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On the Coincidence of Gruenberg–Kegel Graphs of an Almost Simple Group and a Nonsolvable Frobenius Group

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Abstract—Let G be a finite group. Its spectrum $\omega(G)$ is the set of all element orders of G. The prime spectrum $\pi(G)$ is the set of all prime divisors of the order of G. The Gruenberg–Kegel graph (or the prime graph) $\Gamma(G)$ is the simple graph with vertex set $\pi(G)$ in which any two vertices p and q are adjacent if and only if $pq \in \omega(G)$. The structural Gruenberg–Kegel theorem implies that the class of finite groups with disconnected Gruenberg–Kegel graphs widely generalizes the class of finite Frobenius groups, whose role in finite group theory is absolutely exceptional. The question of coincidence of Gruenberg–Kegel graphs of a finite Frobenius group and of an almost simple group naturally arises. The answer to the question is known in the cases when the Frobenius group is solvable and when the almost simple group coincides with its socle. In this short note we answer the question in the case when the Frobenius group is nonsolvable and the socle of the almost simple group is isomorphic to $PSL_2(q)$ for some q.

Keywords: finite group, Gruenberg–Kegel graph (prime graph), nonsolvable Frobenius group, almost simple group.

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INTRODUCTION

We consider only finite groups, and henceforth the term "group" means "finite group." Our notation and terminology are mostly standard and can be found in [9, 10, 12, 13, 15, 16].

Denote by $\mathrm{Soc}(G)$ the *socle* of a group G (i.e., the subgroup of G generated by its nontrivial minimal normal subgroups). Recall that G is called *almost simple* if $\mathrm{Soc}(G)$ is a nonabelian simple group. It is well known that a group G is almost simple if and only if there exists a nonabelian simple group S such that $\mathrm{Inn}(S) \trianglelefteq G \leq \mathrm{Aut}(S)$; moreover, since here $\mathrm{Inn}(S) \cong S$, we will identify S with $\mathrm{Inn}(S)$ and write $S \trianglelefteq G \leq \mathrm{Aut}(S)$.

A group G is called a *Frobenius group* if there exists a nontrivial subgroup $C < G$ such that $C \cap C^g = \{1\}$ whenever $g \in G \setminus C$. In this case, the subgroup C is called a *Frobenius complement* in G. Let

$$K = \{1\} \cup (G \setminus \cup_{g \in G} C^g).$$

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As is known, K is a normal subgroup of G; it is called its *Frobenius kernel*. The class of Frobenius groups plays an absolutely exceptional role in finite group theory. A group G is called 2-*Frobenius* if $G = ABC$, where A and AB are normal subgroups of G and AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively. It is well known that any 2-Frobenius group is solvable.

Let G be a group. The set of its element orders is called its *spectrum* and is denoted by $\omega(G)$. The *prime spectrum* $\pi(G)$ of G is the set of all prime divisors of its order (equivalently, the set of all prime elements from $\omega(G)$). The *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ is the graph with vertex set $\pi(G)$ in which any two vertices p and q are adjacent if and only if $pq \in \omega(G)$. Denote the number of connected components of $\Gamma(G)$ by $s(G)$ and the set of its connected components by $\{\pi_i(G) \mid 1 \le i \le s(G)\}$; for a group G of even order, we assume that $2 \in \pi_1(G)$.

The notion of Gruenberg–Kegel graph has appeared in connection with the study of some cohomological questions of the theory of integral group rings: it was established that the augmentation ideal of an integral group ring is decomposable as a module if and only if the Gruenberg–Kegel graph of the group is disconnected (see [14]). K. W. Gruenberg and O. Kegel described the structure of an arbitrary finite group with disconnected Gruenberg–Kegel graph; the corresponding structural theorem was proved in their unpublished manuscript and was published later by Gruenberg's student Williams [18].

