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Finite Groups with ℙ-Subnormal Schmidt Subgroups

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Abstract—A subgroup H of a group G is called ℙ-subnormal in G whenever either $H = G$ or there is a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, 2, \ldots, n$. We study the structure of a finite group G all of whose Schmidt subgroups are ℙ-subnormal. The obtained results complement the answer to Problem 18.30 in the *Kourovka Notebook*.

Keywords: finite group, ℙ-subnormal subgroup, Schmidt subgroup, saturated Fitting formation.

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INTRODUCTION

We consider only finite groups.

A subgroup H of a group G is called ℙ-*subnormal* if it either coincides with the group G or is connected with G by a chain of subgroups all of whose indices are primes. The notion of ℙ-subnormal subgroup was proposed in [1] in connection with the development of the famous Huppert theorem that a group G is supersoluble if and only if any of its proper subgroups can be connected with G by a chain of subgroups with prime indices.

Groups with a system Σ of given ℙ-subnormal subgroups were studied in many papers. In particular, groups in which every Sylow subgroup is ℙ-subnormal were described in [2]. The supersolubility of a group in the cases where Σ is the set of normalizers of all Sylow subgroups of G and Σ is the set of all Hall subgroups of G was proved in [3]. Classes of groups with ℙ-subnormal primary subgroups and ℙ-subnormal primary cyclic subgroups were considered in [4]. The structure of groups representable as a product of ℙ-subnormal subgroups was studied in [5].

A special place in the study of groups with a given system of ℙ-subnormal subgroups is occupied by the case when $\Sigma = \mathrm{Sch}(G)$ is the set of all Schmidt subgroups of G. Recall that a *Schmidt group* is a nonnilpotent group all of whose proper subgroups are nilpotent. A simple check shows that every nonnilpotent group contains at least one *Schmidt subgroup* (i.e., a subgroup that is a Schmidt group). The study of groups with a given system of ℙ-subnormal subgroups was motivated by Problem 18.30 from the *Kourovka Notebook* [6]:

Is a finite group soluble if all its Schmidt subgroups are ℙ-subnormal?

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Tyutyanov used the classification of finite simple groups to obtain a positive answer to this question in [7]. In connection with this result, it is natural to formulate a more general problem:

Investigate the normal structure of a group all of whose Schmidt subgroups are $\mathbb{P}$-subnormal.

Particular aspects of this problem were addressed in [8], where the metanilpotency of a group with $\mathbb{P}$-subnormal generalized Schmidt subgroups was established. A *generalized Schmidt group* was understood as any *B-group*, i.e., a group whose quotient group by the Frattini subgroup is a Schmidt group (the notion of B-group was proposed by Berkovich in [9]). It is clear that any Schmidt group is a B-group. At the same time, a dihedral group of order 18 is a B-group and not a Schmidt group. As follows from the structure of Schmidt groups, G is a B-group if and only if $G/\Phi(G)$ is a biprimary Miller–Moreno group, i.e., a nonnilpotent group all of whose proper subgroups are abelian.

According to the results of [10], groups with $\mathbb{P}$-subnormal Schmidt subgroups are much more complex than groups with $\mathbb{P}$-subnormal B-subgroups.

For a group G, let $\pi(G) = \{p_1, p_2, \dots, p_r\}$ with $p_1 > p_2 > \dots > p_r$, and let P_i be a Sylow p_i-subgroup of G for $i = 1, 2, \dots, r$. We will say that a group G has a *Sylow tower of supersoluble type* (or *G is Ore dispersive*) if the subgroups P_1, P_1P_2, …, $P_1P_2\dots P_{r-1}$ are normal in G. In what follows, we will denote by $\mathfrak{D}$ the class of all groups G having a Sylow tower of supersoluble type. Further, for a given prime p, we denote by $\mathfrak{D}_{\pi(p-1)}$ the class of all Ore dispersive groups G such that $\pi(G) \subseteq \pi(p-1)$, where $\pi(p-1)$ is the set of all prime divisors of $p-1$.

Our main goal is to prove the following theorem.

