε and δ assumed to have the same order of smallness; replacing in (15) δ by ε, we obtain for this case:

$$\frac{\partial h}{\partial t} - \varepsilon \dot{x}(t)\frac{\partial h}{\partial \tilde{x}} = \varkappa \frac{\partial}{\partial \tilde{x}}\left((H + \varepsilon h)\frac{\partial h}{\partial \tilde{x}}\right). \quad (17)$$

PUMPING EFFECT FOR A BED WITH ARBITRARY PROFILE

A method for solving equation (17) is described in detail in [1]. Here, we will give a brief description of the method and use its result.

To solve (17) under conditions (16), we expand the sought-for function $h(x,t)$ in series in the small parameter ε, substitute the obtained sum into (17), (16) and equate the terms with the same smallness order with respect to ε.

In this case, for a zero-order approximation with ε^0, we obtain a linear equation with two boundary conditions:

$$\frac{\partial h_0}{\partial t} = \varkappa H \frac{\partial^2 h_0}{\partial \tilde{x}^2}, \quad h_0(0,t) = a\cos\omega t, \quad \left.\frac{\partial h_0}{\partial \tilde{x}}\right|_{\tilde{x}\to\infty} = 0. \quad (18)$$

For the first-order approximation with respect to ε^1, we obtain a linear nonhomogeneous heat conduction equation with zero boundary conditions:

$$\frac{\partial h_1}{\partial t} - \varkappa H \frac{\partial^2 h_1}{\partial \tilde{x}^2} = \dot{x}(t)\frac{\partial h_0}{\partial \tilde{x}} + \frac{\varkappa}{2}\frac{\partial^2 h_0^2}{\partial \tilde{x}^2}, \quad h_1(0,t) = 0, \quad \left.\frac{\partial h_1}{\partial \tilde{x}}\right|_{\tilde{x}\to\infty} = 0. \quad (19)$$

Further expansions in the small parameter will not be considered here.

The problem for the zero-order approximation is classical, and its solution is well-known:

$$h_0 = ae^{-k\tilde{x}}\cos(\omega t - k\tilde{x}),$$

where k is the wave number:

$$k = \sqrt{\frac{\omega}{2\varkappa H}}, \quad (20)$$

from here it follows that $L = \frac{1}{k}$ is the penetration depth of wave oscillations into the aquifer in the coastal zone which corresponds to (10).

Let us return to the consideration of the first-order approximation with ε^1. As the problem is linear, in the right-hand part of equation (19), we can consider separately the effects of stationary and nonstationary impact under homogeneous boundary conditions. The effect of stationary impact will be denoted by h_1^s; it will correspond to the pumping effect; the transient part will have multiple frequencies with frequency ω and will give 0 after averaging over time. Therefore, let us focus on the stationary part (19), i.e., on the pumping effect. The stationary part of equation (19) can be represented as a sum:

$$h_1^s = h_1^b + h_1^p,$$

where h_1^b is basic pumping, caused by $\frac{\varkappa}{2}\frac{\partial^2 h_0^2}{\partial \tilde{x}^2}$ from (19), and h_1^p is the profile pumping, caused by $\dot{x}(t)\frac{\partial h_0}{\partial \tilde{x}}$ from (19).

The basic component of the pumping effect is described in many studies, e.g., in [1]; therefore, here we will pass to an expression for h_1^b:

$$h_1^b = \frac{a^2}{4H}\left(1 - e^{-2k\tilde{x}}\right).$$

This type of pumping is not related to bed sloping. The contribution of the effect of pumping h_1^p will be produced only by the odd terms in the series:

$$\frac{dx(t)}{dt} = \sum_{n=1}^{\infty} - n\omega m_n \alpha^n \sin\omega t \cos^{n-1}\omega t, \quad (21)$$

and the solution becomes:

$$h_1^p = \left(\frac{1}{2}a^2 k m_1 + \frac{3}{8}a^4 k m_3 + \frac{5}{16}a^6 k m_5 \ldots\right) \times \left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right). \quad (22)$$

As follows from (22), the profile pumping depends on the shape of the bed via coefficients $m_1, m_3, m_5 \ldots$ An expression for the stationary part of the pumping effect, normalized by $\frac{a^2}{H}$, takes the form:

$$\frac{h_1^s}{\frac{a^2}{H}} = \frac{Hk}{2}\left(m_1 + \frac{3}{8}a^2 m_3 + \frac{5}{16}a^4 m_5 \ldots\right) \times \left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right) + \frac{1}{4}\left(1 - e^{-2k\tilde{x}}\right). \quad (23)$$

From here, it can be seen that, in dimensionless form, the basic pumping and the linear term in the profile part of solution does not depend on the amplitude. On the other hand, the nonlinear part of the profile pumping depends on the amplitude and manifests itself the greater, the higher the amplitude.

