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Consecutive Primes in Short Intervals

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On the occasion of the 130th anniversary
of I. M. Vinogradov's birth

Abstract—We obtain a lower bound for $\#\{x/2 < p_n \le x\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le y\}$, where p_n is the nth prime.

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1. INTRODUCTION

Let p_n denote the nth prime. We prove the following result.

Theorem 1.1. *There are positive absolute constants c and C such that the following holds. Let ε be a real number with $0 < \varepsilon < 1$. Then there is a number $c_0(\varepsilon) > 0$, depending only on ε, such that if $x \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ satisfy the conditions*

$$c_0(\varepsilon) \le y \le \ln x, \qquad 1 \le m \le c\varepsilon \ln y, \qquad 1 \le q \le y^{1-\varepsilon}, \qquad (a, q) = 1,$$

then

$$\#\Big\{\frac{x}{2} < p_n \le x\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\Big\} \ge \pi(x)\Big(\frac{y}{2q \ln x}\Big)^{\exp(Cm)}.$$

Theorem 1.1 extends a result of Maynard [5, Theorem 3.3], who established the same result but with $y = \varepsilon \ln x$.

From Theorem 1.1 we obtain

Corollary 1.1. *There is an absolute constant $C > 0$ such that if m is a positive integer and x and y are real numbers satisfying* $\exp(Cm) \le y \le \ln x$, *then*

$$\#\Big\{\frac{x}{2} < p_n \le x\colon\ p_{n+m} - p_n \le y\Big\} \ge \pi(x)\Big(\frac{y}{2 \ln x}\Big)^{\exp(Cm)}.$$

Let us introduce necessary notation. The expression $b \mid a$ means that b divides a. For a fixed a the sum $\sum_{b|a}$ and the product $\prod_{b|a}$ should be interpreted as being over all positive divisors of a.

We will use I. M. Vinogradov's notation: $A \ll B$ means that $|A| \le cB$ with a positive absolute constant c.

We reserve the letter p for primes. In particular, the sum $\sum_{p\le K}$ should be interpreted as being over all prime numbers not exceeding K.

We will also use the following notation:

$\#A$ is the number of elements of a finite set A;

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are the sets of all positive integers, integers, real numbers, and complex numbers;

$\mathbb{P}$ is the set of all prime numbers;

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$[x]$ is the integer part of a number x; i.e., $[x]$ is the largest integer n such that $n \le x$;
$\{x\}$ is the fractional part of a number x; i.e., $\{x\} = x - [x]$;
$\lceil x \rceil$ is the smallest integer n such that $n \ge x$;
$\mathrm{Re}\, s$ and $\mathrm{Im}\, s$ are the real and imaginary parts of a complex number s;
$(a_1, \dots, a_n)$ is the greatest common divisor of integers $a_1, \dots, a_n$;
$[a_1, \dots, a_n]$ is the least common multiple of integers $a_1, \dots, a_n$;
$\varphi(n)$ is the Euler totient function: $\varphi(n) = \#\{1 \le m \le n\colon (m,n) = 1\}$;
$\mu(n)$ is the Möbius function, which is defined as follows:

(i) $\mu(1) = 1$,
(ii) $\mu(n) = 0$ if there is a prime p such that $p^2 \mid n$, and
(iii) $\mu(n) = (-1)^s$ if $n = q_1 \dots q_s$, where $q_1 < \dots < q_s$ are primes;

$\Lambda(n)$ is the von Mangoldt function:

$$\Lambda(n) = \begin{cases} \ln p & \text{if} \;\; n = p^k, \\ 0 & \text{if} \;\; n \ne p^k; \end{cases}$$

$P^-(n)$ is the least prime factor of $n > 1$ (by convention $P^-(1) = +\infty$);
$\binom{n}{k} = n!/(k!\,(n-k)!)$ is the binomial coefficient.

For real numbers a and b we use (a, b) and $[a, b]$ to denote, respectively, the open and closed intervals with endpoints a and b. By $(a_1, \dots, a_n)$ we also denote a vector; the meaning of the notation should be clear from the context.

By definition, we put

$$\sum_{\varnothing} = 0 \qquad \text{and} \qquad \prod_{\varnothing} = 1.$$

We define

$$\mathcal{M} = \{n \in \mathbb{N}\colon \; \mu(n) \ne 0\}.$$

We will use the following functions:

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}, \qquad \Phi(x, z) = \#\bigl\{1 \le n \le x\colon \; P^-(n) > z\bigr\},$$

$$\pi(x) = \sum_{p \le x} 1, \qquad \theta(x) = \sum_{p \le x} \ln p, \qquad \psi(x) = \sum_{n \le x} \Lambda(n),$$

$$\pi(x; q, a) = \sum_{p \le x,\; p \equiv a \pmod q} 1, \qquad \psi(x; q, a) = \sum_{n \le x,\; n \equiv a \pmod q} \Lambda(n).$$

Let $m > 1$ and a be integers. If $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 \pmod m$ (the Fermat–Euler theorem). Let d be the smallest positive value of γ for which $a^\gamma \equiv 1 \pmod m$. We call d the *order* of $a \pmod m$ and say that a *belongs to* $d \pmod m$.

Let q be a positive integer. We recall that a *Dirichlet character modulo* q is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ such that

(1) $\chi(n + q) = \chi(n)$ for all $n \in \mathbb{Z}$ (i.e., χ is a periodic function with period q);
(2) $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$ (i.e., χ is a totally multiplicative function);
(3) $\chi(1) = 1$;
(4) $\chi(n) = 0$ for all $n \in \mathbb{Z}$ such that $(n, q) > 1$.

By X_q we denote the set of all Dirichlet characters modulo q. We recall that $\#X_q = \varphi(q)$ and that the *principal character modulo* q is

$$\chi_0(n) := \begin{cases} 1 & \text{if } (n,q)=1, \\ 0 & \text{if } (n,q)>1. \end{cases}$$

Let $\chi \in X_q$. We say that *the character* χ *restricted by* $(n,q)=1$ *has period* q_1 if it has the property that $\chi(m)=\chi(n)$ for all $m,n\in\mathbb{Z}$ such that $(m,q)=1$, $(n,q)=1$ and $m\equiv n \pmod{q_1}$. Let $c(\chi)$ denote the *conductor* of χ, which is the least positive integer q_1 such that χ restricted by $(n,q)=1$ has period q_1. We say that χ is *primitive* if $c(\chi)=q$, and *imprimitive* if $c(\chi)<q$. By X_q^* we denote the set of all primitive characters modulo q. We observe that the principal character modulo 1 is primitive. On the other hand, any principal character modulo $q>1$ is imprimitive, since its conductor is clearly 1. For $\chi\in X_q$ we put

$$E_{\chi_0}(\chi) := \begin{cases} 1 & \text{if } \chi \text{ is the principal character modulo } q, \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi(x,\chi) = \sum_{n\le x} \Lambda(n)\chi(n), \qquad \psi'(x,\chi) = \psi(x,\chi) - E_{\chi_0}(\chi)x.$$

A character χ is said to be *real* if $\chi(n)\in\mathbb{R}$ for all $n\in\mathbb{Z}$. A character χ is said to be *complex* if there is an integer n such that $\operatorname{Im}(\chi(n))\neq 0$.

We say that characters χ_1 and χ_2 (modulo q_1 and modulo q_2, respectively) are equal and write $\chi_1=\chi_2$ if $\chi_1(n)=\chi_2(n)$ for any integer n. Otherwise, we say that characters χ_1 and χ_2 are not equal and write $\chi_1\neq\chi_2$.

Let χ be a Dirichlet character modulo q. The corresponding L-*function* is defined by the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $s\in\mathbb{C}$ with $\operatorname{Re} s>1$. It is well known that if χ is not the principal character modulo q, then $L(s,\chi)$ can be analytically continued to $\mathbb{C}$. If χ is the principal character modulo q, then $L(s,\chi)$ can be analytically continued to $\mathbb{C}\setminus\{1\}$ with a simple pole at $s=1$.

We say that two linear functions $L_1(n)=a_1n+b_1$ and $L_2(n)=a_2n+b_2$ with integer coefficients are equal and write $L_1=L_2$ if $a_1=a_2$ and $b_1=b_2$. Otherwise, we say that the linear functions L_1 and L_2 are not equal and write $L_1\neq L_2$.

Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be a set of k linear functions with integer coefficients:

$$L_i(n) = a_in+b_i, \qquad i=1,\dots,k.$$

For $L(n)=an+b$, $a,b\in\mathbb{Z}$, we define

$$\Delta_L = |a|\prod_{i=1}^{k} |ab_i - ba_i|.$$

We say that $L(n)=an+b$ belongs to $\mathcal{L}$ ($L\in\mathcal{L}$) if there is an i, $1\le i\le k$, such that $L=L_i$. Otherwise, we say that $L(n)=an+b$ does not belong to $\mathcal{L}$ ($L\notin\mathcal{L}$).

This paper is organized as follows. In Sections 2–4 we give necessary lemmas. In Section 5 we prove Theorem 1.1 and Corollary 1.1.

2. PREPARATORY LEMMAS

In this section we present some well-known lemmas which will be used in the following sections.

Lemma 2.1 (see, for example, [6, Ch. 1]). *Let x be a real number with $x \geq 2$. Then*

$$b_1 \ln x \leq \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \leq b_2 \ln x \qquad \text{and} \qquad b_3 \ln x \leq \prod_{p \leq x} \left(1 + \frac{1}{p}\right) \leq b_4 \ln x,$$

where b_i, $i = 1, \dots, 4$, are positive absolute constants.

Lemma 2.2 (see, for example, [4, Chs. 1, 2]). *The limits $\lim_{x\to+\infty} \psi(x)/x$, $\lim_{x\to+\infty} \theta(x)/x$, $\lim_{x\to+\infty} \pi(x)/(x/\ln x)$, and $\lim_{n\to+\infty} p_n/(n \ln n)$ exist and*

$$\lim_{x\to+\infty} \frac{\psi(x)}{x} = \lim_{x\to+\infty} \frac{\theta(x)}{x} = \lim_{x\to+\infty} \frac{\pi(x)}{x/\ln x} = 1, \qquad \lim_{n\to+\infty} \frac{p_n}{n \ln n} = 1.$$

From Lemma 2.2 we obtain

Lemma 2.3. *It holds that*

$$b_5 x \leq \psi(x) \leq b_6 x, \qquad b_7 x \leq \theta(x) \leq b_8 x \qquad \text{for} \quad x \geq 2, \tag{2.1}$$

$$b_9 \frac{x}{\ln x} \leq \pi(x) \leq b_{10} \frac{x}{\ln x} \qquad \text{for} \quad x \geq 2, \tag{2.2}$$

$$b_{11} n \ln(n+2) \leq p_n \leq b_{12} n \ln(n+2) \qquad \text{for} \quad n \geq 1,$$

where b_i, $i = 5, \dots, 12$, are positive absolute constants.

Lemma 2.4 (see, for example, [7, Ch. 2]). *Let n be an integer with $n > 1$. Then*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

From Lemma 2.4 we readily obtain the following two lemmas.

Lemma 2.5. *Let m and n be integers with $m \geq 1$ and $n \geq 1$. Then*

$$\varphi(mn) \geq \varphi(m)\varphi(n).$$

Lemma 2.6. *Let m and n be integers with $m \geq 1$, $n \geq 1$, and $(m, n) = 1$. Then*

$$\varphi(mn) = \varphi(m)\varphi(n).$$

Lemma 2.7. *Let n be an integer with $n \geq 1$. Then*

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}. \tag{2.3}$$

Proof. For $n = 1$, equality (2.3) holds. Let $n > 1$. Let us express n in the standard form $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, where $q_1 < \dots < q_r$ are prime numbers. Applying Lemmas 2.4 and 2.6, we have

$$\begin{aligned} \frac{n}{\varphi(n)} &= \prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p|n} \left(1 + \frac{1}{p-1}\right) = \left(1 + \frac{1}{q_1 - 1}\right) \dots \left(1 + \frac{1}{q_r - 1}\right) \\ &= \left(1 + \frac{1}{\varphi(q_1)}\right) \dots \left(1 + \frac{1}{\varphi(q_r)}\right) = 1 + \sum_{s=1}^{r} \sum_{1 \leq i_1 < \dots < i_s \leq r} \frac{1}{\varphi(q_{i_1}) \dots \varphi(q_{i_s})} \\ &= 1 + \sum_{s=1}^{r} \sum_{1 \leq i_1 < \dots < i_s \leq r} \frac{1}{\varphi(q_{i_1} \dots q_{i_s})} = \sum_{d|n,\, d \in \mathcal{M}} \frac{1}{\varphi(d)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}. \qquad \square \end{aligned}$$

Lemma 2.8 (see, for example, [6, Ch. 1]). *Let n be an integer with $n \geq 1$. Then*

$$\varphi(n) \geq c\frac{n}{\ln\ln(n+2)},$$

where $c > 0$ is an absolute constant.

Lemma 2.9 (see, for example, [1, Ch. 28]). *Let x be a real number with $x \geq 2$. Then*

$$\sum_{1\leq n\leq x}\frac{1}{\varphi(n)} \leq c\ln x,$$

where $c > 0$ is an absolute constant.

Lemma 2.10 (see, for example, [2, Ch. 5]). *Let n be an integer with $n \geq 1$. Then*

$$\sum_{p|n}\frac{\ln p}{p} \leq c\ln\ln(3n),$$

where $c > 0$ is an absolute constant.

Lemma 2.11. *Let a, b, and c be integers such that $(a,b) \mid c$. Then the equation*

$$ax + by = c \tag{2.4}$$

has a solution in integers.

Proof. We put $d = (a,b)$. Then $c = dl$ for some $l \in \mathbb{Z}$. It is well known (see, for example, [7, Ch. 1, Exercise 1]) that the equation

$$ax + by = d \tag{2.5}$$

has a solution in integers. Let $x_0 \in \mathbb{Z}$ and $y_0 \in \mathbb{Z}$ be a solution of (2.5). Then the integers lx_0 and ly_0 satisfy (2.4). $\square$

Lemma 2.12. *Let n and k be integers such that $1 \leq k \leq n$. Then*

$$\binom{n}{k} \geq k^{-k}(n-k)^k. \tag{2.6}$$

Proof. For $k = n$ inequality (2.6) holds. Let $1 \leq k < n$. Then

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!} \geq \frac{(n-k)^k}{k!} \geq k^{-k}(n-k)^k. \qquad \square$$

Lemma 2.13 (see [3, Ch. 0]). *Let x and z be real numbers such that $2 \leq z \leq x/2$. Then*

$$\Phi(x,z) \geq c_0\frac{x}{\ln z},$$

where $c_0 > 0$ is an absolute constant.

3. LEMMAS ON DIRICHLET CHARACTERS

In this section we give some well-known lemmas on Dirichlet characters which will be used in the following sections.

Lemma 3.1. *Let a, b, and n be integers such that $1 \leq a < b$, $a \mid b$, and $(n,a) = 1$. Then there is an integer t such that $(n + ta, b) = 1$.*

Proof. If $(n,b) = 1$, we take $t = 0$. Let $(n,b) > 1$. Then the set $\Omega = \{p \mid b \colon p \nmid a\}$ is nonempty. Let $\Omega = \{q_1,\dots,q_r\}$ with $q_1 < \ldots < q_r$. Let $1 \leq i \leq r$. Since $(a,q_i) = 1$, the congruence

$$n + ta \equiv 1 \pmod{q_i}$$

has a solution; i.e., there is an integer m_i such that $n + am_i \equiv 1 \pmod{q_i}$. Consider the system

$$\begin{cases} t \equiv m_1 \pmod{q_1}, \\ \dots\dots\dots\dots\dots\dots \\ t \equiv m_r \pmod{q_r}. \end{cases} \tag{3.1}$$

Since the numbers $q_1, \dots, q_r$ are coprime, the system has a solution. Let an integer t_0 satisfy system (3.1). We claim that t_0 is a desired number, i.e., that $(n + t_0 a, b) = 1$. Assume the contrary: $(n + t_0 a, b) > 1$. Then there is a prime p such that $p \mid b$ and $p \mid (n + t_0 a)$. If $p \nmid a$, then $p \in \Omega$, i.e., $p = q_i$ for some $1 \le i \le r$. However,

$$n + t_0 a \equiv 1 \pmod{q_i}$$

and hence $p \nmid (n + t_0 a)$. We arrive at a contradiction. Thus this case is impossible. Hence, $p \mid a$. Since $p \mid (n + t_0 a)$, we see that $p \mid n$. Hence, $(n, a) > 1$. This contradicts the hypothesis of the lemma. Therefore, the assumption $(n + t_0 a, b) > 1$ is false. Hence, $(n + t_0 a, b) = 1$. □

Lemma 3.2. *Let $q \ge 2$ be an integer and $\chi \in X_q$. Suppose that χ restricted by $(n, q) = 1$ has period q_1. Then χ restricted by $(n, q) = 1$ also has period (q, q_1).*

Proof. We put $\delta = (q, q_1)$. Let m and n be integers such that $(m, q) = 1$, $(n, q) = 1$, and $m \equiv n \pmod{\delta}$. We need to prove that $\chi(m) = \chi(n)$. By Lemma 2.11, there are integers k and l such that

$$m + q_1 k = n + ql.$$

We put $A = m + q_1 k = n + ql$. Since $(n, q) = 1$, we have $(n + ql, q) = 1$. Hence, $(A, q) = 1$. Since χ has period q, it follows that

$$\chi(A) = \chi(n + ql) = \chi(n).$$

Since $(A, q) = 1$, $(m, q) = 1$, and $A \equiv m \pmod{q_1}$, we have $\chi(A) = \chi(m)$. Hence, $\chi(m) = \chi(n)$. □

Lemma 3.3. *Let $q \ge 1$ and $\chi \in X_q$. Then $c(\chi)$ divides q.*

Proof. If $q = 1$, then $c(\chi) = 1$ and the statement is obvious. Let $q \ge 2$. By Lemma 3.2, χ restricted by $(n, q) = 1$ has period $\delta = (c(\chi), q)$. If $c(\chi)$ is not a divisor of q, then $\delta < c(\chi)$, which contradicts the definition of the conductor. □

Lemma 3.4. *Let $q \ge 1$ and $\chi \in X_q$. Then there exists a unique Dirichlet character $\chi_1 \in X_{c(\chi)}$ such that*

$$\chi(n) = \begin{cases} \chi_1(n) & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1. \end{cases} \tag{3.2}$$

Furthermore, χ_1 is primitive.

We say that χ_1 *induces* χ.

Proof of Lemma 3.4. I. Let $q = 1$. Then $c(\chi) = 1$, $\#X_1 = 1$, and $\chi_1 = \chi$, so the statement is obvious.

II. Let $q \ge 2$ and χ be a primitive character modulo q. Then $c(\chi) = q$ and we can take $\chi_1 = \chi$. Let us prove the uniqueness. Suppose that there are two different characters $\chi_1, \chi_2 \in X_q$ satisfying (3.2). Then for any n such that $(n, q) > 1$ we have $\chi_1(n) = 0 = \chi_2(n)$. For any n such that $(n, q) = 1$, we have $\chi_1(n) = \chi(n) = \chi_2(n)$. Therefore, $\chi_1(n) = \chi_2(n)$ for any integer n; i.e., $\chi_1 = \chi_2$, a contradiction.