Theorem 1. *Let $\mathfrak{F} = \{H \mid \mathrm{Sch}(H) \subseteq \mathfrak{U}\}$, where $\mathfrak{U}$ is the class of all supersoluble groups. Then the following statements hold*:

(1) *$\mathfrak{F}$ is a local formation with canonical local definition F such that $F(p) = \mathfrak{N}_p\mathfrak{D}_{\pi(p-1)}$ for each prime p*;

(2) *if each Schmidt subgroup of G is $\mathbb{P}$-subnormal, then $G/F(G) \in \mathfrak{F}$*;

(3) *if each Schmidt subgroup of G is $\mathbb{P}$-subnormal and H is its nonsupersoluble Schmidt subgroup with normal Sylow p-subgroup P, then $P \subseteq G^{\mathfrak{F}} \cap F(G)$.*

Obviously, every subnormal subgroup of a soluble group is $\mathbb{P}$-subnormal. The groups in which every Schmidt subgroup is subnormal were described in [11].

1. DEFINITIONS AND PRELIMINARY RESULTS

In this paper we use the definitions and notation adopted in [12].

Fix the following notation:

- $\mathfrak{U}$ is the class of all supersoluble groups;
- $\mathfrak{N}$ is the class of all nilpotent groups;
- if $\mathfrak{F}$ is a nonempty class and π is a set of primes, then $\mathfrak{F}_\pi$ is the class of all π-groups from $\mathfrak{F}$;
- if $\mathfrak{F}$ is a formation, then $G^{\mathfrak{F}}$ is the intersection of all normal subgroups N of a group G for which $G/N \in \mathfrak{F}$ (the subgroup $G^{\mathfrak{F}}$ is called the $\mathfrak{F}$-*residual* of G);
- $\mathbb{P}$ is the set of all primes;
- if n is a positive integer, then $\pi(n)$ is the set of all primes dividing n (in particular, $\pi(G) = \pi(|G|)$);
- $\mathrm{Sch}(G)$ is the set of all Schmidt subgroups of a group G;

– if A and B are subgroups of a group G, then $[A]B$ is their semidirect product with the normal subgroup A.

The basic structure of Schmidt groups, which is described in the following lemma, was established in [13, 14].

Lemma 1. *Let S be a Schmidt group. Then the following statements hold*:

(1) $\pi(S) = \{p, q\}$;

(2) $S = [P]\langle a\rangle$, *where P is a normal Sylow p-subgroup of S and $\langle a\rangle$ is its Sylow q-subgroup such that* $\langle a^q\rangle \subseteq Z(S)$;

(3) *P is the $\mathfrak{N}$-residual of S*;

(4) *$P/\Phi(P)$ is a minimal normal subgroup of $S/\Phi(P)$ and, in addition,* $\Phi(P) = P^{'} \subseteq Z(S)$;

(5) $\Phi(S) = Z(S) = P^{'} \times \langle a^q\rangle$;

(6) *if $Z(S) = 1$, then $|S| = p^m q$, where m is the exponent of p modulo q.*

Following [15], we call a Schmidt $\{p, q\}$-group with a normal Sylow p-subgroup and a nonnormal cyclic Sylow q-subgroup an $S_{<p,q>}$-*group*. In addition, we call a Schmidt group $S = [P]\langle a\rangle$ with a normal Sylow p-subgroup P for which $|P/\Phi(P)| = p^m$, where m is the exponent of p modulo q, and a nonnormal cyclic Sylow q-subgroup $\langle a\rangle$ an $S_{<p,q,m>}$-*group*. Note that an $S_{<p,q,m>}$-group S is supersoluble if and only if $m = 1$.

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G if either $H = G$ or there exists a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G$$

such that $|H_i : H_{i-1}| \in \mathbb{P}$ for any $i = 1, 2, \ldots, n$. If H is a $\mathbb{P}$-subnormal subgroup of G, then we write H $\mathbb{P}$-sn G according to [1].

In the following lemma we give the main properties of $\mathbb{P}$-subnormal subgroups.