Gruenberg–Kegel Theorem (see [18, Theorem A]). *If G is a group with disconnected graph $\Gamma(G)$, then one of the following holds*:

(1) *G is a Frobenius group*;

(2) *G is a 2-Frobenius group*;

(3) *G is an extension of a nilpotent group N by a group A such that $S \trianglelefteq A \le \mathrm{Aut}(S)$, where S is a nonabelian simple group, $s(G) \le s(S)$, and $\pi(N) \cup \pi(A/S) \subseteq \pi_1(G)$.*

As seen from the Gruenberg–Kegel theorem, the class of groups with disconnected Gruenberg–Kegel graphs widely generalizes the class of finite Frobenius groups. There naturally arises the question on the coincidence of Gruenberg–Kegel graphs for the groups from different assertions of the Gruenberg–Kegel theorem, in particular, the question on the coincidence of the graphs $\Gamma(G)$ and $\Gamma(H)$ in the cases where G is an almost simple group and H is a Frobenius or 2-Frobenius group. The case where G is a nonabelian simple group was completely studied by Zinov'eva and Mazurov [4], and the corresponding results in the case where the group H is solvable can be extracted from the main results of [2, 3]. Thus, it remains to study the case where the group G is almost simple but not simple and H is a nonsolvable Frobenius group. This case was studied in special situations, for example, in Mahmoudifar's paper [17], which provided an example of a nonsolvable Frobenius group having the same Gruenberg–Kegel graph as the group $PGL_2(49)$. In this paper, we describe all almost simple groups with socle isomorphic to $PSL_2(q)$ and Gruenberg–Kegel graphs coinciding with the Gruenberg–Kegel graphs of some nonsolvable Frobenius groups. We prove the following theorem.

Theorem 1. *Let q be a prime power, and let G be an almost simple group such that $\mathrm{Soc}(G) \cong PSL_2(q)$. Then there exists a nonsolvable Frobenius group H such that $\Gamma(G) = \Gamma(H)$ if and only if G is a group from the following list*: $PSL_2(11).2 \cong PGL_2(11)$, $PSL_2(19).2 \cong PGL_2(19)$, $PSL_2(25).2_2$, $PGL_2(49).2_1 \cong PGL_2(49)$, $PSL_2(81).2_1$, $PSL_2(81).4_1$, *and* $PSL_2(81).4_2$.

1. PRELIMINARY RESULTS

The largest integer nonnegative power of a prime p that divides a positive integer n is called the p-part of n and is denoted by n_p. The set of all prime divisors of a positive integer n is denoted by $\pi(n)$.

The following easily proved results are well known (the corresponding references are given for the completeness of the proof).

Lemma 1.1 (see, for example, [7, Lemma 9]). *Let q and n be odd positive integers. Then $(q^n+1)_2=(q+1)_2$ and $(q^n-1)_2=(q-1)_2$.*

Lemma 1.2 (see, for example, [19, Lemma 6]). *Let q, k, and m be positive integers with $q>1$. Then*

(1) $(q^k-1,q^m-1)=q^{(k,m)}-1$;

(2) $(q^k+1,q^m+1)=q^{(k,m)}+1$ *if* $k_2=m_2$, *and* $(q^k+1,q^m+1)=(2,q+1)$ *otherwise*;

(3) $(q^k-1,q^m+1)=q^{(k,m)}+1$ *if* $k_2>m_2$, *and* $(q^k-1,q^m+1)=(2,q+1)$ *otherwise.*

Lemma 1.3 (see, for example, [8, Lemma 2]). *If K is a normal subgroup of a group L and $r,s\in\pi(K)\setminus\pi(|L:K|)$, then the numbers r and s are not adjacent in the graph $\Gamma(K)$ if and only if they are not adjacent in $\Gamma(L)$.*

The following statements will also be used in the proof of Theorem 1.

Lemma 1.4 (see [6, Lemma 1.3]). *Suppose that $q=p^m$, p is a prime, m is a positive integer, and $|\pi(q^2-1)|=3$. Then one of the following holds*:

(i) $17\neq q=p\geq 11$ *and* $p^2-1=2^a3^bs^c$, *where $s>3$ is a prime, a and b are positive integers, and c is either* 1 *or* 2 *for* $p\in\{97,577\}$;

(ii) $q\in\{16,\ 25,\ 27,\ 49,\ 81\}$;

(iii) $p\in\{2,\ 3\}$, $\dfrac{q-1}{(2,\ q-1)}$ *and m are odd primes, and* $\left|\pi\left(\dfrac{q+1}{p+1}\right)\right|=1$.

Lemma 1.5 (see [11]). *Let p and q be primes such that $p^a-q^b=1$ for some natural a and b. Then $(p^a,q^b)\in\{(3^2,2^3),(p,2^b),(2^a,q)\}$, where a is a prime and b is a power of* 2.