As follows from (23), in the case of a flat sloped shore, i.e., $m_1 = \cot\alpha, m_3 = m_5 = \ldots = 0$, we obtain the following expression for the stationary part of pumping effect:

$$\frac{h_1^s}{\frac{a^2}{H}} = \frac{Hk}{2}\cot\alpha\left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right) + \frac{1}{4}\left(1 - e^{-2k\tilde{x}}\right), \quad (24)$$

which fully coincides with the results obtained before in [1].

Now we give an example of function $F(y)$, which approximates the profile of the shore, such that there will be no slope component m_1.

For example, if we consider a shore profile in the form of a cubic parabola:

$$y = (px)^{\frac{1}{3}},$$

where parameter p determines the slope of the shore, than, the lesser p, the gentler the slope.

In this example, as $m_1 = m_5 = \ldots = 0$, the series (6) will consist of a single term, namely:

$$x = \frac{y^3}{p} = m_3 y^3.$$

Substituting this expression into (23), we come to the stationary part of equation (19) in the form:

$$\frac{h_1^s}{\frac{a^2}{H}} = \frac{3}{8}a^4\frac{Hk}{p}\cot\alpha\left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right) + \frac{1}{4}\left(1 - e^{-2k\tilde{x}}\right), \quad (25)$$

whence it follows that

$$h_1^s(\infty) = \frac{3}{8}a^4\frac{k}{p} + \frac{1}{4}\frac{a^2}{H}. \quad (26)$$

Expression (26) shows the essence of the process of seawater flow into the permeable soil of the coastal zone. Because of the nonlinear effects, the time-averaged groundwater level is higher than the average sea level. In [1], this effect was called *base pumping*, i.e., the effect of additional pumping of water into the soil; it is described by the term $\frac{1}{4}\left(1 - e^{-2k\tilde{x}}\right)$ in the sum (23). However, in the motion of water edge by a shore profile, the edge will move in both vertical and horizontal directions. Such motion causes an effect analogous to nonlinearity, which was called *profile pumping* in [1]. It is represented by the term $\frac{Hk}{2}\left(m_1 + \frac{3}{8}a^2 m_3 + \frac{5}{16}a^4 m_5 \ldots\right) \times \left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right)$ in the sum (23).

With fixed parameters of sea waves, the magnitude of profile pumping is the greater, the gentler the shore profile, as can be seen, in particular, from (26). This feature can be explained by the fact that the part of shore surface getting inundated because of the motion of water edge in horizontal direction as well is greater than that in the case of vertical profile, resulting in a greater surface of water seepage.

In studying groundwater flow in the coastal zone, most processes can be adequately described by the first term of series (6); however, in some cases, one has to incorporate nonlinear terms of series (6) to obtain a full description of the pumping effect, as shown in the example.

Now, a question arises of whether an example can be found in which the contribution of the first term of the approximating function to the pumping effect will be less than the effect of the subsequent terms of series (6). To do this, we will use (23) and consider the following dimensional expression:

$$h_1^p = \frac{Hkm_1}{2}\left(1 + \frac{3}{4}\alpha^2\frac{m_3}{m_1} + \ldots\right) = \frac{1}{2}a^2 k m_1 + \frac{3}{8}a^4 k m_3 + \ldots.$$

Now, to compare the contributions, we write the ratio of m_1 to m_3 in the following manner:

$$\frac{4m_1}{3a^2 m_3} < 1$$

or

$$\frac{m_1}{m_3} < \frac{3}{4}a^2.$$

This inequality holds at large enough amplitudes a. This means that the nonlinearity of the shore profile manifests itself at large amplitudes.