Lemma 2. *Suppose that H, K, and N are subgroups of G, and N is normal in G. Then*:

(1) *if H $\mathbb{P}$-sn G, then $H \cap N$ $\mathbb{P}$-sn N and HN/N $\mathbb{P}$-sn G/N*;

(2) *if $N \subseteq H$ and H/N $\mathbb{P}$-sn G/N, then H $\mathbb{P}$-sn G*;

(3) *if H $\mathbb{P}$-sn K and K $\mathbb{P}$-sn G, then H $\mathbb{P}$-sn G*;

(4) *if $G^{\mathfrak{U}} \subseteq H$, then H $\mathbb{P}$-sn G*;

(5) *if H $\mathbb{P}$-sn G and $H \subseteq K$, then H $\mathbb{P}$-sn K*;

(6) *if the group G is soluble and H $\mathbb{P}$-sn G, then the subgroup $H^{\mathfrak{U}}$ is subnormal in G*;

(7) *if H is a Schmidt subgroup of a soluble group G and H $\mathbb{P}$-sn G, then either the subgroup H is supersoluble or $H^{\mathfrak{U}} \subseteq F(G)$.*

Proof. Statements (1)–(4) are proved in Lemma 3.1 from [2], and statement (5) is proved in Lemma 3.4 from [2].

Let us prove statement (6). If $H = G$, then it is obvious. Therefore, we can assume that $H \neq G$ and there exists a subgroup chain

$$H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G$$

such that $|H_i : H_{i-1}| \in \mathbb{P}$ for any $i = 1, 2, \ldots, n$. Since the group G is soluble, we have $H_i^{\mathfrak{U}} \subseteq H_{i-1}$ for all $i = 1, 2, \ldots, n$ (see, e.g., Lemma 3.3 from [2]). Since the formation $\mathfrak{U}$ is hereditary, we have $H_{i-1}H_i^{\mathfrak{U}}/H_i^{\mathfrak{U}} \in \mathfrak{U}$. Therefore, since

$$H_{i-1}H_i^{\mathfrak{U}}/H_i^{\mathfrak{U}} \cong H_{i-1}/H_{i-1} \cap H_i^{\mathfrak{U}},$$

we have $H^{\mathfrak{U}}_{i-1} \subseteq H^{\mathfrak{U}}_i$. Consequently, $H^{\mathfrak{U}}_{i-1} \subseteq H^{\mathfrak{U}}_i \subseteq H_i$ and $H^{\mathfrak{U}}_{i-1} \trianglelefteq H^{\mathfrak{U}}_i$. Thus, the subgroup $H^{\mathfrak{U}}$ is subnormal in G.

Let us prove statement (7). Let H be a $\mathbb{P}$-subnormal Schmidt subgroup of a soluble group G. Then, by statement (6), the supersoluble residual $H^{\mathfrak{U}}$ of the subgroup H is subnormal in G. By Lemma 1, $\pi(H) = \{p, q\}$ and $H = [P]\langle a\rangle$, where P is a normal Sylow p-subgroup of H and $\langle a\rangle$ is its Sylow q-subgroup. In addition, P is the $\mathfrak{N}$-residual of H and $P/\Phi(P)$ is a minimal normal subgroup of $H/\Phi(P)$. Then either $H^{\mathfrak{U}} = P$ or $H^{\mathfrak{U}} \subseteq \Phi(P)$. If $H^{\mathfrak{U}} = P$, then it follows from statement (6) that the subgroup P is subnormal in G. Then, obviously, $P = H^{\mathfrak{N}} \subseteq F(G)$. Let $H^{\mathfrak{U}} \subseteq \Phi(P)$. Since the subgroup P is normal in H, we have $H^{\mathfrak{U}} \subseteq \Phi(H)$. Now it follows from the facts that the formation $\mathfrak{U}$ of all supersoluble groups is saturated (see Example IV.3.4.(f) in [12]) and $H/\Phi(H) \in \mathfrak{U}$ that H is a supersoluble subgroup.

The lemma is proved.

Remark 1. The requirement of solubility of the group G in statements (6) and (7) of Lemma 2 is essential and cannot be discarded in the general case. For example, in the group $PSL_2(7)$, the subgroup $H \cong S_4$ is $\mathbb{P}$-subnormal, but its $\mathfrak{U}$-residual, obviously, is not a subnormal subgroup. In the alternating group A_5, the subgroup A_4 is a $\mathbb{P}$-subnormal Schmidt subgroup, but its supersoluble residual is not contained in the Fitting subgroup of A_5.

Recall that a *formation* is a class of groups closed under taking homomorphic images and finite subdirect products. A formation $\mathfrak{F}$ is called

– *saturated* if, for any group G, the membership $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$;
– *hereditary* if it is closed under taking subgroups.