III. Let $q \ge 2$ and χ be an imprimitive character modulo q. Then $1 \le c(\chi) < q$ and by Lemma 3.3 we have $c(\chi) \mid q$. We define χ_1. Let $n \in \mathbb{Z}$. Consider several cases.

If $(n, c(\chi)) > 1$, then we put $\chi_1(n) = 0$.

If $(n,c(\chi))=1$, then by Lemma 3.1 there is an integer t such that $(n+tc(\chi),q)=1$. We put

$$\chi_1(n)=\chi(n+tc(\chi)).$$

The choice of t subject to the indicated condition is immaterial, since χ restricted by $(n,q)=1$ has period $c(\chi)$. Thus, $\chi_1(n)$ is defined for any integer n. We claim that χ_1 is a character modulo $c(\chi)$. By construction,

$$\chi_1(n)=0 \qquad \text{for any } n\in\mathbb{Z} \text{ such that } (n,c(\chi))>1.$$

By Lemma 3.1, there is an integer t such that $(1+tc(\chi),q)=1$. Since the choice of such a t is immaterial, we take $t=0$. We have $\chi_1(1)=\chi(1)=1$. Now we prove that

$$\chi_1(n+c(\chi))=\chi_1(n) \qquad \text{for all} \quad n\in\mathbb{Z}. \tag{3.3}$$

If $(n,c(\chi))>1$, then we have $(n+c(\chi),c(\chi))>1$. Hence,

$$\chi_1(n+c(\chi))=0=\chi_1(n).$$

Let $(n,c(\chi))=1$. Then we have $(n+c(\chi),c(\chi))=1$. By Lemma 3.1, there are integers t_1 and t_2 such that $(n+t_1c(\chi),q)=1$ and $(n+c(\chi)+t_2c(\chi),q)=1$. By construction, we have

$$\chi_1(n)=\chi(n+t_1c(\chi)) \qquad \text{and} \qquad \chi_1(n+c(\chi))=\chi(n+c(\chi)+t_2c(\chi)).$$

Since χ restricted by $(n,q)=1$ has period $c(\chi)$, we have $\chi(n+t_1c(\chi))=\chi(n+c(\chi)+t_2c(\chi))$. Hence, $\chi_1(n)=\chi_1(n+c(\chi))$ and (3.3) is proved. Now we prove that

$$\chi_1(mn)=\chi_1(m)\chi_1(n) \qquad \text{for all} \quad m,n\in\mathbb{Z}. \tag{3.4}$$

If $(m,c(\chi))>1$, then we have $(mn,c(\chi))>1$. Hence, $\chi_1(mn)=0$ and $\chi_1(m)=0$. Therefore, relation (3.4) holds. Similarly, (3.4) holds if $(n,c(\chi))>1$. Let $(m,c(\chi))=1$ and $(n,c(\chi))=1$. Then $(mn,c(\chi))=1$. By Lemma 3.1, there are integers t_1, t_2, and t_3 such that $(m+t_1c(\chi),q)=1$, $(n+t_2c(\chi),q)=1$, and $(mn+t_3c(\chi),q)=1$. We put $m_1=m+t_1c(\chi)$, $n_1=n+t_2c(\chi)$, and $u=mn+t_3c(\chi)$. By construction,

$$\chi_1(mn)=\chi(u), \qquad \chi_1(m)=\chi(m_1), \qquad \text{and} \qquad \chi_1(n)=\chi(n_1).$$

Since χ is a totally multiplicative function, it follows that

$$\chi_1(m)\chi_1(n)=\chi(m_1)\chi(n_1)=\chi(m_1n_1).$$

Since $(m_1,q)=1$ and $(n_1,q)=1$, we have $(m_1n_1,q)=1$. It is clear that $m_1n_1\equiv u \pmod{c(\chi)}$. Since χ restricted by $(n,q)=1$ has period $c(\chi)$, we find that $\chi(u)=\chi(m_1n_1)$. Therefore, $\chi_1(mn)=\chi_1(m)\chi_1(n)$ and (3.4) is proved. Thus, we have proved that χ_1 is a character modulo $c(\chi)$, i.e., $\chi_1\in X_{c(\chi)}$.

Now we prove that χ_1 satisfies (3.2). It suffices to show that

$$\chi_1(n)=\chi(n) \qquad \text{if} \quad (n,q)=1. \tag{3.5}$$

Since $(n,q)=1$, we have $(n,c(\chi))=1$ (see Lemma 3.3). By Lemma 3.1, there is an integer t such that $(n+tc(\chi),q)=1$. By construction $\chi_1(n)=\chi(n+tc(\chi))$. Since $(n+tc(\chi),q)=1$, $(n,q)=1$, and $n+tc(\chi)\equiv n \pmod{c(\chi)}$, we have $\chi(n+tc(\chi))=\chi(n)$. Hence, $\chi_1(n)=\chi(n)$ and (3.5) is proved.

Now we prove that χ_1 is a primitive character. Suppose that there is a positive integer q_2 such that χ_1 restricted by $(n,c(\chi))=1$ has period q_2. Let m and n be integers such that $(m,q)=1$,

$(n,q)=1$, and $m\equiv n \pmod{q_2}$. By Lemma 3.3, we have $(m,c(\chi))=1$ and $(n,c(\chi))=1$. Then (see (3.5))

$$\chi(m)=\chi_1(m)=\chi_1(n)=\chi(n).$$

Hence, χ restricted by $(n,q)=1$ has period q_2. From the definition of a conductor it follows that $q_2\geq c(\chi)$. Hence, χ_1 is a primitive character.

Now we prove the uniqueness. Suppose that there are two different characters $\chi_1,\chi_2\in X_{c(\chi)}$ satisfying (3.2). If $(n,c(\chi))>1$, then $\chi_1(n)=0=\chi_2(n)$. Let $(n,c(\chi))=1$. By Lemma 3.1, there is an integer t such that $(n+tc(\chi),q)=1$. Since χ_1 and χ_2 are periodic functions with period $c(\chi)$, we have

$$\chi_1(n)=\chi_1(n+tc(\chi))=\chi(n+tc(\chi))=\chi_2(n+tc(\chi))=\chi_2(n).$$

Thus, $\chi_1(n)=\chi_2(n)$ for any $n\in\mathbb{Z}$, and so $\chi_1=\chi_2$. We obtain a contradiction. The uniqueness is proved. $\square$

Lemma 3.5. *Let $q>1$ be an integer expressed in the standard form as $q=q_1^{\alpha_1}\dots q_r^{\alpha_r}$, where $q_1<\dots<q_r$ are primes and $\alpha_1,\dots,\alpha_r$ are positive integers. Let χ be a Dirichlet character modulo q. Then there exist unique characters χ_i modulo $q_i^{\alpha_i}$, $i=1,\dots,r$, such that*

$$\chi(n)=\chi_1(n)\dots\chi_r(n) \qquad \text{for all} \quad n. \tag{3.6}$$

Furthermore, if the character χ is real, then all characters χ_i, $i=1,\dots,r$, are real. If the character χ is primitive, then all characters χ_i, $i=1,\dots,r$, are primitive.

Proof. For any $1\leq i\leq r$ we take A_i such that

$$A_i\equiv 1 \pmod{q_i^{\alpha_i}} \qquad \text{and} \qquad A_i\equiv 0 \pmod{q_j^{\alpha_j}} \quad \text{for any} \quad j\neq i, \quad 1\leq j\leq r. \tag{3.7}$$

Since the moduli of these congruences are coprime, the system has a solution (see, for example, [7, Ch. 4]). Thus, integers $A_1,\dots,A_r$ are defined.

Let $1\leq i\leq r$ and $n\in\mathbb{Z}$. We put

$$\chi_i(n)=\chi\Biggl(nA_i+\sum_{1\leq j\leq r,\, j\neq i}A_j\Biggr). \tag{3.8}$$

It is easy to show that χ_i is a Dirichlet character modulo $q_i^{\alpha_i}$.

Now we prove that (3.6) holds. Let $n\in\mathbb{Z}$. Setting

$$n_i=nA_i+\sum_{1\leq j\leq r,\, j\neq i}A_j, \qquad i=1,\dots,r,$$

we have

$$\chi_1(n)\dots\chi_r(n)=\chi(n_1)\dots\chi(n_r)=\chi(n_1\dots n_r).$$

From (3.7) we obtain

$$n_1\dots n_r\equiv n \pmod{q_s^{\alpha_s}} \qquad \text{for any} \quad 1\leq s\leq r.$$

Hence, $n_1\dots n_r-n$ is divisible by q, i.e.,

$$n_1\dots n_r\equiv n \pmod{q}.$$

Hence, $\chi(n_1\dots n_r)=\chi(n)$ and (3.6) is proved.

Now we prove the uniqueness of the representation of χ in the form (3.6). Suppose that

$$\chi(n)=\widetilde{\chi}_1(n)\dots\widetilde{\chi}_r(n), \tag{3.9}$$

where $\widetilde{\chi}_i$ is a Dirichlet character modulo $q_i^{\alpha_i}$, $i = 1, \dots, r$. Let $1 \le i \le r$ and $n \in \mathbb{Z}$. We have (see (3.7))

$$nA_i + \sum_{1\le j\le r,\, j\ne i} A_j \equiv 1 \pmod{q_s^{\alpha_s}} \qquad \text{for any} \quad 1 \le s \le r, \quad s \ne i,$$

and

$$nA_i + \sum_{1\le j\le r,\, j\ne i} A_j \equiv n \pmod{q_i^{\alpha_i}}.$$

Hence,

$$\widetilde{\chi}_s\Biggl(nA_i + \sum_{1\le j\le r,\, j\ne i} A_j\Biggr) = 1 \qquad \text{for any} \quad 1 \le s \le r, \quad s \ne i,$$

and

$$\widetilde{\chi}_i\Biggl(nA_i + \sum_{1\le j\le r,\, j\ne i} A_j\Biggr) = \widetilde{\chi}_i(n).$$

From (3.9) we obtain

$$\chi\Biggl(nA_i + \sum_{1\le j\le r,\, j\ne i} A_j\Biggr) = \widetilde{\chi}_i(n).$$

Therefore (see (3.8)), $\widetilde{\chi}_i(n) = \chi_i(n)$. Since this equation holds for any $n \in \mathbb{Z}$, we have $\widetilde{\chi}_i = \chi_i$, $i = 1, \dots, r$. Thus, the uniqueness of the representation of χ in the form (3.6) is proved.

We see from (3.8) that if the character χ is real, then all characters χ_i, $i = 1, \dots, r$, are real. We claim that if the character χ is primitive, then all characters χ_i, $i = 1, \dots, r$, are primitive. Assume the contrary: there is an i, $1 \le i \le r$, such that the character χ_i is imprimitive. Then $c(\chi_i) < q_i^{\alpha_i}$. Since $c(\chi_i) \mid q_i^{\alpha_i}$ (see Lemma 3.3), we have

$$c(\chi_i) = q_i^{\beta}, \qquad \beta < \alpha_i.$$

We put

$$\widetilde{q} = q_i^{\beta} \prod_{1\le j\le r,\, j\ne i} q_j^{\alpha_j}.$$

Let us show that the character χ restricted by $(n, q) = 1$ has period $\widetilde{q}$. Take integers m and n such that $(m, q) = (n, q) = 1$ and $m \equiv n \pmod{\widetilde{q}}$. Let $1 \le j \le r$, $j \ne i$. Since

$$m \equiv n \pmod{q_j^{\alpha_j}},$$

we have $\chi_j(m) = \chi_j(n)$. Since $(m, q_i^{\alpha_i}) = (n, q_i^{\alpha_i}) = 1$,

$$m \equiv n \pmod{q_i^{\beta}},$$

and χ_i restricted by $(n, q_i^{\alpha_i}) = 1$ has period q_i^{β}, we have $\chi_i(m) = \chi_i(n)$. This implies

$$\chi(m) = \chi_i(m) \prod_{1\le j\le r,\, j\ne i} \chi_j(m) = \chi_i(n) \prod_{1\le j\le r,\, j\ne i} \chi_j(n) = \chi(n).$$

We have proved that χ restricted by $(n, q) = 1$ has period $\widetilde{q}$. But then $c(\chi) \le \widetilde{q} < q$. This contradicts the fact that the character χ is primitive. Hence, all characters χ_i, $i = 1, \dots, r$, are primitive. Lemma 3.5 is proved. $\square$

Lemma 3.6. *Let q be a positive integer such that there exists a real primitive character χ modulo q. Then the number q is of the form $2^\alpha k$, where $\alpha \in \{0,\dots,3\}$ and $k \geq 1$ is an odd square-free integer.*

Proof. Modulo $q=1$ there exists a real primitive character; namely, $\chi(n)=1$ for all $n\in\mathbb{Z}$. The number 1 is of the form $2^\alpha k$; namely, $\alpha=0$ and $k=1$.

Let $q>1$ be an integer such that there exists a real primitive character χ modulo q. Suppose that $q=p^rs$, where $p\geq 3$ is a prime number, $(p,s)=1$, and $r\geq 2$. Let $\widetilde{q}=p^{r-1}s$. We claim that the character χ restricted by $(n,q)=1$ has period $\widetilde{q}$. Let m and n be integers such that $(m,q)=(n,q)=1$ and $m\equiv n \pmod{\widetilde{q}}$. We have $m=n+\widetilde{q}t$, $t\in\mathbb{Z}$, and

$$m^{p^{r-1}}=(n+\widetilde{q}t)^{p^{r-1}}=n^{p^{r-1}}+\sum_{i=1}^{p^{r-1}}\binom{p^{r-1}}{i}(\widetilde{q}t)^in^{p^{r-1}-i}=n^{p^{r-1}}+\sum_{i=1}^{p^{r-1}}A_it^in^{p^{r-1}-i}, \tag{3.10}$$

where

$$A_i=\binom{p^{r-1}}{i}(\widetilde{q}\,)^i.$$

Let $2\leq i\leq p^{r-1}$. Then

$$A_i=\binom{p^{r-1}}{i}(p^{r-1}s)^i=p^rs\binom{p^{r-1}}{i}p^{(i-1)r-i}s^{i-1}.$$

It is clear that $i-1\geq 1$. We claim that

$$(i-1)r-i\geq 0 \tag{3.11}$$

or, which is equivalent, $i(r-1)\geq r$. Indeed, since $i\geq 2$ and $r\geq 2$, we have

$$i(r-1)\geq 2(r-1)\geq r.$$

Hence, $A_i=p^rsN$, where $N\in\mathbb{N}$. Thus, for any $2\leq i\leq p^{r-1}$,

$$A_i\equiv 0 \pmod{q}.$$

We have $A_1=p^{r-1}(p^{r-1}s)=p^rsp^{r-2}$. Since $r\geq 2$, we obtain

$$A_1\equiv 0 \pmod{q}.$$

Hence (see (3.10)),

$$m^{p^{r-1}}\equiv n^{p^{r-1}} \pmod{q}.$$

Using the properties of a character, we obtain

$$(\chi(m))^{p^{r-1}}=(\chi(n))^{p^{r-1}}.$$

Since $(m,q)=(n,q)=1$ and the character χ is real, we have $\chi(m),\chi(n)\in\{-1,1\}$. Since $p\geq 3$ is a prime number and $r\geq 2$ is an integer, it follows that p^{r-1} is an odd positive integer. Therefore, if $\chi(m)=1$, then $\chi(n)=1$, while if $\chi(m)=-1$, then $\chi(n)=-1$ as well. Thus, $\chi(m)=\chi(n)$. We have proved that the character χ restricted by $(n,q)=1$ has period $\widetilde{q}$. Consequently,

$$c(\chi)\leq\widetilde{q}<q.$$

This contradicts the fact that χ is a primitive character. Hence, the number q is of the form $2^\alpha k$, where $\alpha\geq 0$ is an integer and $k\geq 1$ is an odd square-free integer.

We claim that $\alpha \le 3$. Assume the contrary: $\alpha \ge 4$. Let $k = q_1 \dots q_r$, where $q_1 < \dots < q_r$ are odd primes. By Lemma 3.5, we have

$$\chi(n) = \chi_1(n)\chi_2(n)\dots\chi_{r+1}(n), \tag{3.12}$$

where χ_1 is a real primitive character modulo 2^α and χ_i is a real primitive character modulo q_{i-1}, $i = 2, \dots, r+1$ (if $k = 1$, then $\chi_2, \dots, \chi_{r+1}$ are omitted in (3.12)). It is well known (see, for example, [7, Ch. 6]) that if numbers ν and γ run independently through the sets $\{0,1\}$ and $\{0, \dots, 2^{\alpha-2} - 1\}$ respectively, then $(-1)^\nu \cdot 5^\gamma$ runs (without repetitions) through a reduced residue system modulo 2^α. Hence, for any n with $(n,2) = 1$ there are unique numbers $\nu(n) \in \{0,1\}$ and $\gamma(n) \in \{0, \dots, 2^{\alpha-2} - 1\}$ such that

$$n \equiv (-1)^{\nu(n)} \cdot 5^{\gamma(n)} \pmod{2^\alpha}. \tag{3.13}$$

Since $(-1)^2 = 1$, we have $(\chi_1(-1))^2 = 1$. Thus,

$$\chi_1(-1) = (-1)^a, \qquad a \in \{0,1\}.$$

It is well known (see, for example, [7, Ch. 6]) that the number 5 belongs to $2^{\alpha-2} \pmod{2^\alpha}$; in particular, $5^{2^{\alpha-2}} \equiv 1 \pmod{2^\alpha}$. Hence,

$$(\chi_1(5))^{2^{\alpha-2}} = 1.$$

We obtain

$$\chi_1(5) = \exp\biggl(2\pi i \frac{b}{2^{\alpha-2}}\biggr), \qquad b \in \{0, \dots, 2^{\alpha-2} - 1\}.$$

We see from (3.13) that if n is such that $(n,2) = 1$, then

$$\chi_1(n) = (-1)^{a\nu(n)} \exp\biggl(2\pi i \frac{b\gamma(n)}{2^{\alpha-2}}\biggr). \tag{3.14}$$

We claim that $(b,2) = 1$. Indeed, assume the contrary: $(b,2) > 1$. We show that then χ_1 restricted by $(n, 2^\alpha) = 1$ has period $2^{\alpha-1}$. Let m and n be integers such that $(m, 2^\alpha) = (n, 2^\alpha) = 1$ and $m \equiv n \pmod{2^{\alpha-1}}$. We have

$$m \equiv (-1)^{\nu(m)} \cdot 5^{\gamma(m)} \pmod{2^\alpha} \qquad \text{and} \qquad n \equiv (-1)^{\nu(n)} \cdot 5^{\gamma(n)} \pmod{2^\alpha}.$$

Since these congruences also hold modulo $2^{\alpha-1}$, we have

$$(-1)^{\nu(m)} \cdot 5^{\gamma(m)} \equiv (-1)^{\nu(n)} \cdot 5^{\gamma(n)} \pmod{2^{\alpha-1}}. \tag{3.15}$$