PUMPING EFFECT FOR SMALL ANGLES

In (14) a dimensionless parameter δ was introduced, defined as the product $\varepsilon\beta$, where β is a dimensionless parameter, related with the slope of the linear part of (6). Basing on the conditions of the experimental ground, it was assumed that parameter β is small in order to make δ also small. However, as β depends on the slope of shore profile, a question arises as to what is the critical value of shore profile slope α_{cr} to which the method proposed in this study can be applied?

At small ε and $\beta < 1$, equation (12) was examined in the previous chapter, and it was studied in more detail in [1], which corresponded to a limitation on the angle $\alpha > \alpha_{cr}$. However, if we have to study the situation with angles lesser than α_{cr}, the parameter β is to be considered large.

The test large value of β will be taken as $\beta = \frac{1}{\sqrt{\varepsilon}}$. This leads to a new restriction on angle α, $\alpha < \alpha_{cr}$, where α_{cr} is determined by the relationship $\tan\alpha_{cr1}$. Recall that for α_{cr} from the relationship $\tan\alpha_{cr} = \frac{H}{L}$, for the conditions of the experimental ground, it was found that $\beta \sim 0.94, \alpha = 6^\circ$, now $\alpha_{cr1} = 2^\circ$.

At the new value of parameter β in (15), the order of smallness of summands will change, leading to the relationship:

$$\frac{\partial h}{\partial t} - \sqrt{\varepsilon}\,\dot{x}(t)\frac{\partial h}{\partial \tilde{x}} = \varkappa\frac{\partial}{\partial \tilde{x}}\left((H + \varepsilon h)\frac{\partial h}{\partial \tilde{x}}\right) \tag{27}$$

under boundary conditions (16).

Let us expand equation (27) and the required solution in a series in powers of $\sqrt{\varepsilon}$:

$$h = h_0 + \sqrt{\varepsilon}h_1 + \varepsilon h_2 + \dots. \tag{28}$$

Substituting this expansion in (27), (16) and equaling the terms with equal orders of smallness in $\sqrt{\varepsilon}$, we obtain an appropriate equation with boundary conditions for each approximation.

For the zero-order approximation with the order of smallness of $\sqrt{\varepsilon}^0$:

$$\frac{\partial h_0}{\partial t} = \varkappa H\frac{\partial^2 h_0}{\partial \tilde{x}^2}, \quad h_0(0,t) = a\cos\omega t, \quad \frac{\partial h_0}{\partial \tilde{x}}_{\tilde{x}\to\infty} = 0.$$

The solution for this approximation can be written as:

$$h_0 = ae^{-k\tilde{x}}\cos(\omega t - k\tilde{x}),$$

hereafter, k is a wave number defined in (20).

Relationship (20) can be used to obtain an estimate of the horizontal size L as a characteristic size for the penetration of the wave perturbation into the soil of the coastal zone:

$$k = \frac{1}{L} = 0.02 \text{ for the experimental ground.}$$

For the first-order approximation with the smallness order of $\sqrt{\varepsilon}^1$, we obtain a relationship

$$\frac{\partial h_1}{\partial t} - \varkappa H\frac{\partial^2 h_1}{\partial \tilde{x}^2} = \dot{x}(t)\frac{\partial h_0}{\partial \tilde{x}}, \quad h_1(0,t) = 0, \quad \frac{\partial h_1}{\partial \tilde{x}}_{\tilde{x}\to\infty} = 0.$$

Now, using the linearity, we can show that the solution of this system of equations is the sum of the stationary h_1^s and wave h_1^w components, where

$$h_1^s = \frac{a^2 k m_1}{2}\left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right), \tag{29}$$

$$\begin{aligned} h_1^w = {} & \frac{a^2 k m_1}{2}e^{k\tilde{x}}\sin(2\omega t - k\tilde{x}) \\ & - \frac{\alpha^2 k m_1}{2}e^{-k\tilde{x}}\cos(2\omega t - k\tilde{x}) \\ & - \frac{a^2 k m_1}{2}e^{-\sqrt{2}k\tilde{x}}\sin\left(2\omega t - \sqrt{2}k\tilde{x}\right) \\ & + \frac{a^2 k m_1}{2}e^{-\sqrt{2}k\tilde{x}}\cos\left(2\omega t - \sqrt{2}k\tilde{x}\right), \end{aligned} \tag{30}$$

for the linear bed slope, the coefficient $m_1 = \cot\alpha$.