A class $\mathfrak{F}$ is called a *Fitting class* if it satisfies the following requirements:

(1) $\mathfrak{F}$ is a normal hereditary class;
(2) if $G = AB$, where $A \trianglelefteq G$, $B \trianglelefteq G$, $A \in \mathfrak{F}$, and $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

A *Fitting formation* is a formation that is a Fitting class.

A *minimal supplement* to a normal subgroup N of a group G is a subgroup L of G such that $LN = G$, but $L_1N \neq G$ for any proper subgroup L_1 of L.

Lemma 3. *If K and D are subgroups of a group G, the subgroup D is normal in G, and K/D is an $S_{<p,q>}$-subgroup, then a minimal supplement L to the subgroup D in K has the following properties*:

(1) *L is a p-closed $\{p, q\}$-subgroup*;
(2) *all proper normal subgroups of L are nilpotent*;
(3) *the subgroup L contains an $S_{<p,q>}$-subgroup $[P]Q$ such that Q is not contained in D and $L = ([P]Q)^L = Q^L$*;
(4) *if an $S_{<p,q>}$-subgroup $[P]Q$ is $\mathbb{P}$-subnormal in G, then one of the following holds*:
   (i) *$K/D = ([P]Q)D/D$; in particular, the subgroup K/D is $\mathbb{P}$-subnormal in G/D*;
   (ii) *the subgroup K/D is supersoluble, has a Sylow p-subgroup of order p, q divides $p - 1$, and a Sylow q-subgroup of K/D is $\mathbb{P}$-subnormal in G/D*;
(5) *K/D is an $S_{<p,q,m>}$-group if and only if $[P]Q$ is an $S_{<p,q,m>}$-group.*

Proof. Statements (1)–(3) are proved in Lemma 2 from [17].

Let us prove statement (4). Assume that an $S_{<p,q>}$-subgroup $[P]Q$ is $\mathbb{P}$-subnormal in G. By Lemma 11.1 from [16], $LD = K$ and $L \cap D \subseteq \Phi(L)$. The Frattini subgroup consists of nongenerating elements; therefore, if $([P]Q)(L \cap D) = L$, then $[P]Q = L$. In this case, by statement (1) of Lemma 2,

$$([P]Q)D/D = LD/D = K/D$$

is a $\mathbb{P}$-subnormal subgroup of G/D.

Now, let $[P]Q$ be a proper subgroup of L. Then it follows from $L \cap D \subseteq \Phi(L)$ that $([P]Q)(L \cap D)$ is a proper subgroup of L. If the subgroup $([P]Q)(L \cap D)$ is contained in some subnormal subgroup of L, then by statement (2) it is nilpotent, which is impossible, since $([P]Q)(L \cap D)$ contains a Schmidt subgroup $[P]Q$. Consequently, $([P]Q)(L \cap D)$ is not contained in any normal maximal subgroup of L. From

$$K/D = LD/D \cong L/L \cap D$$

we conclude that $L/L \cap D$ is a Schmidt group. Now, according to statement (2) of Lemma 1, $Q(L \cap D)/(L \cap D)$ is a Sylow q-subgroup of $L/L \cap D$. Further, $([P]Q)(L \cap D)/L \cap D$ is a proper subgroup of $L/L \cap D$; hence, by statement (5) of Lemma 1,

$$([P]Q)(L \cap D)/L \cap D \subseteq \Phi(L/L \cap D)(Q(L \cap D)/L \cap D).$$

By statement (4) of Lemma 1, $\Phi(L/L \cap D)(Q(L \cap D)/L \cap D)$ is a maximal subgroup of $L/L \cap D$. Since the subgroup $[P]Q$ is $\mathbb{P}$-subnormal in G by the hypothesis, it follows in view of Lemma 2 that the subgroup $\Phi(L/L \cap D)(Q(L \cap D)/L \cap D)$ is $\mathbb{P}$-subnormal in $L/L \cap D$. Hence, the index of the maximal subgroup

$$\Phi(L/L \cap D)(Q(L \cap D)/L \cap D)$$

in the group $L/L \cap D$ is a prime p, and therefore the group $(L/L \cap D)/\Phi(L/L \cap D)$ is supersoluble. However, since the formation $\mathfrak{U}$ is saturated, it follows that the group $L/L \cap D$ is supersoluble. This and the isomorphism $K/D \cong L/L \cap D$ imply that K/D is a supersoluble group. Now, by Lemma 1 from [18], the group K/D has a Sylow p-subgroup of order p and q divides $p - 1$. Since $|K/D| = pq^n$, $([P]Q)(L \cap D)/L \cap D$ is a Sylow q-subgroup of $L/L \cap D$. Using Lemma 2, we conclude that a Sylow q-subgroup of K/D is $\mathbb{P}$-subnormal in G/D.