Since $\alpha \ge 4$, we obtain

$$(-1)^{\nu(m)} \cdot 5^{\gamma(m)} \equiv (-1)^{\nu(n)} \cdot 5^{\gamma(n)} \pmod{4}.$$

It is clear that

$$(-1)^{\nu(m)} \cdot 5^{\gamma(m)} \equiv (-1)^{\nu(m)} \pmod{4} \qquad \text{and} \qquad (-1)^{\nu(n)} \cdot 5^{\gamma(n)} \equiv (-1)^{\nu(n)} \pmod{4}.$$

Hence,

$$(-1)^{\nu(m)} \equiv (-1)^{\nu(n)} \pmod{4}.$$

If $\nu(m) = 0$, then $\nu(n) = 0$; if $\nu(m) = 1$, then $\nu(n) = 1$. Thus,

$$\nu(m) = \nu(n). \tag{3.16}$$

Therefore (see (3.15)),

$$5^{\gamma(m)} \equiv 5^{\gamma(n)} \pmod{2^{\alpha-1}}.$$

Suppose, for definiteness, that $\gamma(m) \geq \gamma(n)$. We have

$$5^{\gamma(n)}\big(5^{\gamma(m)-\gamma(n)} - 1\big) \equiv 0 \pmod{2^{\alpha-1}}.$$

Since $(5^{\gamma(n)}, 2^{\alpha-1}) = 1$, we obtain

$$5^{\gamma(m)-\gamma(n)} - 1 \equiv 0 \pmod{2^{\alpha-1}}.$$

Hence,

$$5^{\gamma(m)-\gamma(n)} \equiv 1 \pmod{2^{\alpha-1}}.$$

Since 5 belongs to $2^{\alpha-3} \pmod{2^{\alpha-1}}$, we have (see [7, Ch. 6])

$$\gamma(m) - \gamma(n) \equiv 0 \pmod{2^{\alpha-3}}.$$

Therefore,

$$\gamma(m) = \gamma(n) + 2^{\alpha-3}t, \tag{3.17}$$

where $t \geq 0$ is an integer. Since $(b, 2) > 1$, we have

$$b = 2\widetilde{b}, \tag{3.18}$$

where $\widetilde{b} \geq 0$ is an integer. We obtain (see (3.14) and (3.16)–(3.18))

$$\begin{aligned}\chi_1(m) &= (-1)^{a\nu(m)} \exp\left(2\pi i \frac{\widetilde{b}\gamma(m)}{2^{\alpha-3}}\right) = (-1)^{a\nu(n)} \exp\left(2\pi i \frac{\widetilde{b}(\gamma(n) + 2^{\alpha-3}t)}{2^{\alpha-3}}\right)\\ &= (-1)^{a\nu(n)} \exp\left(2\pi i \frac{\widetilde{b}\gamma(n)}{2^{\alpha-3}}\right) \exp(2\pi i \widetilde{b}t) = (-1)^{a\nu(n)} \exp\left(2\pi i \frac{\widetilde{b}\gamma(n)}{2^{\alpha-3}}\right) = \chi_1(n).\end{aligned}$$

Thus, we have proved that χ_1 restricted by $(n, 2^\alpha) = 1$ has period $2^{\alpha-1}$. Hence,

$$c(\chi_1) \leq 2^{\alpha-1} < 2^\alpha.$$

This contradicts the fact that χ_1 is a primitive character. Hence, $(b, 2) = 1$.

For $n = 5$ we have $\nu(5) = 0$ and $\gamma(5) = 1$. Therefore (see (3.14)),

$$\chi_1(5) = \exp\left(2\pi i \frac{b}{2^{\alpha-2}}\right) = \exp\left(\pi i \frac{b}{2^{\alpha-3}}\right).$$

Since $\alpha \geq 4$ and $(b, 2) = 1$, we have $\mathrm{Im}(\chi_1(5)) \neq 0$. This contradicts the fact that χ_1 is a real character. Hence, $0 \leq \alpha \leq 3$. Lemma 3.6 is proved. □

Lemma 3.7. *Let q_1 and q_2 be positive integers with $q_1 \neq q_2$, χ_1 be a primitive character modulo q_1, and χ_2 be a primitive character modulo q_2. Then $\chi_1 \neq \chi_2$.*

Proof. Assume the contrary: $\chi_1 = \chi_2$. Let m and n be integers such that $(m, q_1) = (n, q_1) = 1$ and $m \equiv n \pmod{q_2}$. Then

$$\chi_1(m) = \chi_2(m) = \chi_2(n) = \chi_1(n).$$

Hence, χ_1 restricted by $(n, q_1) = 1$ has period q_2. Hence, $c(\chi_1) \leq q_2$. Since χ_1 is a primitive character modulo q_1, we have $c(\chi_1) = q_1$. Thus, $q_1 \leq q_2$. Similarly, it can be proved that $q_2 \leq q_1$. Hence, $q_1 = q_2$. We have arrived at a contradiction, which means that $\chi_1 \neq \chi_2$. □

4. LEMMAS ON $\psi(x,\chi)$

In this section we present some lemmas on $\psi(x,\chi)$. Most of these lemmas are well known. The proof of Lemma 4.6 is based on Maynard's ideas (see the proof of Theorem 3.2 in [5]). The proof of Lemma 4.9 follows a standard proof of the Bombieri–Vinogradov theorem (see, for example, [1, Ch. 28]).

Lemma 4.1. *Let $u \ge 2$ be a real number, and let $Q \ge 2$ and W be integers with $(W,Q)=1$. Then*

$$\psi(u;Q,W) - \frac{u}{\varphi(Q)} = \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \psi'(u,\chi)$$

(*the overbar denotes complex conjugation*).

Proof. We define

$$I_{Q,W}(n) = \begin{cases} 1 & \text{if } n \equiv W \pmod{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Since (see, for example, [1, Ch. 4])

$$\frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \chi(n) = I_{Q,W}(n),$$

we have

$$\psi(u;Q,W) = \sum_{\substack{n \le u \\ n \equiv W \ (\mathrm{mod}\ Q)}} \Lambda(n) = \sum_{n \le u} \Lambda(n) I_{Q,W}(n) = \sum_{n \le u} \Lambda(n) \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \chi(n)$$

$$= \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \left(\sum_{n \le u} \Lambda(n) \chi(n) \right) = \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \psi(u,\chi).$$

Let χ_0 be the principal character modulo Q. Since $(W,Q)=1$, it follows that $\chi_0(W)=1$. We have

$$\sum_{\chi \in X_Q} \overline{\chi(W)} E_{\chi_0}(\chi) u = \overline{\chi_0(W)} u = u.$$

Hence,

$$\psi(u;Q,W) - \frac{u}{\varphi(Q)} = \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \big(\psi(u,\chi) - E_{\chi_0}(\chi) u\big) = \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)} \psi'(u,\chi).$$

Lemma 4.1 is proved. □

Lemma 4.2 (see, for example, [1, Ch. 14])**.** *There is a positive absolute constant $a > 0$ such that if χ is a complex character modulo q, then $L(s,\chi)$ has no zeros in the region*

$$\Omega\colon \qquad \sigma \ge \begin{cases} 1 - \dfrac{a}{\ln(q|t|)} & \text{if } |t| \ge 1, \\[2ex] 1 - \dfrac{a}{\ln q} & \text{if } |t| < 1 \end{cases}$$

(*here $s = \sigma + it$, $\sigma = \operatorname{Re} s$, and $t = \operatorname{Im} s$*). *If χ is a real nonprincipal character modulo q, the only possible zero of $L(s,\chi)$ in this region is a single* (*simple*) *real zero. Furthermore, $L(s,\chi)$ can have a zero in the region Ω for at most one of the real nonprincipal characters χ* (mod q).

Remark. It is easy to see that the constant a can be replaced by any constant a^* such that $0 < a^* < a$.

Lemma 4.3 (see [1, Ch. 20]). *Let χ be a nonprincipal character modulo q and $2 \le T \le u$. Then*

$$\psi(u,\chi) = -\frac{u^{\beta_1}}{\beta_1} + R_4(u,T),$$

where

$$|R_4(u,T)| \le C\biggl(u\ln^2(qu)\exp\biggl(-\frac{a\ln u}{\ln(qT)}\biggr) + uT^{-1}\ln^2(qu) + u^{1/4}\ln u\biggr).$$

Here $C > 0$ is an absolute constant and $a > 0$ is the absolute constant in Lemma 4.2. *The term $-u^{\beta_1}/\beta_1$ should be omitted unless χ is a real character for which $L(s,\chi)$ has a zero β_1 (which is necessarily unique, real, and simple) satisfying*

$$\beta_1 > 1 - \frac{a}{\ln q}.$$

Lemma 4.4 (Page's theorem; see, for example, [1, Ch. 14]). *There are absolute constants $a_1 > 0$ and $a_1' > 0$ such that the following holds. Let $z \ge 3$ be a real number. Then there is at most one real primitive character χ to a modulus q_0, $3 \le q_0 \le z$, for which $L(s,\chi)$ has a real zero β satisfying*

$$\beta > 1 - \frac{a_1}{\ln z}.$$

If such a character χ exists, then

$$q_0 \ge \frac{a_1'(\ln z)^2}{(\ln\ln z)^4}.$$

Such a modulus q_0 is said to be an *exceptional modulus* in the interval $[3,z]$.

Lemma 4.5. *Let $z \ge 3$ be a real number. If an exceptional modulus q_0 in the interval $[3,z]$ exists, then the number q_0 is of the form $2^\alpha k$, where $\alpha \in \{0,\dots,3\}$ and $k \ge 1$ is an odd square-free integer.*

Proof. Suppose an exceptional modulus q_0 in the interval $[3,z]$ exists. In particular, this means that there exists a real primitive character χ modulo q_0. By Lemma 3.6, the number q_0 is of the form $2^\alpha k$ with $\alpha \in \{0,\dots,3\}$ and an odd square-free integer $k \ge 1$. □

Lemma 4.6. *There are positive absolute constants c_0, c_1, γ_0, and C such that the following holds. Let $x \ge c_0$ be a real number, q_0 be an exceptional modulus in the interval $[3, \exp(2c_1\sqrt{\ln x})]$, Q be an integer such that $3 \le Q \le \exp(2c_1\sqrt{\ln x})$ and $Q \ne q_0$ (the last inequality should be interpreted as follows: if q_0 exists, then $Q \ne q_0$; if q_0 does not exist, then Q is any integer in the indicated interval), and χ be a primitive character modulo Q. Then*

$$\max_{2\le u\le x^{1+\gamma_0/\sqrt{\ln x}}} |\psi(u,\chi)| \le Cx\exp(-3c_1\sqrt{\ln x}).$$

Proof. We will choose c_1 and γ_0 later. The number c_0 depends on c_1 and γ_0 and is large enough, and $x \ge c_0(c_1,\gamma_0)$. We put

$$z = \exp(2c_1\sqrt{\ln x}).$$

We have $z \ge 3$ if the number $c_0(c_1,\gamma_0)$ is chosen large enough. By Lemma 4.4, there is at most one real primitive χ to a modulus q_0, $3 \le q_0 \le z$, for which $L(s,\chi)$ has a real zero β satisfying

$$\beta > 1 - \frac{a_1}{\ln z} = 1 - \frac{a_1}{2c_1\sqrt{\ln x}}. \tag{4.1}$$

If such a character χ exists, then

$$q_0 \geq \frac{a_1'(\ln z)^2}{(\ln\ln z)^4} = \frac{a_1'(2c_1\sqrt{\ln x})^2}{((1/2)\ln\ln x + \ln(2c_1))^4} \geq \frac{a_1' c_1^2 \ln x}{(\ln\ln x)^4} \tag{4.2}$$

provided that $c_0(c_1, \gamma_0)$ is chosen large enough. Let Q be an integer such that $3 \leq Q \leq \exp(2c_1\sqrt{\ln x})$ and $Q \neq q_0$, and let χ be a primitive character modulo Q. Since $Q > 1$, we see that χ is a nonprincipal character. By Lemma 4.3, if $2 \leq T \leq u$, then

$$\psi(u,\chi) = -\frac{u^{\beta_1}}{\beta_1} + R_4(u,T), \tag{4.3}$$

where

$$\begin{aligned} |R_4(u,T)| &\leq C\biggl(u\ln^2(Qu)\exp\biggl(-\frac{a\ln u}{\ln(QT)}\biggr) + uT^{-1}\ln^2(Qu) + u^{1/4}\ln u\biggr) \\ &= C(\Delta_1 + \Delta_2 + \Delta_3). \end{aligned} \tag{4.4}$$

The term $-u^{\beta_1}/\beta_1$ is to be omitted unless χ is a real character modulo Q for which $L(s,\chi)$ has a zero β_1 (which is necessarily unique, real, and simple) satisfying

$$\beta_1 > 1 - \frac{a}{\ln Q}.$$

Let

$$2 \leq u \leq x^{1+\gamma_0/\sqrt{\ln x}}.$$

Let $u \geq c_2(c_1)$, where $c_2(c_1) > 0$ is a number depending only on c_1. We choose

$$T = \exp(4c_1\sqrt{\ln u}). \tag{4.5}$$

Then $2 \leq T \leq u$ if $c_2(c_1)$ is chosen large enough.

I. Now we estimate the quantity

$$\Delta_1 = u\ln^2(Qu)\exp\biggl(-\frac{a\ln u}{\ln(QT)}\biggr).$$

If $c_0(c_1, \gamma_0)$ is chosen large enough, then

$$1 + \frac{\gamma_0}{\sqrt{\ln x}} \leq 2. \tag{4.6}$$

Hence,

$$\begin{gathered} \ln u \leq \biggl(1 + \frac{\gamma_0}{\sqrt{\ln x}}\biggr)\ln x \leq 2\ln x, \\ QT \leq \exp\bigl(2c_1\sqrt{\ln x} + 4c_1\sqrt{\ln u}\,\bigr) \leq \exp\bigl(10c_1\sqrt{\ln x}\,\bigr), \\ \ln(QT) \leq 10c_1\sqrt{\ln x}, \qquad -\frac{a\ln u}{\ln(QT)} \leq -\frac{a\ln u}{10c_1\sqrt{\ln x}}. \end{gathered} \tag{4.7}$$

If $c_0(c_1, \gamma_0)$ is chosen large enough, then

$$\ln Q \leq 2c_1\sqrt{\ln x} \leq \ln x.$$

Therefore,

$$\ln^2(Qu) \leq 2\bigl(\ln^2 Q + \ln^2 u\bigr) \leq 10\ln^2 x = 10\exp(2\ln\ln x). \tag{4.8}$$

We have

$$\Delta_1 \le 10u\exp\biggl(-\frac{a\ln u}{10c_1\sqrt{\ln x}} + 2\ln\ln x\biggr).$$

Consider two cases.

(1) Let $x^{1/4} \le u \le x^{1+\gamma_0/\sqrt{\ln x}}$. Then

$$\frac{\ln x}{4} \le \ln u \le \biggl(1 + \frac{\gamma_0}{\sqrt{\ln x}}\biggr)\ln x \le 2\ln x.$$

Let

$$0 < c_1 \le \sqrt{\frac{a}{160}} \qquad \Rightarrow \qquad -\frac{a}{40c_1} \le -4c_1.$$

Hence,

$$-\frac{a\ln u}{10c_1\sqrt{\ln x}} \le -\frac{(a/4)\ln x}{10c_1\sqrt{\ln x}} = -\frac{a\sqrt{\ln x}}{40c_1} \le -4c_1\sqrt{\ln x}$$

and

$$-\frac{a\ln u}{10c_1\sqrt{\ln x}} + 2\ln\ln x \le -4c_1\sqrt{\ln x} + 2\ln\ln x \le -\frac{7}{2}c_1\sqrt{\ln x}$$

provided that $c_0(c_1,\gamma_0)$ is chosen large enough. If $0 < \gamma_0 \le c_1/2$, then

$$\Delta_1 \le 10x^{1+\gamma_0/\sqrt{\ln x}}\exp\biggl(-\frac{7}{2}c_1\sqrt{\ln x}\biggr) = 10x\exp\biggl(-\frac{7}{2}c_1\sqrt{\ln x} + \gamma_0\sqrt{\ln x}\biggr) \le 10x\exp(-3c_1\sqrt{\ln x}).$$

(2) Let $c_2(c_1) \le u < x^{1/4}$ (we may assume that $c_0(c_1,\gamma_0) > (c_2(c_1))^4$ and $c_2(c_1) \ge 10$). We have

$$\begin{aligned}\Delta_1 &\le 10u\exp\biggl(-\frac{a\ln u}{10c_1\sqrt{\ln x}} + 2\ln\ln x\biggr) \le 10u\exp(2\ln\ln x)\\ &\le 10x^{1/4}\exp(2\ln\ln x) \le 10x\exp(-3c_1\sqrt{\ln x})\end{aligned}$$

provided that $c_0(c_1,\gamma_0)$ is chosen large enough.

Thus, if $0 < c_1 < \sqrt{a/160}$, $0 < \gamma_0 \le c_1/2$, $x \ge c_0(c_1,\gamma_0)$, and $c_2(c_1) \le u \le x^{1+\gamma_0/\sqrt{\ln x}}$, then

$$\Delta_1 \le 10x\exp(-3c_1\sqrt{\ln x}).$$

II. Now we estimate the quantity

$$\Delta_2 = uT^{-1}\ln^2(Qu).$$

From (4.5) and (4.8) we obtain

$$\Delta_2 \le 10u\exp\bigl(-4c_1\sqrt{\ln u} + 2\ln\ln x\bigr).$$

Consider two cases.

(1) Let $x^{9/10} \le u \le x^{1+\gamma_0/\sqrt{\ln x}}$. Then

$$\frac{9}{10}\ln x \le \ln u \le \biggl(1 + \frac{\gamma_0}{\sqrt{\ln x}}\biggr)\ln x \le 2\ln x, \qquad -4c_1\sqrt{\ln u} \le -4c_1\sqrt{\frac{9}{10}\ln x} < -\frac{15}{4}c_1\sqrt{\ln x}.$$

Since $0 < \gamma_0 \le c_1/2$, we have

$$\Delta_2 \le 10x^{1+\gamma_0/\sqrt{\ln x}} \exp\biggl(-\frac{15}{4}c_1\sqrt{\ln x} + 2\ln\ln x\biggr) = 10x\exp\biggl(-\frac{15}{4}c_1\sqrt{\ln x} + 2\ln\ln x + \gamma_0\sqrt{\ln x}\biggr)$$
$$\le 10x\exp\biggl(-\frac{13}{4}c_1\sqrt{\ln x} + 2\ln\ln x\biggr) \le 10x\exp(-3c_1\sqrt{\ln x})$$

provided that $c_0(c_1,\gamma_0)$ is chosen large enough.

(2) Let $c_2(c_1) \le u < x^{9/10}$. Then

$$\Delta_2 \le 10u\exp\bigl(-4c_1\sqrt{\ln u} + 2\ln\ln x\bigr) \le 10u\exp(2\ln\ln x)$$
$$\le 10x^{9/10}\exp(2\ln\ln x) \le 10x\exp(-3c_1\sqrt{\ln x})$$

provided that $c_0(c_1,\gamma_0)$ is chosen large enough.