If r is an odd prime and $q>1$ is a positive integer coprime to r, then define

$$e(q,r)=\min\{k\in\mathbb{N}\mid q^k\equiv 1 \pmod r\}.$$

Lemma 1.6 (Zsigmondy's theorem; see, for example, [20]). *Let q and m be positive integers greater than 1. Then there exists an odd prime r such that $e(q,r)=m$, except for the following cases*:

(1) $q=2$ *and* $m=6$;

(2) $q=2^l-1$ *for some* $l>1$ *and* $m=2$.

Lemma 1.7. *Suppose that $q=p^k$, where p is a prime and k is a positive integer. If $\pi(q^2-1)=\{2,3,5\}$, then $q\in\{11,19,49\}$.*

Proof. Let us apply Lemma 1.4. As follows from Lemma 1.6, q cannot be a number from assertion (iii) of Lemma 1.4; if q is a number from assertion (ii) of this lemma, then $q=49$.

Now, let $q=p$ is a number from assertion (i) of Lemma 1.4, and let $p^2-1=2^a3^bs^c$. Obviously, if $s=5$, then $p\notin\{97,577\}$; hence, $c=1$.

Assume that $q \equiv 1 \pmod 4$. Then either $p+1 = 2 \times 3^u$ and $p-1 = 2^w \times 5$ or $p+1 = 2 \times 5$ and $p-1 = 2^w \times 3^v$ for some positive integers u, v, and w. The second case is impossible, since in this case $p = 9$ is not a prime. Let $p+1 = 2 \times 3^u$ and $p-1 = 2^w \times 5$. Then

$$p = 2 \times 3^u - 1 = 2^w \times 5 + 1; \text{ hence, } 3^u - 1 = 2^{w-1} \times 5.$$

Assume that u is even. Then $3^u - 1 = (3^{u/2}-1)(3^{u/2}+1)$ and $(3^{u/2}-1, 3^{u/2}+1) = 2$. Hence, either $3^{u/2}-1$ or $3^{u/2}+1$ is a power of 2. By Lemma 1.5, either $3^u - 1 = 8$ or $3^u - 1 = 80$. In the first case $p = 2\times(3^u-1)+1 = 17$ and $\pi(p^2-1) = \{2,3\}$; in the second case $p = 2\times(3^u-1)+1 = 161 = 7\times 23$ is not a prime. For odd u, the number $3^u - 1$ has remainder 1 or 2 when divided by 5; hence, the equality is not true.

Assume that $q \equiv 3 \pmod 4$. Then either $p-1 = 2 \times 3^u$ and $p+1 = 2^w \times 5$ or $p-1 = 2 \times 5$ and $p+1 = 2^w \times 3^v$ for some positive integers u, v, and w. In the second case, $p = 11$ and $\pi(p^2-1) = \{2,3,5\}$. Let $p-1 = 2 \times 3^u$ and $p+1 = 2^w \times 5$. Then

$$p = 2 \times 3^u + 1 = 2^w \times 5 - 1; \text{ hence, } 3^u + 1 = 2^{w-1} \times 5.$$

Let u be odd. Then $3^u + 1$ gives remainder 4 or 3 when divided by 5; hence, the equality is not true. Thus, u is even and $u = 2u_0$. Obviously, $((3^{u_0})^2+1)_2 = 2$, whence $2^{w-1}{=}2$ and $(3^{u_0})^2 + 1 = 2 \times 5 = 10$. Therefore, $3^u = 9$, whence $p = 19$ is a prime with $\pi(p^2-1) = \{2,3,5\}$. □

Lemma 1.8 (see [4, Lemma 3, Proposition 1])**.**

(1) *If G is a nonsolvable Frobenius group, then $\Gamma(G)$ has two connected components; one of them is a complete graph, and the other contains the vertices* 2, 3, *and* 5 *and is a complete graph without the edge* $\{3,5\}$.

(2) *Finite disjoint sets π_1 and π_2 consisting of primes are connected components of the graph $\Gamma(G)$ for some nonsolvable Frobenius group G if and only if one of these sets contains* 2, 3, *and* 5.