For the second-order approximation with respect to $\sqrt{\varepsilon}^2$, similar relationships have the form:

$$\begin{aligned} & \frac{\partial h_2}{\partial t} - \varkappa H\frac{\partial^2 h_2}{\partial \tilde{x}^2} = \dot{x}(t)\frac{\partial h_1}{\partial \tilde{x}} + \frac{\varkappa}{2}\frac{\partial^2 h_0^2}{\partial \tilde{x}^2}, \\ & h_2(0,t) = 0, \quad \frac{\partial h_2}{\partial \tilde{x}}_{\tilde{x}\to\infty} = 0. \end{aligned} \tag{31}$$

Under this approximation, we will study only the stationary part of the solution, because the wave part vanishes when averaged over an oscillation period; therefore, its contribution to the result is zero.

$$h_2^s = \frac{a^2}{4H}\left(1 - e^{-2k\tilde{x}}\right). \tag{32}$$

Further expansion in the small parameter $\sqrt{\varepsilon}$ will not be considered as negligible.

Thus, the stationary part of the solution of equation (27) with boundary conditions (16) can be written as a sum:

$$\begin{aligned} h^s = {} & \frac{a^2 k m_1}{2}\left(1 - e^{-k\tilde{x}}(\sin(k\tilde{x}) + \cos(k\tilde{x}))\right) \\ & + \frac{a^2}{4H}\left(1 - e^{-2k\tilde{x}}\right). \end{aligned} \tag{33}$$

Relationship (29) shows profile pumping, in which restrictions on the smallness of shore profile slope are removed. In this case, the major portion of the basic pumping remains unchanged and corresponds to (32). The total effect of pumping, comprising its basic and profile components, remains the same. This implies that relationship (33) can be used, but in (33), the restrictions on the smallness of bed profile slopes are eased. Therefore, relationship (33) can be used for

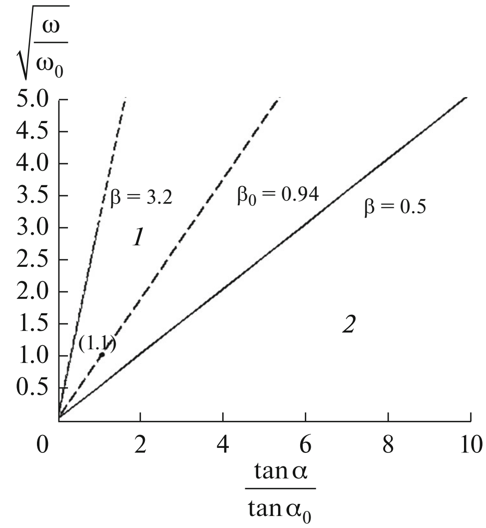

Fig. 1. The significance of the contributions of the basic and profile components to the pumping effect. Domain *1* is for the predominance of the profile over basic pumping effect. Domain *2* is for the predominance of the basic over profile pumping effect. Equilibrium line of the effects is the straight line $\beta = 0.5$. Point (1, 1) lies on the straight line $\beta = 0.5$ and corresponds to the conditions of the experimental ground on Spitsbergen Isl.

angles $\alpha_{\text{cr1}} < \alpha < \alpha_{\text{cr}}$. Qualitatively, this means that the profile pumping will dominate.

With $\tilde{x}$ in (29) tending to infinity, using (13), we obtain the expression:

$$\frac{h_1^s(\infty)}{\frac{a^2}{H}} = \frac{1}{2}\beta. \quad (34)$$

The analysis of the effect for small slopes of shore profile shows profile pumping to be the dominating component. However, it should be mentioned that for a shore profile with an arbitrary shape, in the case of small β, i.e., for $\alpha > \alpha_{\text{cr}}$, the following result was derived from (24):

$$\frac{h_1^s(\infty)}{\frac{a^2}{H}} = \frac{1}{2}\beta + \frac{1}{4}. \quad (35)$$