Thus, if $0 < c_1 < \sqrt{a/160}$, $0 < \gamma_0 \le c_1/2$, $x \ge c_0(c_1,\gamma_0)$, and $c_2(c_1) \le u \le x^{1+\gamma_0/\sqrt{\ln x}}$, then

$$\Delta_2 \le 10x\exp(-3c_1\sqrt{\ln x}).$$

III. Now we estimate the quantity

$$\Delta_3 = u^{1/4}\ln u.$$

Since (see (4.6) and (4.7))

$$\ln u \le 2\ln x \qquad \text{and} \qquad u^{1/4} \le x^{(1+\gamma_0/\sqrt{\ln x})/4} \le x^{1/2},$$

we have

$$\Delta_3 \le 2x^{1/2}\ln x \le x\exp(-3c_1\sqrt{\ln x})$$

provided that $c_0(c_1,\gamma_0)$ is chosen large enough.

Finally, we obtain the following (see (4.4)): if $0 < c_1 < \sqrt{a/160}$, $0 < \gamma_0 \le c_1/2$, $x \ge c_0(c_1,\gamma_0)$, and $c_2(c_1) \le u \le x^{1+\gamma_0/\sqrt{\ln x}}$, then

$$|R_4(u,T)| \le 21Cx\exp(-3c_1\sqrt{\ln x}), \tag{4.9}$$

where $C > 0$ is an absolute constant.

IV. Now we estimate the quantity (see (4.3))

$$\Delta_4 = \biggl|-\frac{u^{\beta_1}}{\beta_1}\biggr|.$$

If χ is not a real character modulo Q for which $L(s,\chi)$ has a zero β_1 (which is necessarily unique, real, and simple) satisfying

$$\beta_1 > 1 - \frac{a}{\ln Q},$$

then the term $-u^{\beta_1}/\beta_1$ in (4.3) is to be omitted, and there is nothing to estimate. Let χ be such a character. Then χ is a real primitive character modulo Q. Since $Q \ne q_0$, we have (see Lemma 3.7 and (4.1))

$$\beta_1 \le 1 - \frac{a_1}{\ln z} = 1 - \frac{a_1}{2c_1\sqrt{\ln x}}.$$

Hence,

$$|u^{\beta_1}| = u^{\beta_1} \le u^{1-a_1/(2c_1\sqrt{\ln x})} = u\exp\biggl(-\frac{a_1\ln u}{2c_1\sqrt{\ln x}}\biggr).$$

By the remark made after Lemma 4.2, we may assume that $0 < a < 1/2$. Since $Q \geq 3$, we have

$$\beta_1 > 1 - \frac{a}{\ln Q} > 1 - \frac{1}{2\ln 3} > \frac{1}{2}.$$

Hence, $0 < 1/\beta_1 \leq 2$. Thus,

$$\Delta_4 \leq 2u \exp\biggl(-\frac{a_1 \ln u}{2c_1\sqrt{\ln x}}\biggr). \tag{4.10}$$

Consider two cases.

(1) Let $x^{1/2} \leq u \leq x^{1+\gamma_0/\sqrt{\ln x}}$. We have (see (4.6))

$$\frac{\ln x}{2} \leq \ln u \leq \biggl(1 + \frac{\gamma_0}{\sqrt{\ln x}}\biggr)\ln x \leq 2\ln x.$$

We take

$$0 < c_1 < \sqrt{\frac{\min\{a, a_1\}}{160}} \qquad \Rightarrow \qquad -\frac{a_1}{4c_1} \leq -\frac{7}{2}c_1.$$

Then,

$$-\frac{a_1 \ln u}{2c_1\sqrt{\ln x}} \leq -\frac{(a_1/2)\ln x}{2c_1\sqrt{\ln x}} = -\frac{a_1\sqrt{\ln x}}{4c_1} \leq -\frac{7}{2}c_1\sqrt{\ln x}.$$

Since $0 < \gamma_0 \leq c_1/2$, we obtain (see (4.10))

$$\Delta_4 \leq 2x^{1+\gamma_0/\sqrt{\ln x}} \exp\biggl(-\frac{7}{2}c_1\sqrt{\ln x}\biggr) = 2x\exp\biggl(-\frac{7}{2}c_1\sqrt{\ln x} + \gamma_0\sqrt{\ln x}\biggr) \leq 2x\exp(-3c_1\sqrt{\ln x}).$$

(2) Let $c_2(c_1) \leq u < x^{1/2}$. Then (see (4.10))

$$\Delta_4 \leq 2u \leq 2x^{1/2} \leq 2x\exp(-3c_1\sqrt{\ln x})$$

provided that $c_0(c_1, \gamma_0)$ is chosen large enough. Combining the estimates found at steps I–IV together, we obtain the following (see (4.3) and (4.9)): if $0 < c_1 < \sqrt{\min\{a, a_1\}/160}$, $0 < \gamma_0 \leq c_1/2$, $x \geq c_0(c_1, \gamma_0)$, and $c_2(c_1) \leq u \leq x^{1+\gamma_0/\sqrt{\ln x}}$, then

$$|\psi(u, \chi)| \leq (21C + 2)x\exp(-3c_1\sqrt{\ln x}),$$

where $C > 0$ is an absolute constant.

There is a number $d(c_1) > 0$, depending only on c_1, such that

$$t\exp(-3c_1\sqrt{\ln t}) \geq 1 \qquad \text{if} \quad t \geq d(c_1).$$

We may assume that $c_0(c_1, \gamma_0) > d(c_1)$. Hence, if $2 \leq u < c_2(c_1)$, then (see (2.1))

$$|\psi(u, \chi)| = \biggl|\sum_{n \leq u} \Lambda(n)\chi(n)\biggr| \leq \sum_{n \leq u} \Lambda(n) = \psi(u) \leq b_6 u \leq b_6 c_2(c_1) \leq b_6 c_2(c_1) x\exp(-3c_1\sqrt{\ln x}).$$

Thus, if $0 < c_1 < \sqrt{\min\{a, a_1\}/160}$, $0 < \gamma_0 \leq c_1/2$, and $x \geq c_0(c_1, \gamma_0)$, then

$$\max_{2 \leq u \leq x^{1+\gamma_0/\sqrt{\ln x}}} |\psi(u, \chi)| \leq \bigl(21C + 2 + b_6 c_2(c_1)\bigr)x\exp(-3c_1\sqrt{\ln x}),$$

where $C > 0$ is an absolute constant. We take

$$c_1 = \frac{\sqrt{\min\{a, a_1\}}}{16} \qquad \text{and} \qquad \gamma_0 = \frac{c_1}{2} = \frac{\sqrt{\min\{a, a_1\}}}{32}.$$

Since $a > 0$ and $a_1 > 0$ are absolute constants, we see that c_1, γ_0, $c_0(c_1, \gamma_0)$ and $c_2(c_1)$ are positive absolute constants. Lemma 4.6 is proved. $\square$

Lemma 4.7 (see [1, Ch. 19]). *Let $u \geq 2$ be a real number, $Q \geq 2$ be an integer, $\chi \in X_Q$, and χ_1 be a primitive character modulo q_1 inducing χ. Then*

$$|\psi'(u,\chi) - \psi'(u,\chi_1)| \leq \ln^2(Qu).$$

Lemma 4.8 (see [1, Ch. 28]). *Let Q_1, Q_2, and t be real numbers such that $1 \leq Q_1 < Q_2$ and $t \geq 2$. Then*

$$\sum_{Q_1 < Q \leq Q_2} \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q^*} \max_{2 \leq u \leq t} |\psi(u,\chi)| \leq C \ln^4(tQ_2) \left(\frac{t}{Q_1} + t^{5/6} \ln Q_2 + t^{1/2} Q_2 \right),$$

where $C > 0$ is an absolute constant.

Lemma 4.9. *Let ε and δ be real numbers such that $0 < \varepsilon < 1$ and $0 < \delta < 1/2$. Then there exists a number $c(\varepsilon, \delta) > 0$, depending only on ε and δ, such that if $x \in \mathbb{R}$ and $q \in \mathbb{Z}$ satisfy the conditions $x \geq c(\varepsilon, \delta)$ and $1 \leq q \leq (\ln x)^{1-\varepsilon}$, then there is a positive integer B for which the following relations hold*:

$$1 \leq B \leq \exp(c_1 \sqrt{\ln x}), \qquad 1 \leq \frac{B}{\varphi(B)} \leq 2, \qquad (B, q) = 1$$

and

$$\sum_{\substack{1 \leq Q \leq x^{1/2-\delta} \\ (Q,B)=1}} \max_{2 \leq u \leq x^{1+\gamma/\sqrt{\ln x}}} \max_{W \in \mathbb{Z}\colon (W,Q)=1} \left| \psi(u; Q, W) - \frac{u}{\varphi(Q)} \right| \leq c_2 x \exp(-c_3 \sqrt{\ln x}).$$

Here c_1, γ, c_2, and c_3 are positive absolute constants.

Proof. Let c_0, c_1, γ_0 and C be the positive absolute constants in Lemma 4.6. We will choose γ and $c(\varepsilon, \delta) = c(\varepsilon, \delta, \gamma)$ later; they are assumed to be small and large enough, respectively; for now, let $0 < \gamma \leq \gamma_0$, $c(\varepsilon, \delta, \gamma) \geq c_0$, and $x \geq c(\varepsilon, \delta, \gamma)$. Let q_0 be the exceptional modulus in the interval $[3, \exp(2c_1 \sqrt{\ln x})]$. If q_0 does not exist, then we take $B = 1$. If q_0 exists, then (see (4.2))

$$q_0 \geq \frac{a_1' c_1^2 \ln x}{(\ln \ln x)^4} = \frac{c_4 \ln x}{(\ln \ln x)^4},$$

where $c_4 > 0$ is an absolute constant. We have $q_0 \geq 24$ if $c(\varepsilon, \delta, \gamma)$ is chosen large enough. By Lemma 4.5, the number q_0 is of the form $2^\alpha k$, where $\alpha \in \{0, \dots, 3\}$ and $k \geq 3$ is an odd square-free integer. We put

$$M_1 = \frac{q_0}{2^\alpha} \geq \frac{q_0}{8} \geq \frac{c_4 \ln x}{8(\ln \ln x)^4}.$$

Let $\tau = (M_1, q)$ and $M_2 = M_1/\tau$. Then $(M_2, q) = 1$. Since $\tau \leq q \leq (\ln x)^{1-\varepsilon}$, we have

$$M_2 = \frac{M_1}{\tau} \geq \frac{M_1}{(\ln x)^{1-\varepsilon}} \geq \frac{c_4 \ln x}{8(\ln \ln x)^4 (\ln x)^{1-\varepsilon}} = \frac{c_4 (\ln x)^\varepsilon}{8(\ln \ln x)^4}.$$

If $c(\varepsilon, \delta, \gamma)$ is chosen large enough, then $M_2 \geq 3$. Hence, $M_2 \geq 3$ is an odd square-free integer. Furthermore, we have $(M_2, q) = 1$ and M_2 divides q_0. Let B be the largest prime divisor of M_2. Hence, $B \geq 3$ is a prime number and B divides q_0. We have (see Lemma 2.4)

$$\frac{B}{\varphi(B)} = \frac{B}{B(1 - 1/B)} = \frac{1}{1 - 1/B} \leq \frac{1}{1 - 1/3} = \frac{3}{2}.$$

Thus, $1 \leq B \leq \exp(2c_1 \sqrt{\ln x})$ is an integer, $(B, q) = 1$, $1 \leq B/\varphi(B) \leq 2$, and $B \geq 3$ is a prime divisor of q_0 if q_0 exists.

Let u be a real number such that $2 \le u \le x^{1+\gamma/\sqrt{\ln x}}$, and let Q and W be integers such that $2 \le Q \le x^{1/2-\delta}$, $(Q,B)=1$, and $(W,Q)=1$. By Lemma 4.1, we have

$$\psi(u;Q,W) - \frac{u}{\varphi(Q)} = \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \overline{\chi(W)}\psi'(u,\chi).$$

Therefore,

$$\left|\psi(u;Q,W) - \frac{u}{\varphi(Q)}\right| \le \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} |\psi'(u,\chi)|.$$

Since the right-hand side of this inequality does not depend on W, we have

$$\max_{W\in\mathbb{Z}\colon (W,Q)=1}\left|\psi(u;Q,W) - \frac{u}{\varphi(Q)}\right| \le \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} |\psi'(u,\chi)|.$$

Let $\chi \in X_Q$, and let χ_1 be a primitive character modulo q_1 inducing χ. From Lemma 3.4 and the definition of the inducing character (which is given below Lemma 3.4), we have $q_1 = c(\chi)$, and hence $q_1 \mid Q$ (see Lemma 3.3). Applying Lemma 4.7, we find

$$|\psi'(u,\chi)| \le |\psi'(u,\chi_1)| + \ln^2(Qu).$$

Since $\#X_Q = \varphi(Q)$, we obtain

$$\begin{aligned}\max_{W\in\mathbb{Z}\colon (W,Q)=1}\left|\psi(u;Q,W) - \frac{u}{\varphi(Q)}\right| &\le \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \big(|\psi'(u,\chi_1)| + \ln^2(Qu)\big)\\ &= \ln^2(Qu) + \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} |\psi'(u,\chi_1)|.\end{aligned}$$

We can assume that

$$1 + \frac{\gamma}{\sqrt{\ln x}} \le 2 \tag{4.11}$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough. Hence,

$$0 < \ln u \le \left(1 + \frac{\gamma}{\sqrt{\ln x}}\right)\ln x \le 2\ln x, \qquad \ln^2 u \le 4\ln^2 x,$$

$$0 < \ln Q \le \left(\frac{1}{2} - \delta\right)\ln x \le \ln x, \qquad \ln^2 Q \le \ln^2 x,$$

$$\ln^2(Qu) \le 2\big(\ln^2 Q + \ln^2 u\big) \le 10\ln^2 x.$$

We obtain

$$\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{W\in\mathbb{Z}\colon (W,Q)=1}\left|\psi(u;Q,W) - \frac{u}{\varphi(Q)}\right| \le 10\ln^2 x + \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi_1)|.$$

Therefore,

$$\begin{aligned}S &= \sum_{\substack{1\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} A_Q = A_1 + \sum_{\substack{2\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} A_Q\\ &\le A_1 + \sum_{\substack{2\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} \left(10\ln^2 x + \frac{1}{\varphi(Q)} \sum_{\chi \in X_Q} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi_1)|\right)\end{aligned}$$

$$\le 10x^{1/2-\delta}\ln^2 x + A_1 + \sum_{\substack{2\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\ \sum_{\chi\in X_Q}\frac{1}{\varphi(Q)}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi_1)|$$
$$= 10x^{1/2-\delta}\ln^2 x + A_1 + S', \tag{4.12}$$

where

$$A_Q := \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{W\in\mathbb{Z}\colon (W,Q)=1}\left|\psi(u;Q,W)-\frac{u}{\varphi(Q)}\right|.$$

Let us estimate the sum S'. Let Q be an integer with $2\le Q\le x^{1/2-\delta}$ and $(Q,B)=1$, let $\chi\in X_Q$, and let χ_1 be the primitive character modulo q_1 inducing χ. Since $q_1\mid Q$, we have $1\le q_1\le x^{1/2-\delta}$ and $(q_1,B)=1$. Hence,

$$S' = \sum_{\substack{2\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\ \sum_{\chi\in X_Q}\frac{1}{\varphi(Q)}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi_1)|$$
$$\le \sum_{\substack{1\le q_1\le x^{1/2-\delta}\\(q_1,B)=1}}\ \sum_{\chi_1\in X^*_{q_1}}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi_1)|\sum_{1\le m\le x^{1/2-\delta}/q_1}\frac{1}{\varphi(mq_1)}.$$

Applying Lemmas 2.5 and 2.9, we obtain

$$\sum_{1\le m\le x^{1/2-\delta}/q_1}\frac{1}{\varphi(mq_1)}\le\frac{1}{\varphi(q_1)}\sum_{1\le m\le x^{1/2-\delta}/q_1}\frac{1}{\varphi(m)}\le\frac{1}{\varphi(q_1)}\sum_{1\le m\le x^{1/2}}\frac{1}{\varphi(m)}\le\frac{1}{\varphi(q_1)}C\ln x,$$

where $C>0$ is an absolute constant. We have

$$S'\le C\ln x\sum_{\substack{1\le q_1\le x^{1/2-\delta}\\(q_1,B)=1}}\frac{1}{\varphi(q_1)}\sum_{\chi_1\in X^*_{q_1}}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi_1)|.$$

Redenoting q_1 by Q and χ_1 by χ, we find

$$S'\le C\ln x\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\frac{1}{\varphi(Q)}\sum_{\chi\in X^*_Q}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi)| = C\ln x\big(S_1'+S_2'+S_3'\big), \tag{4.13}$$

where

$$S_1'=\sum_{\substack{1\le Q\le\ln x\\(Q,B)=1}}R_Q,\qquad S_2'=\sum_{\substack{\ln x<Q\le\exp(c_1\sqrt{\ln x})\\(Q,B)=1}}R_Q,\qquad S_3'=\sum_{\substack{\exp(c_1\sqrt{\ln x})<Q\le x^{1/2-\delta}\\(Q,B)=1}}R_Q,$$
$$R_Q := \frac{1}{\varphi(Q)}\sum_{\chi\in X^*_Q}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi)|,$$

and $c_1>0$ is the absolute constant in Lemma 4.6.

I. Now we estimate S_1'. We have

$$S_1'=\sum_{\substack{1\le Q\le\ln x\\(Q,B)=1}}R_Q\le R_1+\sum_{2\le Q\le\ln x}R_Q = R_1+S_4'. \tag{4.14}$$

(1) Let us estimate R_1. Since $\#X_1^* = 1$, we have

$$R_1 = \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi)|,$$

where $\chi \in X_1^*$, i.e., $\chi(n) = 1$ for any $n \in \mathbb{Z}$. Since χ is the principal character modulo 1, it follows that

$$\psi'(u,\chi) = \psi(u,\chi) - u.$$

We have

$$\psi(u,\chi) = \sum_{n\le u} \Lambda(n)\chi(n) = \sum_{n\le u} \Lambda(n) = \psi(u), \qquad \psi'(u,\chi) = \psi(u) - u.$$

It is well known (see, for example, [1, Ch. 18]) that

$$|\psi(u) - u| \le Cu \exp(-c\sqrt{\ln u}), \qquad u \ge 2, \tag{4.15}$$

where $C > 0$ and $c > 0$ are absolute constants. Consider two cases.