Let us prove statement (5). Assume that K/D is an $S_{<p,q,m>}$-group. Then, by statement (6) of Lemma 1,

$$|(K/D)/Z(K/D)| = p^m q,$$

where m is the exponent of p modulo q. By statement (4) of Lemma 3, the subgroup $[P]Q$ is an $S_{<p,q,k>}$-group for some natural $k \geq 1$ that is the exponent of p modulo q. This and the definition of the exponent imply that $m = k$. Arguing in the reverse order, we can show that if $[P]Q$ is an $S_{\langle p,q,m \rangle}$-group, then K/D is also an $S_{<p,q,m>}$-group.

The lemma is proved.

Recall the definition of a local formation. A function

$$f : \mathbb{P} \to \{\text{formations of finite groups}\}$$

is called a *formation function*.

For a formation function f, a chief factor A/B of a group G is called *f-central* (*f-excentral*) if

$$G/C_G(A/B) \cong \mathrm{Aut}_G(A/B) \in f(p)$$

for all primes $p \in \pi(A/B)$ ($G/C_G(A/B)$ does not belong to $f(p)$ for at least one prime $p \in \pi(A/B)$, respectively). A class of groups $\mathfrak{F} = LF(f)$ is called a *local formation* if it consists of all the groups G

such that either $G = 1$ or $G \neq 1$ and any chief factor A/B of the group G is f-central. In this case, the local formation $\mathfrak{F}$ is said to be *defined by means of the formation function* f, and f is said to be a *local definition* of the formation $\mathfrak{F}$.

Assume that f is a formation function and $\mathfrak{F} = LF(f)$. Then f is called

(a) *internal* if $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$;

(b) *complete* if $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$;

(c) *canonical* if it is complete and internal.

As shown in Theorem IV.3.7 from [12], for any local formation $\mathfrak{F}$ there exists a unique canonical formation function F such that $\mathfrak{F} = LF(F)$. This function is called the *canonical local definition* of the formation $\mathfrak{F}$.

Note that, according to the Gaschütz–Lubeseder–Schmid theorem ([12], Theorem IV.4.6), a formation $\mathfrak{F}$ is saturated if and only if it is local. Hence, in particular, for any saturated formation $\mathfrak{F}$ there exists a canonical local definition F such that $\mathfrak{F} = LF(F)$.

Lemma 4. *Let* $\mathfrak{F} = \{G \mid \mathrm{Sch}(G) \subseteq \mathfrak{U}\}$. *Then the following statements hold*:

(1) $G \in \mathfrak{F}$ *if and only if the group* G *is soluble and, for any primes* $p, q \in \pi(G)$ *and a Hall* $\{p, q\}$-*subgroup, either this subgroup is nilpotent or it is* p-*closed and* q *divides* $p - 1$;

(2) *if* $G \in \mathfrak{F}$, *then the group* G *has a Sylow tower of supersoluble type*;

(3) *the class* $\mathfrak{F}$ *is a hereditary saturated Fitting formation*;

(4) $\mathfrak{U} \subseteq \mathfrak{F}$;

(5) $\mathfrak{F}$ *is a local formation with a canonical local definition* F *such that* $F(p) = \mathfrak{N}_p \mathfrak{D}_{\pi(p-1)}$.

Proof. Statement (1) follows from Lemma 5 and Theorem 1 in [18], and statements (2)–(5) follow from Lemma 2.3 in [10].

The lemma is proved.

Remark 2. It follows from Proposition 2.6 of paper [10] that if a group G belongs to the class $\mathfrak{F} = \{H \mid \mathrm{Sch}(H) \in \mathfrak{U}\}$, then G can have any nilpotent length greater than 1. In particular, there exist groups $G \in \mathfrak{F}$ that are not supersoluble. Let, for example, M be a nonabelian group of order 21. Then there exists a faithful irreducible M-module N over a field of 43 elements (see, e.g., Corollary B.11.8 in [12]). Obviously, the group $G = [N]M$ is not supersoluble, but it belongs to the class $\{H \mid \mathrm{Sch}(H) \in \mathfrak{U}\}$ by Lemma 4.