Lemma 1.9 (see [13, Theorem 4.5.1, Propositions 2.5.12, 4.9.1, 4.9.2])**.** *Suppose that $S = PSL_2(q)$, $q = p^m$, p is a prime, $q > 3$, x is an element of prime order r in* $\mathrm{Aut}(S) \setminus \mathrm{Inndiag}(S)$, *and $S_x = O^{p'}(C_S(x))$. Then the following statements hold*:

(1) $\mathrm{Aut}(S) = \mathrm{Inndiag}(S) \rtimes \Phi$, *where* $\mathrm{Outdiag}(S) \cong Z_{(2,q-1)} \in \{1, Z_2\}$, $\Phi = \langle f \rangle \cong \mathrm{Aut}(F_q) \cong Z_m$ *is the standard group of field automorphisms of S, and* $\mathrm{Out}(S) = \mathrm{Outdiag}(S) \times \Phi$;

(2) *the number r divides m, $S_x \cong PSL_2(q^{1/r})$, and* $C_{\mathrm{Inndiag}(S)}(x) \cong \mathrm{Inndiag}(S_x)$.

2. PROOF OF THEOREM 1

Let G be an almost simple group such that $S = \mathrm{Soc}(G) \cong PSL_2(q)$, where $q = p^m$ and p is a prime.

The spectrum of S is known (see [1, Corollary 3]). If q is even, then $\Gamma(S)$ consists of three cliques: $\pi_1(S) = \{2\}$ and, without loss of generality, $\pi_2(S) = \pi(q-1)$ and $\pi_3(S) = \pi(q+1)$. If q is odd and $q \equiv \varepsilon 1 \pmod 4$, where $\varepsilon \in \{+,-\}$, then $\Gamma(S)$ consists of three cliques: $\pi_1(S) = \pi(q - \varepsilon 1)$ and, without loss of generality, $\pi_2(S) = \pi(\frac{q+\varepsilon 1}{2})$ and $\pi_3(S) = \{p\}$.

Assume that there exists a nonsolvable Frobenius group H such that $\Gamma(G) = \Gamma(H)$. We apply Lemma 1.8. Obviously, $G \neq S$. Note that the numbers 2 and 3 divide the order of S for any q. Let us show that 5 divides $|S|$. If this is not so, then 5 divides the index $|G:S|$; hence, by Lemma 1.9, there exists $x \in G \setminus S$ of order 5, and in this case the order of the subgroup $C_S(x)$ is a multiple of 3, whence 3 and 5 are adjacent in $\Gamma(G)$, a contradiction.

Let $p = 2$. If $\{3, 5\} \subseteq \pi_2(S)$ or $\{3, 5\} \subseteq \pi_3(S)$, then the numbers 3 and 5 are adjacent in $\Gamma(S)$; consequently, they are adjacent in $\Gamma(G)$, a contradiction. If $3 \in \pi_2(S)$ and $5 \in \pi_3(S)$ or $3 \in \pi_3(S)$ and $5 \in \pi_2(S)$, then, since $\{2, 3, 5\} \subseteq \pi_1(G)$, it follows that the graph $\Gamma(G)$ is connected, which is also a contradiction. Therefore, p is odd.

Let $p \notin \{3, 5\}$. Once again, if $\{3, 5\} \subseteq \pi_1(S)$ or $\{3, 5\} \subseteq \pi_2(S)$, then 3 and 5 are adjacent in $\Gamma(S)$; consequently, they are adjacent in $\Gamma(G)$, a contradiction. Hence, either $3 \in \pi_1(S)$ and $5 \in \pi_2(S)$ or $3 \in \pi_2(S)$ and $5 \in \pi_1(S)$. Assume that the index $|G : S|$ is not a power of 2, and let r be an odd prime divisor of $|G : S|$. Then, by Lemma 1.9, there exists an element $x \in G\backslash S$ of order r, and the order of the subgroup $C_S(x)$ is a multiple of both 2 and p; hence, $p \in \pi_1(G)$. At the same time, $3 \in \pi_1(G)$ and $5 \in \pi_1(G)$; therefore, the graph $\Gamma(G)$ is connected, a contradiction. Thus, the index $|G : S|$ is a power of 2. Assume that there exists a number $u \in \pi(q^2 - 1) \setminus \{2, 3, 5\}$. Then $u \in \pi_1(G)$; hence, u is adjacent to 3 and 5 in $\Gamma(G)$; hence, by Lemma 1.3, u is adjacent to 3 and 5 in $\Gamma(S)$, which implies that 3 and 5 lie in the same connected component of $\Gamma(S)$, a contradiction. Therefore, $\pi(q^2 - 1) = \{2, 3, 5\}$; consequently, by Lemma 1.7, we have $q \in \{11, 19, 49\}$.