(i) Let $x^{1/4} \le u \le x^{1+\gamma/\sqrt{\ln x}}$ (we may assume that $c(\varepsilon,\delta,\gamma) > 16$). Then (see (4.11))

$$\frac{1}{4}\ln x \le \ln u \le \left(1 + \frac{\gamma}{\sqrt{\ln x}}\right)\ln x \le 2\ln x, \qquad -c\sqrt{\ln u} \le -\frac{c}{2}\sqrt{\ln x}.$$

Hence,

$$|\psi'(u,\chi)| \le Cu \exp(-c\sqrt{\ln u}) \le Cx^{1+\gamma/\sqrt{\ln x}} \exp\Big(-\frac{c}{2}\sqrt{\ln x}\Big)$$
$$= Cx\exp\Big(\Big(\gamma - \frac{c}{2}\Big)\sqrt{\ln x}\Big) \le Cx\exp\Big(-\frac{c}{4}\sqrt{\ln x}\Big)$$

provided that $0 < \gamma \le c/4$.

(ii) Let $2 \le u < x^{1/4}$. Then

$$|\psi'(u,\chi)| \le Cu\exp(-c\sqrt{\ln u}) \le Cu \le Cx^{1/4} \le Cx\exp\Big(-\frac{c}{4}\sqrt{\ln x}\Big)$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough.

We obtain

$$R_1 = \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi)| \le Cx\exp\Big(-\frac{c}{4}\sqrt{\ln x}\Big). \tag{4.16}$$

(2) Now we estimate

$$S_4' = \sum_{2\le Q\le \ln x} \frac{1}{\varphi(Q)} \sum_{\chi\in X_Q^*} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi)|. \tag{4.17}$$

Let Q be an integer such that $2 \le Q \le \ln x$, and let $\chi \in X_Q^*$. Then χ is a nonprincipal character modulo Q, and hence $\psi'(u,\chi) = \psi(u,\chi)$. Consider two cases.

(i) Let $x^{1/4} \le u \le x^{1+\gamma/\sqrt{\ln x}}$. Then (see (4.11))

$$\frac{1}{4}\ln x \le \ln u \le \left(1 + \frac{\gamma}{\sqrt{\ln x}}\right)\ln x \le 2\ln x.$$

We may assume that $c(\varepsilon,\delta,\gamma) \ge e^{16}$. Hence, $\ln u \ge (\ln x)/4 \ge 4$. We have

$$2 \le Q \le \ln x \le 4\ln u \le \ln^2 u.$$

Therefore (see, for example, [1, Ch. 22]),

$$|\psi(u,\chi)| \le Cu\exp\bigl(-c(2)\sqrt{\ln u}\bigr),$$

where $C>0$ and $c(2)>0$ are absolute constants. We have

$$-c(2)\sqrt{\ln u} \le -\frac{c(2)}{2}\sqrt{\ln x}$$

and

$$\begin{aligned}|\psi(u,\chi)| &\le Cu\exp\biggl(-\frac{c(2)}{2}\sqrt{\ln x}\biggr) \le Cx^{1+\gamma/\sqrt{\ln x}}\exp\biggl(-\frac{c(2)}{2}\sqrt{\ln x}\biggr)\\ &= Cx\exp\biggl(\biggl(\gamma-\frac{c(2)}{2}\biggr)\sqrt{\ln x}\biggr) \le Cx\exp\biggl(-\frac{c(2)}{4}\sqrt{\ln x}\biggr)\end{aligned}$$

provided that $0<\gamma\le c(2)/4$.

(i) Let $2\le u<x^{1/4}$. Then (see (2.1))

$$\psi(u,\chi)=\sum_{n\le u}\Lambda(n)\chi(n)$$

and

$$|\psi(u,\chi)| \le \sum_{n\le u}\Lambda(n)=\psi(u)\le b_6u\le b_6x^{1/4}\le Cx\exp\biggl(-\frac{c(2)}{4}\sqrt{\ln x}\biggr)$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough. Hence,

$$\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi(u,\chi)| \le Cx\exp\biggl(-\frac{c(2)}{4}\sqrt{\ln x}\biggr).$$

Substituting this estimate into (4.17) and using the fact that $\#X_Q^*\le \#X_Q=\varphi(Q)$, we obtain

$$\begin{aligned}S_4' &\le Cx\exp\biggl(-\frac{c(2)}{4}\sqrt{\ln x}\biggr)\ln x = Cx\exp\biggl(-\frac{c(2)}{4}\sqrt{\ln x}+\ln\ln x\biggr)\\ &\le Cx\exp\biggl(-\frac{c(2)}{8}\sqrt{\ln x}\biggr)\end{aligned}\tag{4.18}$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough.

Substituting (4.16) and (4.18) into (4.14), we find

$$S_1' \le Cx\exp(-c\sqrt{\ln x}),\tag{4.19}$$

where $C>0$ and $c>0$ are absolute constants.

II. Now we estimate the quantity

$$S_3' = \sum_{\substack{\exp(c_1\sqrt{\ln x})<Q\le x^{1/2-\delta}\\ (Q,B)=1}}\frac{1}{\varphi(Q)}\sum_{\chi\in X_Q^*}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}|\psi'(u,\chi)|.$$

Let Q be an integer with $\exp(c_1\sqrt{\ln x})<Q\le x^{1/2-\delta}$ and $(Q,B)=1$, and let $\chi\in X_Q^*$. Since $Q>1$, we see that χ is a nonprincipal character modulo Q. Hence,

$$\psi'(u,\chi)=\psi(u,\chi).$$

We have

$$S_3' = \sum_{\substack{\exp(c_1\sqrt{\ln x})<Q\le x^{1/2-\delta}\\(Q,B)=1}} \frac{1}{\varphi(Q)} \sum_{\chi\in X_Q^*} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi(u,\chi)|$$
$$\le \sum_{\exp(c_1\sqrt{\ln x})<Q\le x^{1/2-\delta}} \frac{1}{\varphi(Q)} \sum_{\chi\in X_Q^*} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi(u,\chi)|.$$

Applying Lemma 4.8 with $Q_1=\exp(c_1\sqrt{\ln x})$, $Q_2=x^{1/2-\delta}$, and $t=x^{1+\gamma/\sqrt{\ln x}}$, we obtain

$$S_3' \le C\ln^4\bigl(x^{3/2-\delta+\gamma/\sqrt{\ln x}}\bigr)\Bigl(x\exp\bigl((\gamma-c_1)\sqrt{\ln x}\bigr)+x^{(5/6)(1+\gamma/\sqrt{\ln x})}\ln(x^{1/2-\delta})+x^{1-\delta+\gamma/(2\sqrt{\ln x})}\Bigr).$$

We can assume that

$$\frac{\gamma}{\sqrt{\ln x}}\le\delta \qquad\text{and}\qquad \frac{5}{6}\biggl(1+\frac{\gamma}{\sqrt{\ln x}}\biggr)\le\frac{9}{10}$$

if $c(\varepsilon,\delta,\gamma)$ is chosen large enough. Increasing C if necessary, we have

$$S_3' \le C\ln^4 x\,\bigl(x\exp\bigl((\gamma-c_1)\sqrt{\ln x}\bigr)+x^{9/10}\ln x+x^{1-\delta/2}\bigr).$$

Then

$$(\gamma-c_1)\sqrt{\ln x}\le-\frac{c_1}{2}\sqrt{\ln x}$$

provided that $0<\gamma\le c_1/2$. We obtain

$$x\exp\bigl((\gamma-c_1)\sqrt{\ln x}\bigr)\ln^4 x\le x\exp\Bigl(-\frac{c_1}{2}\sqrt{\ln x}+4\ln\ln x\Bigr)\le x\exp\Bigl(-\frac{c_1}{4}\sqrt{\ln x}\Bigr),$$
$$x^{9/10}\ln^5 x\le x\exp\Bigl(-\frac{c_1}{4}\sqrt{\ln x}\Bigr),\qquad x^{1-\delta/2}\ln^4 x\le x\exp\Bigl(-\frac{c_1}{4}\sqrt{\ln x}\Bigr)$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough. Redenoting $3C$ by C and $c_1/4$ by c, we arrive at

$$S_3'\le Cx\exp(-c\sqrt{\ln x}), \tag{4.20}$$

where $C>0$ and $c>0$ are absolute constants.

III. Now we estimate the quantity

$$S_2' = \sum_{\substack{\ln x<Q\le\exp(c_1\sqrt{\ln x})\\(Q,B)=1}} \frac{1}{\varphi(Q)} \sum_{\chi\in X_Q^*} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi'(u,\chi)|.$$

Let Q be an integer with $\ln x<Q\le\exp(c_1\sqrt{\ln x})$ and $(Q,B)=1$, and let $\chi\in X_Q^*$. Since $Q>1$, we see that χ is a nonprincipal character modulo Q, and hence $\psi'(u,\chi)=\psi(u,\chi)$. We recall that if an exceptional modulus q_0 in the interval $[3,\exp(2c_1\sqrt{\ln x})]$ does not exist, then $B=1$; if q_0 exists, then $B\ge3$ is a prime divisor of q_0, and so $Q\ne q_0$. Since $0<\gamma\le\gamma_0$ and $c(\varepsilon,\delta,\gamma)\ge c_0$, we see from Lemma 4.6 that

$$\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} |\psi(u,\chi)|\le Cx\exp(-3c_1\sqrt{\ln x}).$$

Since $\#X_Q^*\le\#X_Q=\varphi(Q)$, we obtain

$$S_2'\le\sum_{\substack{\ln x<Q\le\exp(c_1\sqrt{\ln x})\\(Q,B)=1}} Cx\exp(-3c_1\sqrt{\ln x})\le Cx\exp(-3c_1\sqrt{\ln x})\exp(c_1\sqrt{\ln x})$$
$$=Cx\exp(-2c_1\sqrt{\ln x}). \tag{4.21}$$

From (4.19)–(4.21) we find

$$S_1' + S_2' + S_3' \le \widetilde{C} x \exp(-\widetilde{c}\sqrt{\ln x}), \tag{4.22}$$

where $\widetilde{C} > 0$ and $\widetilde{c} > 0$ are absolute constants. Substituting (4.22) into (4.13), we obtain

$$S' \le C' x \exp\bigl(-\widetilde{c}\sqrt{\ln x} + \ln\ln x\bigr) \le C' x \exp\Bigl(-\frac{\widetilde{c}}{2}\sqrt{\ln x}\Bigr)$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough. Redenoting C' by C and $\widetilde{c}/2$ by c, we arrive at

$$S' \le C x \exp(-c\sqrt{\ln x}), \tag{4.23}$$

where $C > 0$ and $c > 0$ are absolute constants.

IV. We have

$$x^{1/2-\delta} \ln^2 x \le x^{1/2} \ln^2 x \le x \exp(-c\sqrt{\ln x}) \tag{4.24}$$

provided that $c(\varepsilon,\delta,\gamma)$ is chosen large enough (here $c > 0$ is the absolute constant in (4.23)).

V. Now we estimate the quantity

$$A_1 = \max_{2 \le u \le x^{1+\gamma/\sqrt{\ln x}}} \max_{W \in \mathbb{Z}} |\psi(u;1,W) - u|.$$

Let $W \in \mathbb{Z}$. We have

$$\psi(u;1,W) = \sum_{n \le u,\ n \equiv W \pmod 1} \Lambda(n) = \sum_{n \le u} \Lambda(n) = \psi(u).$$

Hence,

$$A_1 = \max_{2 \le u \le x^{1+\gamma/\sqrt{\ln x}}} |\psi(u) - u|.$$

Using (4.15) and arguing as in cases I(1), (i) and I(1), (ii), we obtain

$$A_1 \le C x \exp(-c\sqrt{\ln x}), \tag{4.25}$$

where $C > 0$ and $c > 0$ are absolute constants.

Substituting (4.23)–(4.25) into (4.12), we find

$$S \le C x \exp(-c\sqrt{\ln x}),$$

where $C > 0$ and $c > 0$ are absolute constants. Thus, if γ is a sufficiently small positive absolute constant, $x \ge c(\varepsilon,\delta,\gamma)$ is a real number, and q is an integer such that $1 \le q \le (\ln x)^{1-\varepsilon}$, then there is an integer B such that

$$1 \le B \le \exp(2c_1\sqrt{\ln x}), \qquad 1 \le \frac{B}{\varphi(B)} \le 2, \qquad (B,q) = 1$$

and

$$\sum_{\substack{1 \le Q \le x^{1/2-\delta} \\ (Q,B)=1}} \max_{2 \le u \le x^{1+\gamma/\sqrt{\ln x}}} \max_{W \in \mathbb{Z}\colon (W,Q)=1} \left|\psi(u;Q,W) - \frac{u}{\varphi(Q)}\right| \le C x \exp(-c\sqrt{\ln x}),$$

where c_1, C, and c are positive absolute constants. Let us redenote $2c_1$ by c_1, C by c_2, and c by c_3. Since γ is an absolute constant, we see that the positive number $c(\varepsilon,\delta,\gamma) = c(\varepsilon,\delta)$ depends only on ε and δ. Lemma 4.9 is proved. ☐

Lemma 4.10. *Let ε and δ be real numbers such that $0<\varepsilon<1$ and $0<\delta<1/2$. Then there is a number $c(\varepsilon,\delta)>0$, depending only on ε and δ, such that if $x\in\mathbb{R}$ and $q\in\mathbb{Z}$ satisfy the conditions $x\ge c(\varepsilon,\delta)$ and $1\le q\le(\ln x)^{1-\varepsilon}$, then there is a positive integer B for which the following relations hold*:

$$1\le B\le\exp(c_1\sqrt{\ln x}),\qquad 1\le\frac{B}{\varphi(B)}\le 2,\qquad (B,q)=1$$

and

$$\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{W\in\mathbb{Z}\colon (W,Q)=1}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\le c_2x\exp(-c_3\sqrt{\ln x}).$$

Here c_1, γ, c_2, and c_3 are positive absolute constants.

Proof. We will choose the number $\widetilde{c}(\varepsilon,\delta)$ later; it is assumed to be large enough. Let $\widetilde{c}(\varepsilon,\delta)\ge c(\varepsilon,\delta)$, where $c(\varepsilon,\delta)$ is the number in Lemma 4.9. Let $x\in\mathbb{R}$ and $q\in\mathbb{Z}$ be such that $x\ge\widetilde{c}(\varepsilon,\delta)$ and $1\le q\le(\ln x)^{1-\varepsilon}$. Then, by Lemma 4.9, there is a positive integer B such that

$$1\le B\le\exp(c_1\sqrt{\ln x}),\qquad 1\le\frac{B}{\varphi(B)}\le 2,\qquad (B,q)=1 \tag{4.26}$$

and

$$\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{W\in\mathbb{Z}\colon (W,Q)=1}|R(u;Q,W)|\le c_2x\exp(-c_3\sqrt{\ln x}), \tag{4.27}$$

where

$$R(u;Q,W):=\psi(u;Q,W)-\frac{u}{\varphi(Q)}$$

and c_1, γ, c_2, and c_3 are positive absolute constants.

We put

$$R_1(u;Q,W):=\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}. \tag{4.28}$$

Let $Q\in\mathbb{Z}$, $W\in\mathbb{Z}$, and $u\in\mathbb{Z}$ be such that $1\le Q\le x^{1/2-\delta}$, $(Q,B)=1$, $(W,Q)=1$, and $3\le u\le x^{1+\gamma/\sqrt{\ln x}}$. We claim that

$$|R_1(u;Q,W)|\le C_1u^{1/2}+|R(u;Q,W)|+\sum_{2\le n\le u-1}\frac{|R(n;Q,W)|}{n\ln^2 n}, \tag{4.29}$$

where $C_1>0$ is an absolute constant. We define

$$\alpha(n)=\begin{cases}1 & \text{if } n\equiv W \pmod{Q},\\ 0 & \text{otherwise}\end{cases}\qquad\text{and}\qquad \pi_1(u;Q,W)=\sum_{n\le u}\frac{\Lambda(n)\alpha(n)}{\ln n}.$$

Let us show that

$$\pi(u;Q,W)=\pi_1(u;Q,W)+\widetilde{R}(u;Q,W),\qquad |\widetilde{R}(u;Q,W)|\le Cu^{1/2}, \tag{4.30}$$

where $C>0$ is an absolute constant. Let $u\ge 8$. Then

$$\pi_1(u;Q,W)=\sum_{p^m\le u}\frac{\alpha(p^m)\ln p}{m\ln p}=\sum_{1\le m\le\ln u/\ln 2}\ \sum_{p\le u^{1/m}}\frac{\alpha(p^m)}{m}$$

$$=\sum_{p\le u}\alpha(p)+\sum_{2\le m\le\ln u/\ln 2}\frac{1}{m}\sum_{p\le u^{1/m}}\alpha(p^m)=S_1+S_2.$$

We have

$$S_1 = \sum_{p\le u,\ p\equiv W \pmod Q} 1 = \pi(u;Q,W)$$

and

$$S_2 \le \sum_{2\le m\le \ln u/\ln 2} \frac{u^{1/m}}{m} = \frac{1}{2}u^{1/2} + \sum_{3\le m\le \ln u/\ln 2} \frac{u^{1/m}}{m} \le \frac{1}{2}u^{1/2} + \frac{1}{3}u^{1/3}\frac{\ln u}{\ln 2}$$
$$\le u^{1/2} + u^{1/3}\ln u \le C' u^{1/2},$$

where $C' > 0$ is an absolute constant. If $3 \le u < 8$, then

$$\biggl|\sum_{2\le m\le \ln u/\ln 2} \frac{1}{m} \sum_{p\le u^{1/m}} \alpha(p^m)\biggr| \le \frac{1}{2}\sum_{p\le 8^{1/2}} 1 + \frac{1}{3}\sum_{p\le 8^{1/3}} 1 = C'' \le C'' u^{1/2}.$$

Thus, (4.30) is proved.