2. PROOF OF THEOREM 1

(1) By statement (5) of Lemma 4, the local formation $\mathfrak{F} = \{G \mid \mathrm{Sch}(G) \subseteq \mathfrak{U}\}$ has a canonical local definition F such that $F(p) = \mathfrak{N}_p \mathfrak{D}_{\pi(p-1)}$.

(2) Let each Schmidt subgroup of G be $\mathbb{P}$-subnormal. Then the group G is soluble by the main result of [7].

Let D be a minimal supplement to $F(G)$ in the group G. In this case, in particular, $DF(G) = G$ and $D \cap F(G) \subseteq \Phi(D)$. Let $K/D \cap F(G)$ be an arbitrary Schmidt subgroup of $D/D \cap F(G)$. Without loss of generality, we can assume that $K/D \cap F(G)$ is an $S_{<p,q>}$-subgroup for some primes p and q. By statement (4) of Lemma 3, a minimal supplement L to the subgroup $D \cap F(G)$ in K contains an $S_{<p,q>}$-subgroup $[P]Q$ such that Q is not contained in $D \cap F(G)$.

The subgroup $[P]Q$ is $\mathbb{P}$-subnormal in G by the hypothesis. Then, by statement (3) of Lemma 3, we have one of the following statements:

(i) $K/D \cap F(G) = ([P]Q)(D \cap F(G))/D \cap F(G)$; in particular, the subgroup $K/D \cap F(G)$ is $\mathbb{P}$-subnormal in $D/D \cap F(G)$;

(ii) the subgroup $K/D \cap F(G)$ is supersoluble, has a Sylow p-subgroup of order p, q divides $p-1$, and a Sylow q-subgroup of $K/D \cap F(G)$ is $\mathbb{P}$-subnormal in $D/D \cap F(G)$.

Consider case (i). Assume that the group $K/D \cap F(G)$ is not supersoluble. Then it follows from

$$K/D \cap F(G) = ([P]Q)(D \cap F(G))/D \cap F(G)$$

and Lemma 1 that the subgroup $[P]Q$ is not supersoluble, and hence $P = ([P]Q)^{\mathfrak{N}} = ([P]Q)^{\mathfrak{U}}$ by statement (3) of Lemma 1. By statement (7) of Lemma 2, we have $P \subseteq D \cap F(G)$. Then, however, $K/D \cap F(G)$ is a q-group, which is impossible.

Thus, all Schmidt subgroups of $D/D \cap F(G)$ are supersoluble; i.e., $D/D \cap F(G) \in \mathfrak{F}$. Therefore, by the isomorphism

$$G/F(G) = DF(G)/F(G) \cong D/D \cap F(G),$$

we have $G/F(G) \in \mathfrak{F}$.

(3) Let H be a nonsupersoluble $\{p, q\}$-Schmidt subgroup of a group G with a normal Sylow p-subgroup P. Then H is an $S_{<p,q,m>}$-group for some natural $m > 1$. Assume that P is not contained in $F(G)$. By statement (3) of Lemma 1, P is the $\mathfrak{N}$-residual of the subgroup H. Therefore, a Sylow q-subgroup Q of H is not contained in $F(G)$. Now, applying statement (5) of Lemma 1, we find that $HF(G)/F(G)$ is an $S_{<p,q>}$-subgroup of $G/F(G)$. Since $G/F(G) \in \mathfrak{F}$, it follows that $HF(G)/F(G)$ is an $S_{<p,q,1>}$-group, which is impossible according to statement (5) of Lemma 3. Hence, P is contained in $F(G)$. It also follows from $G/G^{\mathfrak{F}} \in \mathfrak{F}$ that $H \subseteq G^{\mathfrak{F}}$.

The theorem is proved.

Corollary 1. *Let $\mathfrak{F} = \{H \mid \mathrm{Sch}(H) \in \mathfrak{U}\}$. If $\Phi(G) = 1$ and each Schmidt subgroup of G is $\mathbb{P}$-subnormal in G, then the group G can be presented in the form $G = [F(G)]M$, where $M \in \mathfrak{F}$.*

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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