To evaluate the contributions of the basic and profile components to the total pumping effect, we choose some value of angle α_0 and the appropriate parameters β_0 and ω_0 as characteristics for comparison with other analogous α, β, and ω. In particular, for the conditions of the experimental ground on Spitsbergen Island, ω_0 is the frequency of semidiurnal tide, then $\tan\alpha_0 = \sqrt{\frac{\omega_0 H}{2\varkappa}}$, where $H = 6$ m, $\varkappa = 0.03\ \frac{\text{m}}{\text{s}}$, $\beta_0 = 0.94$. Now, we have a relationships combining all the characteristics mentioned above:

$$\frac{\beta}{\beta_0}\frac{\tan\alpha}{\tan\alpha_0} = \sqrt{\frac{\omega}{\omega_0}}, \quad (36)$$

which can be used to evaluate the pumping effect for other slopes and wave frequencies.

In coordinates $\left(\frac{\tan\alpha}{\tan\alpha_0}, \sqrt{\frac{\omega}{\omega_0}}\right)$, isolines $\beta = \text{const}$ are straight lines running from the origin of coordinates. Figure 1 gives an example of isolines for three cases with β equal to 0.5, 0.94, and 3.2, respectively. The straight line $\beta = 0.5$ corresponds to the case of additional pumping of groundwater from the basic and profile pumping effects. Above this line (domain *1* in Fig. 1), the profile pumping effect is greater than the basic one. The situation in the lower part (domain *2* in the figure) is opposite.

As can be seen from Fig. 1, for the conditions of the experimental ground in Spitsbergen Island, the isoline $\beta = 0.94$ lies in zone *1*, implying that the profile pumping is more important than the basic pumping. A further decrease in α will result in that parameter β will increase and, hence, the relative contribution of the profile pumping effect will also increase. An increase in frequency ω with conserved α will lead to a similar result.

RESTRICTIONS ON WAVE FREQUENCIES

In addition to the effect of shore profile, as mentioned above, the oscillation frequency of sea waves also has its effect on seawater pumping into the soil of the coastal zone. According to (36), the higher the frequency of wave oscillation ω, the stronger the manifestation of the profile pumping effect. However, the described mathematical model cannot be used for large frequencies. The Boussinesq equation used here has been derived for the case $kH \ll 1$, whence a restriction on ω in the model can be derived.

Let us use a restriction

$$kH \ll 1. \quad (37)$$

Substituting the expression for wavenumber k from (20) into (37), we obtain

$$\sqrt{\frac{\omega}{2\varkappa H}}H \ll 1. \quad (38)$$

The final restriction for wave periods can be obtained from (38):

$$\frac{\pi H}{\varkappa} \ll T. \quad (39)$$

For the parameters of the experimental ground on Spitsbergen Isl., the coefficient $\varkappa = 0.03\ \frac{\text{m}}{\text{s}}$; such value is typical of sand, $H = 6$ m. Substituting these values into (39), we obtain the following restriction on the period of wave oscillation: $T \gg 10$ min. This means that, under some conditions at the experimental ground, the mathematical model considered here describes the impact produced on groundwater level by waves, such as tidal and seiche, as well as any low-frequency water level variations of any nature. Waves with higher frequencies cannot be considered in this model.

CONCLUSIONS

Some results were obtained, describing the behavior of groundwater in the coastal zone of a sea.

A mathematical model was presented, describing groundwater table rise in the case of an arbitrary bed profile. It was shown that, as well as in the case of a flat sloped shore, the total pumping effect comprises a basic component, resulting from the nonlinearity of the process, and a profile component, resulting from bed profile. To approximate the bed profile by a nonlinear function, the dependence of profile pumping on the coefficients of function expansion is given.

In addition, it was shown that, in a dimensionless form, the linear component of profile pumping does not depend on the amplitude of water oscillation, while nonlinear terms show a power dependence on the amplitude.

Estimates of pumping effect are given for power functions approximating bed profile.

For the particular case of bed shape—a flat sloped shore—a critical value of bed slope is given for which the mathematical model is true.

It was shown that, at the shoreline slope $2^\circ < \alpha_{cr} < 7^\circ$ the total pumping at the experimental ground on Spitsbergen Isl. is mostly due to profile pumping effect.

Restrictions are given on the types of waves which are considered under the mathematical model described here for the experimental ground on Spitsbergen Isl. The proposed methods were shown to be applicable to waves with periods not exceeding 10 min.

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