Since

$$\psi(x;Q,W) = \sum_{m\le x} \Lambda(m)\alpha(m),$$

we have

$$\pi_1(u;Q,W) = \sum_{2\le n\le u} \frac{\psi(n;Q,W) - \psi(n-1;Q,W)}{\ln n}$$
$$= \sum_{2\le n\le u-1} \psi(n;Q,W)\biggl(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\biggr) + \frac{\psi(u;Q,W)}{\ln u}$$
$$= \sum_{2\le n\le u-1} \biggl(\frac{n}{\varphi(Q)} + R(n;Q,W)\biggr)\biggl(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\biggr) + \frac{u}{\varphi(Q)\ln u} + \frac{R(u;Q,W)}{\ln u}.$$

Further,

$$\sum_{2\le n\le u-1} \frac{n}{\varphi(Q)}\biggl(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\biggr) = \sum_{2\le n\le u-1} \frac{n}{\varphi(Q)} \int_n^{n+1} \frac{dt}{t\ln^2 t} = \frac{1}{\varphi(Q)} \sum_{2\le n\le u-1} \int_n^{n+1} \frac{t-\{t\}}{t\ln^2 t}\,dt$$
$$= \frac{1}{\varphi(Q)}\Biggl(\int_2^u \frac{dt}{\ln^2 t} - \int_2^u \frac{\{t\}\,dt}{t\ln^2 t}\Biggr).$$

Since

$$\int_2^u \frac{dt}{\ln^2 t} = \int_2^u t\,d\biggl(-\frac{1}{\ln t}\biggr) = -\frac{t}{\ln t}\biggr|_2^u + \int_2^u \frac{dt}{\ln t} = -\frac{u}{\ln u} + \frac{2}{\ln 2} + \operatorname{li}(u),$$

we obtain

$$\pi_1(u;Q,W) = \frac{u}{\varphi(Q)\ln u} + \frac{R(u;Q,W)}{\ln u} - \frac{u}{\varphi(Q)\ln u} + \frac{2}{\varphi(Q)\ln 2} + \frac{\operatorname{li}(u)}{\varphi(Q)}$$
$$- \frac{1}{\varphi(Q)}\int_2^u \frac{\{t\}}{t\ln^2 t}\,dt + \sum_{2\le n\le u-1} R(n;Q,W)\biggl(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\biggr).$$

We have (see (4.30))

$$\pi(u;Q,W)=\frac{\mathrm{li}(u)}{\varphi(Q)}+R_1(u;Q,W),$$

where

$$R_1(u;Q,W)=\frac{2}{\varphi(Q)\ln 2}-\frac{1}{\varphi(Q)}\int\limits_2^u\frac{\{t\}}{t\ln^2 t}\,dt+\widetilde{R}(u;Q,W)+\frac{R(u;Q,W)}{\ln u}$$
$$+\sum_{2\le n\le u-1}R(n;Q,W)\biggl(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\biggr).$$

We can estimate this quantity as

$$|R_1(u;Q,W)|\le\frac{2}{\ln 2}+\left|\int\limits_2^u\frac{\{t\}}{t\ln^2 t}\,dt\right|+|\widetilde{R}(u;Q,W)|+\frac{|R(u;Q,W)|}{\ln u}$$
$$+\sum_{2\le n\le u-1}|R(n;Q,W)|\biggl(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\biggr). \tag{4.31}$$

Since $u\ge 3$, we have

$$\frac{|R(u;Q,W)|}{\ln u}\le|R(u;Q,W)|. \tag{4.32}$$

Since

$$\left|\int\limits_2^u\frac{\{t\}}{t\ln^2 t}\,dt\right|\le\int\limits_2^u\frac{dt}{t\ln^2 t}=-\frac{1}{\ln t}\bigg|_2^u=\frac{1}{\ln 2}-\frac{1}{\ln u}\le\frac{1}{\ln 2},$$

it follows (see (4.30)) that

$$\frac{2}{\ln 2}+\left|\int\limits_2^u\frac{\{t\}}{t\ln^2 t}\,dt\right|+|\widetilde{R}(u;Q,W)|\le\frac{3}{\ln 2}+Cu^{1/2}\le\biggl(C+\frac{3}{\ln 2}\biggr)u^{1/2}. \tag{4.33}$$

Let $f(x)=-\ln^{-1}x$ and $n\ge 2$ be an integer. By the mean value theorem, there is a $\xi\in(n,n+1)$ such that

$$\frac{1}{\ln n}-\frac{1}{\ln(n+1)}=f(n+1)-f(n)=f'(\xi)=\frac{1}{\xi\ln^2\xi}\le\frac{1}{n\ln^2 n}. \tag{4.34}$$

Substituting (4.32)–(4.34) into (4.31), we obtain (4.29). Hence,

$$\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}|R_1(u;Q,W)|$$
$$\le\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}|R(u;Q,W)|+\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\bigl|C_1u^{1/2}\bigr|$$
$$+\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\sum_{2\le n\le u-1}\frac{|R(n;Q,W)|}{n\ln^2 n}$$
$$=S_1+S_2+S_3. \tag{4.35}$$

I. Now we estimate S_1. We have (see (4.27))

$$S_1 \le \sum_{\substack{1\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{W\in\mathbb{Z}\\ (W,Q)=1}} |R(u;Q,W)| \le c_2 x\exp(-c_3\sqrt{\ln x}). \qquad (4.36)$$

II. Let us estimate S_2. We can assume that

$$\frac{\gamma}{\sqrt{\ln x}} \le \delta$$

provided that $\widetilde{c}(\varepsilon,\delta)$ is chosen large enough. We have

$$S_2 \le C_1 x^{1-\delta+\gamma/(2\sqrt{\ln x})} \le C_1 x^{1-\delta/2} \le x\exp(-c_3\sqrt{\ln x}) \qquad (4.37)$$

provided that $\widetilde{c}(\varepsilon,\delta)$ is chosen large enough.

III. Now we estimate S_3. Let Q, W, u, and n be integers such that $1\le Q\le x^{1/2-\delta}$, $(Q,B)=1$, $(W,Q)=1$, $3\le u\le x^{1+\gamma/\sqrt{\ln x}}$, and $2\le n\le u-1$. Then

$$|R(n;Q,W)| \le \max_{2\le m\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{V\in\mathbb{Z}\\ (V,Q)=1}} |R(m;Q,V)|.$$

Hence,

$$\sum_{2\le n\le u-1} \frac{|R(n;Q,W)|}{n\ln^2 n} \le \max_{2\le m\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{V\in\mathbb{Z}\\ (V,Q)=1}} |R(m;Q,V)| \sum_{2\le n\le u-1} \frac{1}{n\ln^2 n}$$

$$\le c_0 \max_{2\le m\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{V\in\mathbb{Z}\\ (V,Q)=1}} |R(m;Q,V)|, \qquad \text{where} \quad c_0 := \sum_{n=2}^{\infty} \frac{1}{n\ln^2 n} < +\infty.$$

We have

$$\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\ u\in\mathbb{Z}}} \max_{\substack{W\in\mathbb{Z}\\ (W,Q)=1}} \sum_{2\le n\le u-1} \frac{|R(n;Q,W)|}{n\ln^2 n} \le c_0 \max_{2\le m\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{V\in\mathbb{Z}\\ (V,Q)=1}} |R(m;Q,V)|.$$

Therefore (see (4.27)),

$$S_3 \le c_0 \sum_{\substack{1\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} \max_{2\le m\le x^{1+\gamma/\sqrt{\ln x}}} \max_{\substack{V\in\mathbb{Z}\\ (V,Q)=1}} |R(m;Q,V)| \le c_0c_2 x\exp(-c_3\sqrt{\ln x}). \qquad (4.38)$$

Substituting (4.36)–(4.38) into (4.35), we obtain (see (4.28))

$$\sum_{\substack{1\le Q\le x^{1/2-\delta}\\ (Q,B)=1}} \max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\ u\in\mathbb{Z}}} \max_{\substack{W\in\mathbb{Z}\\ (W,Q)=1}} \left|\pi(u;Q,W) - \frac{\mathrm{li}(u)}{\varphi(Q)}\right| \le c_4 x\exp(-c_3\sqrt{\ln x}), \qquad (4.39)$$

where $c_4 = c_2 + 1 + c_0c_2 > 0$ is an absolute constant.

Let Q and W be integers such that $1\le Q\le x^{1/2-\delta}$, $(Q,B)=1$, and $(W,Q)=1$, and let u be a real number with $2\le u\le x^{1+\gamma/\sqrt{\ln x}}$. Consider two cases.

(1) Let $2\le u\le 3$. Then

$$|\pi(u;Q,W)| \le \pi(u) \le 2, \qquad \left|\frac{\mathrm{li}(u)}{\varphi(Q)}\right| \le \mathrm{li}(u) \le \mathrm{li}(3),$$

and so

$$\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\le|\pi(u;Q,W)|+\left|\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\le 2+\mathrm{li}(3). \tag{4.40}$$

(2) Let $3<u\le x^{1+\gamma/\sqrt{\ln x}}$. Then

$$\left|\frac{\mathrm{li}(u)-\mathrm{li}([u])}{\varphi(Q)}\right|\le\int\limits_{[u]}^{[u]+1}\frac{dt}{\ln t}\le\int\limits_{2}^{3}\frac{dt}{\ln t}=\mathrm{li}(3).$$

Hence,

$$\begin{aligned}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|&=\left|\pi([u];Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}-\frac{\mathrm{li}([u])}{\varphi(Q)}+\frac{\mathrm{li}([u])}{\varphi(Q)}\right|\\&\le\left|\pi([u];Q,W)-\frac{\mathrm{li}([u])}{\varphi(Q)}\right|+\left|\frac{\mathrm{li}(u)-\mathrm{li}([u])}{\varphi(Q)}\right|\le\mathrm{li}(3)+\left|\pi([u];Q,W)-\frac{\mathrm{li}([u])}{\varphi(Q)}\right|. \end{aligned}\tag{4.41}$$

From (4.40) and (4.41) we obtain

$$\begin{aligned}\max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\le\max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\ \max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\\+2\,\mathrm{li}(3)+2.\end{aligned}\tag{4.42}$$

We can assume that

$$x^{1/2}\le x\exp(-c_3\sqrt{\ln x}) \tag{4.43}$$

provided that $\widetilde{c}(\varepsilon,\delta)$ is chosen large enough. From (4.39), (4.42), and (4.43) we obtain

$$\begin{aligned}&\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\ \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\\&\qquad\le\sum_{\substack{1\le Q\le x^{1/2-\delta}\\(Q,B)=1}}\ \max_{\substack{3\le u\le x^{1+\gamma/\sqrt{\ln x}}\\u\in\mathbb{Z}}}\ \max_{\substack{W\in\mathbb{Z}\\(W,Q)=1}}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|+(2\,\mathrm{li}(3)+2)x^{1/2}\\&\qquad\le\big(c_4+2\,\mathrm{li}(3)+2\big)x\exp(-c_3\sqrt{\ln x}).\end{aligned}\tag{4.44}$$

Thus, if $x\ge\widetilde{c}(\varepsilon,\delta)$ is a real number and q is an integer such that $1\le q\le(\ln x)^{1-\varepsilon}$, then there is a positive integer B for which (4.26) and (4.44) hold. Let us redenote $\widetilde{c}(\varepsilon,\delta)$ by $c(\varepsilon,\delta)$ and $c_4+2\,\mathrm{li}(3)+2$ by c_2. Lemma 4.10 is proved. ☐

5. PROOF OF THEOREM 1.1 AND COROLLARY 1.1

Let us introduce some additional notation. Let $\mathcal{A}$ be a set of integers, $\mathcal{P}$ a set of primes, and $L(n)=l_1n+l_2$ a linear function with integer coefficients. We define

$$\mathcal{A}(x)=\big\{n\in\mathcal{A}\colon\ x\le n<2x\big\},\qquad\mathcal{A}(x;q,a)=\big\{n\in\mathcal{A}(x)\colon\ n\equiv a\pmod q\big\},$$

$$L(\mathcal{A})=\{L(n)\colon\ n\in\mathcal{A}\},\qquad\mathcal{P}_{L,\mathcal{A}}(x)=L(\mathcal{A}(x))\cap\mathcal{P},\qquad\mathcal{P}_{L,\mathcal{A}}(x;q,a)=L(\mathcal{A}(x;q,a))\cap\mathcal{P},$$

$$\varphi_L(q)=\frac{\varphi(|l_1|q)}{\varphi(|l_1|)}.$$

Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be a set of distinct linear functions $L_i(n)=a_in+b_i$, $i=1,\dots,k$, with positive integer coefficients. We say such a set is *admissible* if for every prime p there is an integer n_p such that $\big(\prod_{i=1}^k L_i(n_p),p\big)=1$.

We focus on sets satisfying the following hypothesis, which is given in terms of $(\mathcal{A},\mathcal{L},\mathcal{P},B,x,\theta)$, where $\mathcal{L}$ is an admissible set of linear functions, $B\in\mathbb{N}$, x is a large real number, and $0<\theta<1$.

Hypothesis 1. For $(\mathcal{A},\mathcal{L},\mathcal{P},B,x,\theta)$ and $k=\#\mathcal{L}$, the following holds.

(1) $\mathcal{A}$ is well distributed in arithmetic progressions:

$$\sum_{1\le q\le x^\theta}\max_{a\in\mathbb{Z}}\left|\#\mathcal{A}(x;q,a)-\frac{\#\mathcal{A}(x)}{q}\right|\ll\frac{\#\mathcal{A}(x)}{(\ln x)^{100k^2}}.$$

(2) The primes in $L(\mathcal{A})\cap\mathcal{P}$ are well distributed in most arithmetic progressions: for any $L\in\mathcal{L}$ we have

$$\sum_{\substack{1\le q\le x^\theta\\(q,B)=1}}\max_{\substack{a\in\mathbb{Z}\\(L(a),q)=1}}\left|\#\mathcal{P}_{L,\mathcal{A}}(x;q,a)-\frac{\#\mathcal{P}_{L,\mathcal{A}}(x)}{\varphi_L(q)}\right|\ll\frac{\#\mathcal{P}_{L,\mathcal{A}}(x)}{(\ln x)^{100k^2}}.$$

(3) $\mathcal{A}$ is not too concentrated in any arithmetic progression: for any $1\le q<x^\theta$ we have

$$\max_{a\in\mathbb{Z}}\#\mathcal{A}(x;q,a)\ll\frac{\#\mathcal{A}(x)}{q}.$$

Maynard proved the following result (see [5, Proposition 6.1]).

Proposition 5.1. *Let α and θ be real numbers such that $\alpha>0$ and $0<\theta<1$. Let $\mathcal{A}$ be a set of integers, $\mathcal{P}$ a set of primes, and $\mathcal{L}=\{L_1,\dots,L_k\}$ an admissible set of k linear functions, and let B and x be integers. Let the coefficients of $L_i(n)=a_in+b_i\in\mathcal{L}$ satisfy $1\le a_i,b_i\le x^\alpha$ for all $1\le i\le k$, and let $k\le(\ln x)^{1/5}$ and $1\le B\le x^\alpha$. Let $x^{\theta/10}\le R\le x^{\theta/3}$. Let ρ and ξ satisfy $k(\ln\ln x)^2/\ln x\le\rho,\xi\le\theta/10$, and define*

$$\mathcal{S}(\xi;D)=\big\{n\in\mathbb{N}\colon\ p\mid n\ \Rightarrow\ (p>x^\xi\ \text{or}\ p\mid D)\big\}.$$

Then there is a number $C>0$ depending only on α and θ such that the following holds. If $k\ge C$ and $(\mathcal{A},\mathcal{L},\mathcal{P},B,x,\theta)$ satisfy Hypothesis 1, then there exist nonnegative weights $w_n=w_n(\mathcal{L})$ satisfying

$$w_n\ll(\ln R)^{2k}\prod_{i=1}^k\prod_{p\mid L_i(n),\ p\nmid B}4 \tag{5.1}$$

such that the following statements hold.

(1) *We have*

$$\sum_{n\in\mathcal{A}(x)}w_n=\left(1+O\left(\frac{1}{(\ln x)^{1/10}}\right)\right)\frac{B^k}{\varphi(B)^k}\,\mathfrak{S}_B(\mathcal{L})\,\#\mathcal{A}(x)(\ln R)^kI_k. \tag{5.2}$$

(2) *For $L(n)=a_Ln+b_L\in\mathcal{L}$ we have*

$$\sum_{n\in\mathcal{A}(x)}\mathbf{1}_{\mathcal{P}}(L(n))w_n\ge\left(1+O\left(\frac{1}{(\ln x)^{1/10}}\right)\right)\frac{B^{k-1}}{\varphi(B)^{k-1}}\,\mathfrak{S}_B(\mathcal{L})\frac{\varphi(a_L)}{a_L}\,\#\mathcal{P}_{L,\mathcal{A}}(x)(\ln R)^{k+1}J_k$$
$$+O\left(\frac{B^k}{\varphi(B)^k}\,\mathfrak{S}_B(\mathcal{L})\,\#\mathcal{A}(x)(\ln R)^{k-1}I_k\right). \tag{5.3}$$

(3) *For* $L(n)=a_0n+b_0\notin\mathcal{L}$ *and* $D\le x^\alpha$, *if* $\Delta_L\ne 0$, *we have*

$$\sum_{n\in\mathcal{A}(x)}\mathbf{1}_{\mathcal{S}(\xi;D)}(L(n))w_n\ll \xi^{-1}\frac{\Delta_L}{\varphi(\Delta_L)}\frac{D}{\varphi(D)}\frac{B^k}{\varphi(B)^k}\,\mathfrak{S}_B(\mathcal{L})\,\#\mathcal{A}(x)(\ln R)^{k-1}I_k, \tag{5.4}$$

where

$$\Delta_L=|a_0|\prod_{i=1}^{k}|a_0b_i-b_0a_i|.$$

(4) *For* $L\in\mathcal{L}$ *we have*

$$\sum_{n\in\mathcal{A}(x)}w_n\sum_{p\mid L(n),\,p<x^\rho,\,p\nmid B}1\ll \rho^2k^4(\ln k)^2\frac{B^k}{\varphi(B)^k}\,\mathfrak{S}_B(\mathcal{L})\,\#\mathcal{A}(x)(\ln R)^kI_k. \tag{5.5}$$

Here I_k *and* J_k *are quantities depending only on* k, *and* $\mathfrak{S}_B(\mathcal{L})$ *is a quantity depending only on* $\mathcal{L}$, *and these satisfy*

$$\mathfrak{S}_B(\mathcal{L})=\prod_{p\nmid B}\biggl(1-\frac{\#\{1\le n\le p\colon\ p\mid \prod_{i=1}^k L_i(n)\}}{p}\biggr)\biggl(1-\frac{1}{p}\biggr)^{-k}\ge \exp(-ck), \tag{5.6}$$

$$I_k=\int_0^\infty\dots\int_0^\infty F^2(t_1,\dots,t_k)\,dt_1\dots dt_k\gg(2k\ln k)^{-k}, \tag{5.7}$$

$$J_k=\int_0^\infty\dots\int_0^\infty\biggl(\int_0^\infty F(t_1,\dots,t_k)\,dt_k\biggr)^2dt_1\dots dt_{k-1}\gg\frac{\ln k}{k}I_k \tag{5.8}$$

for a smooth function $F=F_k\colon\mathbb{R}^k\to\mathbb{R}$ *depending only on* k. *The implied constants here depend only on* α, θ, *and the implied constants from Hypothesis* 1. *The constant* c *in inequality* (5.6) *is positive and absolute.*

Proof of Theorem 1.1. First we prove the following

Lemma 5.1. *Let* k *be a positive integer. Let* a, q, *and* $b_1,\dots,b_k$ *be positive integers such that* $b_1<\dots<b_k$ *and* $(a,q)=1$. *Let* $L_i(n)=qn+a+qb_i$, $i=1,\dots,k$. *Then* $\mathcal{L}=\{L_1,\dots,L_k\}$ *is an admissible set if and only if for any prime* p *such that* $p\nmid q$ *there is an integer* m_p *with* $m_p\not\equiv b_i\pmod p$ *for all* $1\le i\le k$.

Proof. (1) Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be an admissible set. Let p be a prime such that $p\nmid q$. Since $\mathcal{L}$ is an admissible set, there is an integer n_p such that $\bigl(\prod_{i=1}^k L_i(n_p),p\bigr)=1$. Since $(q,p)=1$, there is an integer q' such that $qq'\equiv 1\pmod p$. We put $m_p=-(n_p+q'a)$. Let i be an integer with $1\le i\le k$. Since $(q',p)=1$ and $(L_i(n_p),p)=1$, it follows that $(q'L_i(n_p),p)=1$. We have

$$q'L_i(n_p)\equiv -m_p+b_i\pmod p.$$

Hence, $m_p\not\equiv b_i\pmod p$.