Let $p = 5$. Since $5 \equiv 1 \pmod 4$, we have $q \equiv 1 \pmod 4$ and $\pi_1(S) = \pi(q - 1)$. If $3 \in \pi_2(S)$, then, since $\{2, 3, 5\} \subseteq \pi_1(G)$, the graph $\Gamma(G)$ is connected, a contradiction. Hence, $3 \in \pi_1(S)$; i.e., the number 3 divides $5^m - 1$, which implies that m is even. Assume that the index $|G : S|$ is not a power of 2, and let r be an odd prime divisor of $|G : S|$. Then, by Lemma 1.9, there exists an element $x \in G \setminus S$ of order r, and in this case $C_S(x) \geq PSL_2(q^{1/r})$ and $|PSL_2(q^{1/r})| = \dfrac{q^{1/r}}{2}(q^{2/r} - 1)$. Hence, by Lemma 1.2, the graph $\Gamma(G)$ is connected, a contradiction. Thus, the index $|G : S|$ is a power of 2. Assume that there exists a number $u \in \pi(q - 1) \setminus \{2, 3\}$. Then $u \in \pi_1(S) \subseteq \pi_1(G)$, and, since $5 \in \pi_3(S)$, by Lemma 1.3, u is nonadjacent to 5 in $\Gamma(G)$, a contradiction. Therefore, $\pi(q - 1) = \{2, 3\}$, which implies that either $q^{m/2} - 1$ or $q^{m/2} + 1$ is a power of 2; consequently, by Lemma 1.5, $m/2 = 1$, whence $q = 25$.

Let $p = 3$. If $5 \in \pi_2(S)$, then, since $\{2, 3, 5\} \subseteq \pi_1(G)$, the graph $\Gamma(G)$ is connected, a contradiction. Therefore, $5 \in \pi_1(S)$. If m is odd, we have $q \equiv -1 \pmod 4$, $\pi_1(S) = \pi(q + 1)$, and $5 \notin \pi_1(S)$, a contradiction. Therefore, m is even, $q \equiv 1 \pmod 4$, and $\pi_1(S) = \pi(q - 1)$. Assume that the index $|G : S|$ is not a power of 2, and let r be an odd prime divisor of $|G : S|$. Then, by Lemma 1.9, there exists an element $x \in G \setminus S$ of order r, and in this case $C_S(x) \geq PSL_2(q^{1/r})$ and $|PSL_2(q^{1/r})| = \dfrac{q^{1/r}}{2}(q^{2/r} - 1)$; hence, by Lemma 1.2, the graph $\Gamma(G)$ is connected, a contradiction. Thus, the index $|G : S|$ is a power of 2. Assume that there exists a number $u \in \pi(q - 1) \setminus \{2, 5\}$. Then $u \in \pi_1(S) \subseteq \pi_1(G)$ and, since $3 \in \pi_3(S)$, by Lemma 1.3, u is nonadjacent to 3 in $\Gamma(G)$, a contradiction. Therefore, $\pi(q - 1) = \{2, 5\}$, which implies that either $q^{m/2} - 1$ or $q^{m/2} + 1$ is a power of 2; consequently, by Lemma 1.5, we have $m/2 \in \{1, 2\}$, whence $q \in \{9, 81\}$. However, $\pi(PSL_2(9)) = \{2, 3, 5\}$; but, if $S \cong PSL_2(9)$, then the graph $\Gamma(G)$ is connected, a contradiction. Therefore, $q = 81$.

Thus, $q \in \{11, 19, 25, 49, 81\}$ and, consequently, $|\pi(S)| = |\pi(\mathrm{Aut}(S))| = 4$. The Gruenberg–Kegel graphs of almost simple 4-primary groups are known [5, 6]. As follows from Lemma 1.8 and [6, Table 1] with the corrections made in [5], there exists a nonsolvable Frobenius group H with the property $\Gamma(G) = \Gamma(H)$ if and only if G is one of the following groups: $PSL_2(11).2 \cong PGL_2(11)$, $PSL_2(19).2 \cong PGL_2(19)$, $PSL_2(25).2_2$, $PGL_2(49).2_1 \cong PGL_2(49)$, $PSL_2(81).2_1$, $PSL_2(81).4_1$, and $PSL_2(81).4_2$.

The theorem is proved.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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