(2) Suppose that for any prime p with $p\nmid q$ there is an integer m_p such that $m_p\not\equiv b_i\pmod p$ for all $1\le i\le k$. Let us show that then $\mathcal{L}$ is an admissible set. First we observe that $\mathcal{L}=\{L_1,\dots,L_k\}$ is a set of distinct linear functions $L_i(n)=qn+l_i$, $i=1,\dots,k$, with positive integer coefficients. Thus, we need to prove that for any prime p there is an integer n_p such that $\bigl(\prod_{i=1}^k L_i(n_p),p\bigr)=1$. Let p be a prime number. Consider two cases.

(i) Let $p \mid q$. Since $(a,q)=1$, we have $(a,p)=1$. Let i be an integer with $1\le i\le k$. For any integer n we have

$$L_i(n)\equiv a \pmod p,$$

and so $L_i(n)\not\equiv 0 \pmod p$. Hence, $\bigl(\prod_{i=1}^k L_i(n),p\bigr)=1$. Therefore, in this case we may take any integer as n_p.

(ii) Let $p\nmid q$. Then $(q,p)=1$, and so there is an integer c such that

$$qc\equiv a \pmod p. \tag{5.9}$$

By assumption, there is an integer m_p such that $m_p\not\equiv b_i \pmod p$ for all $1\le i\le k$. We put $n_p=-m_p-c$. Let i be an integer with $1\le i\le k$. We have

$$n_p+c+b_i\not\equiv 0 \pmod p.$$

Since $(q,p)=1$, we obtain

$$qn_p+qc+qb_i\not\equiv 0 \pmod p.$$

In view of (5.9) this yields $L_i(n_p)\not\equiv 0 \pmod p$. Hence, $(L_i(n_p),p)=1$. Since this holds for all $1\le i\le k$, we have $\bigl(\prod_{i=1}^k L_i(n_p),p\bigr)=1$. Lemma 5.1 is proved. □

The proof of the following lemma is based on Maynard's ideas used in the proof of Lemma 8.1 in [5] (the notation $L\in\mathcal{L}$ was explained in the Introduction).

Lemma 5.2. *There are positive absolute constants c and C such that the following holds. Let x and η be real numbers with $x\ge c$ and $(\ln x)^{-9/10}\le\eta\le 1$. Let k and a be positive integers. Let $b_1,\dots,b_k$ be integers with $1\le b_i\le \ln x$, $i=1,\dots,k$. Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be a set of k linear functions $L_i(n)=an+b_i$, $i=1,\dots,k$. For $L(n)=an+b$, $b\in\mathbb{Z}$, we define*

$$\Delta_L=a^{k+1}\prod_{i=1}^k|b_i-b|.$$

Then

$$\sum_{\substack{1\le b\le\eta\ln x\\ L=an+b\notin\mathcal{L}}}\frac{\Delta_L}{\varphi(\Delta_L)}\le C\eta\ln\ln(a+2)\ln(k+1)\ln x.$$

Proof. Consider two cases.

(1) Let $k>\ln\ln x$. We can assume that $\ln\ln x\ge 100$ provided that c is chosen large enough. Therefore, $k\ge 100$. Let b be an integer such that $1\le b\le\eta\ln x$ and $L=an+b\notin\mathcal{L}$. Then $\Delta_L\in\mathbb{N}$. Applying Lemma 2.8, we see that

$$\frac{\Delta_L}{\varphi(\Delta_L)}\le c_0\ln\ln(\Delta_L+2), \tag{5.10}$$

where $c_0>0$ is an absolute constant. Further,

$$\ln\Delta_L=(k+1)\ln a+\sum_{i=1}^k\ln|b_i-b|.$$

For any $1\le i\le k$ we have $|b_i-b|\le\ln x$. Hence,

$$\ln\Delta_L\le(k+1)\ln a+k\ln\ln x\le 2k\ln a+k^2.$$

Since

$$2k\ln a\le k^2\ln(a+2) \qquad\text{and}\qquad k^2\le k^2\ln(a+2),$$

we have

$$\ln \Delta_L \le 2k^2 \ln(a+2).$$

We observe that if $u \ge 2$ and $v \ge 2$ are real numbers, then

$$u + v \le uv. \tag{5.11}$$

Applying (5.11), we obtain

$$\ln(\Delta_L + 2) \le \ln(3\Delta_L) = \ln \Delta_L + \ln 3 \le 2k^2 \ln(a+2) + 3 \le 6k^2 \ln(a+2).$$

Applying (5.11) again, we have

$$\begin{aligned}\ln\ln(\Delta_L + 2) &\le \ln 6 + 2\ln k + \ln\ln(a+2) \le 2 + 2\ln k + 25\ln\ln(a+2)\\ &\le 4\ln k + 25\ln\ln(a+2) \le 100\ln k \ln\ln(a+2) \le 100\ln(k+1)\ln\ln(a+2).\end{aligned}$$

Substituting this estimate into (5.10), we obtain

$$\frac{\Delta_L}{\varphi(\Delta_L)} \le 100c_0 \ln\ln(a+2)\ln(k+1) = c_1 \ln\ln(a+2)\ln(k+1),$$

where $c_1 = 100c_0 > 0$ is an absolute constant. Thus,

$$\begin{aligned}\sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \frac{\Delta_L}{\varphi(\Delta_L)} &\le c_1 \ln\ln(a+2)\ln(k+1) \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} 1 \le c_1 \ln\ln(a+2)\ln(k+1)[\eta\ln x]\\ &\le c_1\eta \ln\ln(a+2)\ln(k+1)\ln x. \end{aligned} \tag{5.12}$$

(2) Let $1 \le k \le \ln\ln x$. For an integer b we define

$$\Delta(b) := \prod_{i=1}^{k} |b - b_i|.$$

Let b be an integer such that $1 \le b \le \eta\ln x$ and $L = an + b \notin \mathcal{L}$. Applying Lemmas 2.5 and 2.4, we obtain

$$\frac{\Delta_L}{\varphi(\Delta_L)} = \frac{a^{k+1}\Delta(b)}{\varphi(a^{k+1}\Delta(b))} \le \frac{a^{k+1}}{\varphi(a^{k+1})}\frac{\Delta(b)}{\varphi(\Delta(b))} = \frac{a}{\varphi(a)}\frac{\Delta(b)}{\varphi(\Delta(b))}.$$

Hence,

$$S = \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \frac{\Delta_L}{\varphi(\Delta_L)} \le \frac{a}{\varphi(a)} \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \frac{\Delta(b)}{\varphi(\Delta(b))} = \frac{a}{\varphi(a)}\widetilde{S}. \tag{5.13}$$

Applying Lemma 2.7, we have

$$\begin{aligned}\widetilde{S} &= \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \frac{\Delta(b)}{\varphi(\Delta(b))} = \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \sum_{d\mid\Delta(b)} \frac{\mu^2(d)}{\varphi(d)}\\ &= \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \sum_{\substack{1\le d\le \eta\ln x\\ d\mid\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} + \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \sum_{\substack{d>\eta\ln x\\ d\mid\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} = S_1 + S_2.\end{aligned} \tag{5.14}$$

First we estimate the sum S_2. Let b and d be positive integers such that $1 \le b \le \eta \ln x$, $L = an + b \notin \mathcal{L}$, $d > \eta \ln x$, and $d \mid \Delta(b)$. We claim that

$$\frac{\mu^2(d)}{\varphi(d)} \le \frac{\mu^2(d) \sum_{p|d} \ln p}{\varphi(d) \ln(\eta \ln x)}. \tag{5.15}$$

We can assume that

$$d > \eta \ln x \ge (\ln x)^{1/10} \ge 100$$

provided that c is chosen large enough. If $\mu^2(d) = 0$, then inequality (5.15) holds. Let $\mu^2(d) \ne 0$. Then d is square-free. Therefore, $\sum_{p|d} \ln p = \ln d$. Inequality (5.15) is equivalent to the inequality

$$\ln(\eta \ln x) \le \sum_{p|d} \ln p = \ln d,$$

which obviously holds. Thus, (5.15) is proved. We have

$$S_2 = \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \sum_{\substack{d > \eta \ln x \\ d|\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} \le \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \sum_{\substack{d > \eta \ln x \\ d|\Delta(b)}} \frac{\mu^2(d) \sum_{p|d} \ln p}{\varphi(d) \ln(\eta \ln x)}$$
$$= \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \sum_{p|\Delta(b)} \frac{\ln p}{\ln(\eta \ln x)} \sum_{\substack{d > \eta \ln x \\ p|d,\ d|\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)}.$$

Let $b \in \mathbb{N}$, $d \in \mathbb{N}$, and $p \in \mathbb{P}$ be such that $1 \le b \le \eta \ln x$, $L = an + b \notin \mathcal{L}$, $p \mid \Delta(b)$, $d > \eta \ln x$, d is a multiple of p, and $d \mid \Delta(b)$. Then $d = pt$, where $t \in \mathbb{N}$, $t > (\eta \ln x)/p$, and $t \mid \Delta(b)$. We have (see Lemmas 2.5 and 2.4)

$$\varphi(d) = \varphi(pt) \ge \varphi(p)\varphi(t) = (p-1)\varphi(t) \ge \frac{p}{2}\varphi(t).$$

Hence,

$$\frac{\mu^2(d)}{\varphi(d)} = \frac{\mu^2(pt)}{\varphi(pt)} \le \frac{2\mu^2(pt)}{p\varphi(t)} \le \frac{2\mu^2(t)}{p\varphi(t)}.$$

We obtain (see Lemma 2.7)

$$\sum_{\substack{d > \eta \ln x \\ p|d,\ d|\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} \le \frac{2}{p} \sum_{\substack{t > (\eta \ln x)/p \\ t|\Delta(b)}} \frac{\mu^2(t)}{\varphi(t)} \le \frac{2}{p} \sum_{t|\Delta(b)} \frac{\mu^2(t)}{\varphi(t)} = \frac{2}{p} \frac{\Delta(b)}{\varphi(\Delta(b))}.$$

Therefore,

$$S_2 \le \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \sum_{p|\Delta(b)} \frac{\ln p}{\ln(\eta \ln x)} \frac{2}{p} \frac{\Delta(b)}{\varphi(\Delta(b))} = \frac{2}{\ln(\eta \ln x)} \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \frac{\Delta(b)}{\varphi(\Delta(b))} \sum_{p|\Delta(b)} \frac{\ln p}{p}.$$

Since $\eta \ge (\ln x)^{-9/10}$, we have

$$\frac{2}{\ln(\eta \ln x)} \le \frac{2}{\ln((\ln x)^{1/10})} = \frac{20}{\ln \ln x}.$$

Thus,

$$S_2 \le \frac{20}{\ln \ln x} \sum_{\substack{1 \le b \le \eta \ln x \\ L = an+b \notin \mathcal{L}}} \frac{\Delta(b)}{\varphi(\Delta(b))} \sum_{p|\Delta(b)} \frac{\ln p}{p}. \tag{5.16}$$

Let b be an integer such that $1 \le b \le \eta \ln x$ and $L = an + b \notin \mathcal{L}$. Applying Lemmas 2.8 and 2.10, we obtain

$$\frac{\Delta(b)}{\varphi(\Delta(b))} \le c_2 \ln\ln(\Delta(b)+2) \le c_2 \ln\ln(3\Delta(b)), \qquad \sum_{p\mid\Delta(b)} \frac{\ln p}{p} \le c_3 \ln\ln(3\Delta(b)), \tag{5.17}$$

where $c_2 > 0$ and $c_3 > 0$ are absolute constants. We have

$$\ln \Delta(b) = \sum_{i=1}^{k} \ln|b_i - b| \le k \ln\ln x \le (\ln\ln x)^2. \tag{5.18}$$

Hence,

$$\ln\ln(3\Delta(b)) = \ln(\ln 3 + \ln \Delta(b)) \le \ln(\ln 3 + (\ln\ln x)^2) \le 3 \ln\ln\ln x \tag{5.19}$$

provided that c is chosen large enough. It follows from (5.17) and (5.19) that

$$\frac{\Delta(b)}{\varphi(\Delta(b))} \sum_{p\mid\Delta(b)} \frac{\ln p}{p} \le 9c_2c_3(\ln\ln\ln x)^2 = c_4(\ln\ln\ln x)^2,$$

where $c_4 = 9c_2c_3 > 0$ is an absolute constant. Substituting this estimate into (5.16), we obtain

$$S_2 \le \frac{20c_4(\ln\ln\ln x)^2}{\ln\ln x}\eta \ln x.$$

We can assume that

$$\frac{20c_4(\ln\ln\ln x)^2}{\ln\ln x} \le 1$$

provided that c is chosen large enough. Hence,

$$S_2 \le \eta \ln x \le \frac{1}{\ln 2}\ln(k+1)\eta\ln x \le 2\ln(k+1)\eta\ln x. \tag{5.20}$$

Now we estimate S_1. We have

$$S_1 = \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}}} \sum_{\substack{1\le d\le \eta\ln x\\ d\mid\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} = \sum_{1\le d\le\eta\ln x} \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}\\ d\mid\Delta(b)}} \frac{\mu^2(d)}{\varphi(d)} = \sum_{1\le d\le\eta\ln x} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}\\ d\mid\Delta(b)}} 1$$
$$= \sum_{1\le d\le\eta\ln x} \frac{\mu^2(d)}{\varphi(d)} N_0(d) = \sum_{\substack{1\le d\le\eta\ln x\\ d\in\mathcal{M}}} \frac{1}{\varphi(d)} N_0(d). \tag{5.21}$$

Let d be an integer such that $1 \le d \le \eta\ln x$ and $d \in \mathcal{M}$. We claim that

$$N_0(d) \le \frac{2\eta\ln x}{d} \prod_{p\mid d} \min\{p,k\}. \tag{5.22}$$

If $d = 1$, then the inequality is obvious. Let $d > 1$. We define

$$R(b) = (b-b_1)\dots(b-b_k).$$

Then $\Delta(b) = |R(b)|$. We have

$$N_0(d) = \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}\\ d\mid\Delta(b)}} 1 = \sum_{\substack{1\le b\le \eta\ln x\\ L=an+b\notin\mathcal{L}\\ R(b)\equiv 0 \pmod d}} 1.$$

Let d be expressed in the standard form as $d = q_1 \dots q_r$, where $q_1 < \dots < q_r$ are prime numbers. It is well known (see, for example, [7, Ch. 4]) that the congruence $R(b) \equiv 0 \pmod{d}$ is equivalent to the system of congruences

$$\begin{cases} R(b) \equiv 0 \pmod{q_1}, \\ \dots\dots\dots\dots\dots\dots \\ R(b) \equiv 0 \pmod{q_r}. \end{cases} \tag{5.23}$$

Let $1 \le j \le r$. Let Ω_j be the set of numbers of a complete system of residues modulo q_j satisfying the congruence $R(b) \equiv 0 \pmod{q_j}$. Since $R(b_1) = 0$, we see that $\Omega_j \ne \varnothing$. Since the leading coefficient of the polynomial $R(b)$ is 1 and the degree of the polynomial $R(b)$ is k, we have $\#\Omega_j \le k$ (see, for example, [7, Ch. 4]). It is also clear that $\#\Omega_j \le q_j$. Thus,

$$\#\Omega_j \le \min\{q_j, k\}.$$

System (5.23) is equivalent to the union of $T = \#\Omega_1 \dots \#\Omega_r$ systems

$$\begin{cases} b \equiv \tau_1 \pmod{q_1}, \\ \dots\dots\dots\dots\dots\dots \\ b \equiv \tau_r \pmod{q_r}, \end{cases} \tag{5.24}$$

where $\tau_1 \in \Omega_1, \dots, \tau_r \in \Omega_r$. It is well known (see, for example, [7, Ch. 4]) that the system of congruences (5.24) is equivalent to the congruence

$$b \equiv x_0 \pmod{d},$$

where $x_0 = x_0(\tau_1, \dots, \tau_r)$. It is also known that the numbers $x_0(\tau_1, \dots, \tau_r)$, $\tau_1 \in \Omega_1, \dots, \tau_r \in \Omega_r$, are incongruent modulo d. Thus,

$$\big\{b \in \mathbb{Z}\colon\ R(b) \equiv 0 \ (\mathrm{mod}\ d)\big\} = \bigsqcup_{\tau_1 \in \Omega_1, \dots, \tau_r \in \Omega_r} \big\{x_0(\tau_1, \dots, \tau_r) + dt\colon\ t \in \mathbb{Z}\big\}.$$

Let $\tau_1 \in \Omega_1, \dots, \tau_r \in \Omega_r$ and $x_0 = x_0(\tau_1, \dots, \tau_r)$. We have

$$\begin{aligned} \#\big\{t \in \mathbb{Z}\colon\ 1 \le x_0 + dt \le \eta \ln x\big\} &= \left[\frac{\eta \ln x - x_0}{d}\right] - \left\lceil \frac{1 - x_0}{d} \right\rceil + 1 \\ &= \frac{\eta \ln x - x_0}{d} - \theta_1 - \left(\frac{1 - x_0}{d} + \theta_2\right) + 1 = \frac{\eta \ln x}{d} + 1 - \theta_1 - \theta_2 - \frac{1}{d}, \end{aligned}$$

where θ_1 and θ_2 are real numbers with $0 \le \theta_1 < 1$ and $0 \le \theta_2 < 1$. Since $1 \le d \le \eta \ln x$, we obtain

$$\#\big\{t \in \mathbb{Z}\colon\ 1 \le x_0 + dt \le \eta \ln x\big\} \le \frac{\eta \ln x}{d} + 1 \le 2\frac{\eta \ln x}{d}.$$

Thus,

$$N_0(d) \le \frac{2\eta \ln x}{d} T \le \frac{2\eta \ln x}{d} \prod_{p \mid d} \min\{p, k\}.$$

Inequality (5.22) is proved.

Substituting (5.22) into (5.21), we obtain

$$S_1 \le \sum_{\substack{1 \le d \le \eta \ln x \\ d \in \mathcal{M}}} \frac{1}{\varphi(d)} \frac{2\eta \ln x}{d} \prod_{p \mid d} \min\{p, k\} = 2\eta \ln x \sum_{\substack{1 \le d \le \eta \ln x \\ d \in \mathcal{M}}} \frac{\prod_{p \mid d} \min\{p, k\}}{d \varphi(d)} = 2\eta \ln x\, S_3. \tag{5.25}$$

Let d be an integer such that $1 \le d \le \eta \ln x$ and $d \in \mathcal{M}$. We have (see Lemmas 2.6 and 2.4)

$$d = \prod_{p|d} p, \qquad \varphi(d) = \prod_{p|d} \varphi(p) = \prod_{p|d} (p-1),$$

$$\frac{\prod_{p|d} \min\{p,k\}}{d\varphi(d)} = \frac{\prod_{p|d} \min\{p,k\}}{\prod_{p|d} p(p-1)} = \prod_{p|d,\ p\le k} \frac{1}{p-1} \prod_{p|d,\ p>k} \frac{k}{p(p-1)}.$$

Hence,

$$S_3 = \sum_{\substack{1\le d\le \eta \ln x \\ d\in\mathcal{M}}} \prod_{p|d,\ p\le k} \frac{1}{p-1} \prod_{p|d,\ p>k} \frac{k}{p(p-1)} \le \prod_{p\le k}\left(1+\frac{1}{p-1}\right) \prod_{p>k}\left(1+\frac{k}{p(p-1)}\right) = AB. \quad (5.26)$$

We have (see Lemma 2.1)

$$A = \prod_{p\le k}\left(1+\frac{1}{p-1}\right) \le \prod_{p\le k+1}\left(1+\frac{1}{p-1}\right) = \prod_{p\le k+1}\left(1-\frac{1}{p}\right)^{-1} \le c_5 \ln(k+1), \quad (5.27)$$

where $c_5 > 0$ is an absolute constant.

Now we estimate B. Since $\ln(1+u) \le u$ and $u > 0$, we get

$$\ln B = \sum_{p>k} \ln\left(1+\frac{k}{p(p-1)}\right) \le \sum_{p>k} \frac{k}{p(p-1)} = k \sum_{p\ge k+1} \frac{1}{p(p-1)} \le k \sum_{n\ge k+1} \frac{1}{n(n-1)}.$$

We define

$$s_m = \sum_{n=k+1}^{m} \frac{1}{n(n-1)} = \sum_{n=k+1}^{m}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \frac{1}{k}-\frac{1}{m}, \qquad m \ge k+1.$$

Then,

$$\sum_{n\ge k+1} \frac{1}{n(n-1)} = \lim_{m\to+\infty} s_m = \frac{1}{k}.$$

We obtain $\ln B \le 1$, i.e.,

$$B \le e < 3. \quad (5.28)$$

It follows from (5.26)–(5.28) that $S_3 \le c_6 \ln(k+1)$, where $c_6 > 0$ is an absolute constant. Substituting this estimate into (5.25), we obtain

$$S_1 \le c_7 \eta \ln(k+1) \ln x, \quad (5.29)$$

where $c_7 > 0$ is an absolute constant. Therefore (see (5.14), (5.20), and (5.29)),

$$\widetilde{S} \le (c_7+2)\eta \ln(k+1)\ln x = c_8 \eta \ln(k+1) \ln x,$$

where $c_8 = c_7 + 2 > 0$ is an absolute constant. We obtain (see (5.13) and Lemma 2.8)

$$S \le c_8 \frac{a}{\varphi(a)} \eta \ln(k+1) \ln x \le c_9 \eta \ln\ln(a+2) \ln(k+1) \ln x, \quad (5.30)$$

where $c_9 > 0$ is an absolute constant. We put $C = c_1 + c_9$, where c_1 is the constant in (5.12). Then $C > 0$ is an absolute constant and in both cases, $1 \le k \le \ln\ln x$ and $k > \ln\ln x$, we have

$$\sum_{\substack{1\le b\le \eta \ln x \\ L=an+b\notin\mathcal{L}}} \frac{\Delta_L}{\varphi(\Delta_L)} \le C\eta \ln\ln(a+2)\ln(k+1)\ln x.$$

Lemma 5.2 is proved. □

Lemma 5.3. *Let $\mathcal{A}=\mathbb{N}$, $\mathcal{P}=\mathbb{P}$, $\alpha=1/5$, and $\theta=1/3$, and let $C_0=C(1/5,1/3)>0$ be the absolute constant in Proposition 5.1. Let ε be a real number with $0<\varepsilon<1$. Then there is a number $c_0(\varepsilon)>0$ such that the following holds. Let $x\in\mathbb{N}$, $y\in\mathbb{R}$, and $q\in\mathbb{N}$ satisfy the conditions $x\ge c_0(\varepsilon)$, $1\le y\le\ln x$, and $1\le q\le y^{1-\varepsilon}$. Then there is a positive integer B such that*

$$1\le B\le\exp(\vartheta\sqrt{\ln x}),\qquad 1\le\frac{B}{\varphi(B)}\le 2,\qquad (B,q)=1. \tag{5.31}$$

Furthermore, let $k\in\mathbb{N}$, $\rho\in\mathbb{R}$, $\xi\in\mathbb{R}$, $R\in\mathbb{R}$, $\eta\in\mathbb{R}$, and $a\in\mathbb{Z}$, be such that

$$C_0\le k\le(\ln x)^{1/5}, \tag{5.32}$$

$$\frac{k(\ln\ln x)^2}{\ln x}\le\rho\le\frac{1}{30},\qquad \xi=\rho, \tag{5.33}$$

$$R=x^{1/9},\qquad 0<\eta\le\frac{1}{2}, \tag{5.34}$$

$$1\le a\le q,\qquad (a,q)=1. \tag{5.35}$$

Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be an admissible set of k linear functions, where $L_i(n)=qn+a+qb_i$, $i=1,\dots,k$, $b_1,\dots,b_k$ are positive integers with $b_1<\ldots<b_k$, and $qb_k\le\eta y$. Then the hypothesis of Proposition 5.1 holds and there exist nonnegative weights $w_n=w_n(\mathcal{L})$ with the properties stated in Proposition 5.1; the implied constants in (5.1)–(5.5) are positive and absolute. In (5.31), $\vartheta>0$ is also an absolute constant.

Proof. We will choose $c_0(\varepsilon)$ later; this number is assumed to be large enough. We take $\delta=1/10$ and let $c_0(\varepsilon)\ge c(\varepsilon,\delta)=c(\varepsilon,1/10)$, where $c(\varepsilon,\delta)$ is the quantity in Lemma 4.10. Let $x\in\mathbb{N}$, $y\in\mathbb{R}$, and $q\in\mathbb{N}$ be such that $x\ge c_0(\varepsilon)$, $1\le y\le\ln x$, and $1\le q\le y^{1-\varepsilon}$. By Lemma 4.10, there is a positive integer B such that

$$1\le B\le\exp(c_1\sqrt{\ln x}),\qquad 1\le\frac{B}{\varphi(B)}\le 2,\qquad (B,q)=1$$

and

$$\sum_{\substack{1\le Q\le x^{2/5}\\ (Q,B)=1}}\ \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}}\ \max_{W\in\mathbb{Z}:\ (W,Q)=1}\left|\pi(u;Q,W)-\frac{\mathrm{li}(u)}{\varphi(Q)}\right|\le c_2x\exp(-c_3\sqrt{\ln x}), \tag{5.36}$$

where c_1, γ, c_2, and c_3 are positive absolute constants. Let (5.32)–(5.35) hold. Let $\mathcal{L}=\{L_1,\dots,L_k\}$ be an admissible set of k linear functions $L_i(n)=qn+a+qb_i$, $i=1,\dots,k$, where $b_1,\dots,b_k$ are positive integers with $b_1<\ldots<b_k$ and $qb_k\le\eta y$. Let us show that the hypothesis of Proposition 5.1 holds. First we show that the set $(\mathcal{A},\mathcal{L},\mathcal{P},B,x,1/3)$ satisfies Hypothesis 1.

I. Let us show that condition (2) of Hypothesis 1 holds. Let $L(n)=l_1n+l_2\in\mathcal{L}$. Clearly, we have

$$1\le l_1\le\ln x\qquad\text{and}\qquad 1\le l_2\le\ln x. \tag{5.37}$$

Let us show that

$$S:=\sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}}\ \max_{\substack{b\in\mathbb{Z}\\ (L(b),r)=1}}\left|\#\mathcal{P}_{L,\mathcal{A}}(x;r,b)-\frac{\#\mathcal{P}_{L,\mathcal{A}}(x)}{\varphi_L(r)}\right|\le\frac{\#\mathcal{P}_{L,\mathcal{A}}(x)}{(\ln x)^{100k^2}}. \tag{5.38}$$

It is not hard to see that

$$\mathcal{P}_{L,\mathcal{A}}(x)=\bigl\{l_1x+l_2\le p<2l_1x+l_2:\ p\equiv l_2\ (\mathrm{mod}\ l_1)\bigr\},$$
$$\mathcal{P}_{L,\mathcal{A}}(x;r,b)=\bigl\{l_1x+l_2\le p<2l_1x+l_2:\ p\equiv l_1b+l_2\ (\mathrm{mod}\ l_1r)\bigr\}$$

and hence

$$\#\mathcal{P}_{L,\mathcal{A}}(x) = \pi(2l_1x + l_2 - 1; l_1, l_2) - \pi(l_1x + l_2 - 1; l_1, l_2), \tag{5.39}$$

$$\#\mathcal{P}_{L,\mathcal{A}}(x; r, b) = \pi(2l_1x + l_2 - 1; l_1r, L(b)) - \pi(l_1x + l_2 - 1; l_1r, L(b)).$$

We obtain

$$S = \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \max_{\substack{b\in\mathbb{Z}\\ (L(b),r)=1}} \biggl|\pi(2l_1x + l_2 - 1; l_1r, L(b)) - \pi(l_1x + l_2 - 1; l_1r, L(b)) - \frac{\pi(2l_1x + l_2 - 1; l_1, l_2) - \pi(l_1x + l_2 - 1; l_1, l_2)}{\varphi(l_1r)/\varphi(l_1)}\biggr| \le S_1 + S_2 + S_3 + S_4, \tag{5.40}$$

where

$$S_1 = \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \max_{\substack{b\in\mathbb{Z}\\ (L(b),r)=1}} \left|\pi(l_1x + l_2 - 1; l_1r, L(b)) - \frac{\mathrm{li}(l_1x + l_2 - 1)}{\varphi(l_1r)}\right|,$$

$$S_2 = \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \left|\frac{\pi(l_1x + l_2 - 1; l_1, l_2)}{\varphi(l_1r)/\varphi(l_1)} - \frac{\mathrm{li}(l_1x + l_2 - 1)}{\varphi(l_1r)}\right|,$$

$$S_3 = \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \max_{\substack{b\in\mathbb{Z}\\ (L(b),r)=1}} \left|\pi(2l_1x + l_2 - 1; l_1r, L(b)) - \frac{\mathrm{li}(2l_1x + l_2 - 1)}{\varphi(l_1r)}\right|,$$

$$S_4 = \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \left|\frac{\pi(2l_1x + l_2 - 1; l_1, l_2)}{\varphi(l_1r)/\varphi(l_1)} - \frac{\mathrm{li}(2l_1x + l_2 - 1)}{\varphi(l_1r)}\right|.$$

Let us show that

$$(L(b), l_1) = 1 \tag{5.41}$$

for any $b \in \mathbb{Z}$. Assume the contrary: there is an integer b such that $(L(b), l_1) > 1$. Then there is a prime p such that $p \mid l_1$ and $p \mid L(b)$. Hence $p \mid l_2$, and we see that $p \mid L(n)$ for any integer n. Since $L \in \mathcal{L}$, we see that $p \mid L_1(n) \dots L_k(n)$ for any integer n. But this contradicts the fact that $\mathcal{L} = \{L_1, \dots, L_k\}$ is an admissible set. Thus, (5.41) is proved. We also observe that since $(B, q) = 1$ and $l_1 = q$, we have

$$(B, l_1) = 1. \tag{5.42}$$

Let r be an integer with $1 \le r \le x^{1/3}$ and $(r, B) = 1$. Applying (5.37), we have

$$l_1r \le x^{1/3}\ln x \le x^{2/5}, \qquad l_1x + l_2 - 1 \ge l_1x \ge x \ge 2, \qquad l_1x + l_2 - 1 \le 2x\ln x \le x^{1+\gamma/\sqrt{\ln x}}$$

provided that $c_0(\varepsilon)$ is chosen large enough. Hence, we obtain (see (5.41), (5.42) and (5.36))

$$S_1 = \sum_{\substack{r:\ l_1\le l_1r\le l_1x^{1/3}\\ (l_1r,B)=1}} \max_{\substack{b\in\mathbb{Z}\\ (L(b),l_1r)=1}} \left|\pi(l_1x + l_2 - 1; l_1r, L(b)) - \frac{\mathrm{li}(l_1x + l_2 - 1)}{\varphi(l_1r)}\right|$$

$$\le \sum_{\substack{1\le Q\le x^{2/5}\\ (Q,B)=1}} \max_{2\le u\le x^{1+\gamma/\sqrt{\ln x}}} \max_{W\in\mathbb{Z}:\ (W,Q)=1} \left|\pi(u; Q, W) - \frac{\mathrm{li}(u)}{\varphi(Q)}\right| \le c_2x\exp(-c_3\sqrt{\ln x}). \tag{5.43}$$

Applying Lemmas 2.5 and 2.9, we get

$$S_2 = \varphi(l_1)\left|\pi(l_1x+l_2-1;l_1,l_2) - \frac{\mathrm{li}(l_1x+l_2-1)}{\varphi(l_1)}\right| \sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \frac{1}{\varphi(l_1r)}$$
$$\le \left|\pi(l_1x+l_2-1;l_1,l_2) - \frac{\mathrm{li}(l_1x+l_2-1)}{\varphi(l_1)}\right| \sum_{1\le r\le x^{1/3}} \frac{1}{\varphi(r)}$$
$$\le \widetilde{c}\ln x\left|\pi(l_1x+l_2-1;l_1,l_2) - \frac{\mathrm{li}(l_1x+l_2-1)}{\varphi(l_1)}\right|,$$

where $\widetilde{c}>0$ is an absolute constant. Since $l_1x+l_2-1\ge l_1x\ge x$ (see (5.37)), we obtain

$$1\le l_1\le \ln x\le \ln(l_1x+l_2-1).$$

Hence (see, for example, [1, Ch. 22]),

$$\left|\pi(l_1x+l_2-1;l_1,l_2) - \frac{\mathrm{li}(l_1x+l_2-1)}{\varphi(l_1)}\right| \le C(l_1x+l_2-1)\exp\bigl(-c\sqrt{\ln(l_1x+l_2-1)}\,\bigr), \tag{5.44}$$

where C and c are positive absolute constants. We have

$$\exp\bigl(-c\sqrt{\ln(l_1x+l_2-1)}\,\bigr)\le \exp(-c\sqrt{\ln x}), \tag{5.45}$$
$$l_1x+l_2-1\le x\ln x+\ln x\le 2x\ln x. \tag{5.46}$$

We can assume that

$$-c\sqrt{\ln x}+2\ln\ln x\le -\frac{c}{2}\sqrt{\ln x} \tag{5.47}$$

if $c_0(\varepsilon)$ is chosen large enough. Hence,

$$S_2\le \widetilde{C}x\exp\bigl(-c\sqrt{\ln x}+2\ln\ln x\bigr)\le \widetilde{C}x\exp\Bigl(-\frac{c}{2}\sqrt{\ln x}\Bigr), \tag{5.48}$$

where $\widetilde{C}=2\widetilde{c}C$ is a positive absolute constant. Similarly, it can be shown that

$$S_3\le Cx\exp(-c\sqrt{\ln x}) \qquad \text{and} \qquad S_4\le Cx\exp(-c\sqrt{\ln x}), \tag{5.49}$$

where C and c are positive absolute constants. Substituting (5.43), (5.48), and (5.49) into (5.40), we obtain

$$\sum_{\substack{1\le r\le x^{1/3}\\ (r,B)=1}} \max_{\substack{b\in\mathbb{Z}\\ (L(b),r)=1}} \left|\#\mathcal{P}_{L,\mathcal{A}}(x;r,b) - \frac{\#\mathcal{P}_{L,\mathcal{A}}(x)}{\varphi_L(r)}\right| \le c_4x\exp(-c_5\sqrt{\ln x}), \tag{5.50}$$

where c_4 and c_5 are positive absolute constants. Applying (5.44)–(5.47), we have

$$\pi(l_1x+l_2-1;l_1,l_2) = \frac{\mathrm{li}(l_1x+l_2-1)}{\varphi(l_1)} + R_1, \qquad |R_1|\le Cx\exp(-c\sqrt{\ln x}),$$

where C and c are positive absolute constants. Similarly, it can be shown that

$$\pi(2l_1x+l_2-1;l_1,l_2) = \frac{\mathrm{li}(2l_1x+l_2-1)}{\varphi(l_1)} + R_2, \qquad |R_2|\le Cx\exp(-c\sqrt{\ln x}),$$

where C and c are positive absolute constants. Therefore (see (5.39)),

$$\#\mathcal{P}_{L,\mathcal{A}}(x) = \frac{\mathrm{li}(2l_1x + l_2 - 1) - \mathrm{li}(l_1x + l_2 - 1)}{\varphi(l_1)} + R, \tag{5.51}$$

$$|R| \le c_6 x \exp(-c_7\sqrt{\ln x}), \tag{5.52}$$

where c_6 and c_7 are positive absolute constants. We have

$$2l_1x + l_2 - 1 \le 2x\ln x + \ln x \le 3x\ln x,$$

$$\ln(2l_1x + l_2 - 1) \le \ln x + \ln\ln x + \ln 3 \le 2\ln x$$

provided that $c_0(\varepsilon)$ is chosen large enough. Hence,

$$\frac{\mathrm{li}(2l_1x + l_2 - 1) - \mathrm{li}(l_1x + l_2 - 1)}{\varphi(l_1)} = \frac{1}{\varphi(l_1)} \int\limits_{l_1x+l_2-1}^{2l_1x+l_2-1} \frac{dt}{\ln t} \ge \frac{l_1x}{\varphi(l_1)\ln(2l_1x + l_2 - 1)}$$
$$\ge \frac{l_1x}{2\varphi(l_1)\ln x}. \tag{5.53}$$

Let us show that

$$|R| \le \frac{l_1x}{4\varphi(l_1)\ln x}. \tag{5.54}$$

Since $l_1/\varphi(l_1) \ge 1$, we see from (5.52) that it is sufficient to show that

$$c_6 x \exp(-c_7\sqrt{\ln x}) \le \frac{x}{4\ln x}.$$

This inequality holds if $c_0(\varepsilon)$ is chosen large enough. Thus, (5.54) is proved. From (5.51), (5.53), and (5.54) we obtain

$$\#\mathcal{P}_{L,\mathcal{A}}(x) \ge \frac{l_1x}{4\varphi(l_1)\ln x}. \tag{5.55}$$

Now we prove (5.38). Since $l_1/\varphi(l_1) \ge 1$, we see from (5.50) and (5.55) that it suffices to establish the estimate

$$c_4 x \exp(-c_5\sqrt{\ln x}) \le \frac{x}{4(\ln x)^{100k^2+1}}. \tag{5.56}$$

Taking logarithms, we obtain

$$\ln c_4 + \ln x - c_5\sqrt{\ln x} \le \ln x - \ln 4 - 100k^2 \ln\ln x - \ln\ln x$$

or, which is equivalent,

$$100k^2 \ln\ln x \le c_5\sqrt{\ln x} - \ln\ln x - \ln(4c_4).$$

Since $k \le (\ln x)^{1/5}$, we have

$$100k^2 \ln\ln x \le 100(\ln x)^{2/5} \ln\ln x.$$

The inequality

$$100(\ln x)^{2/5} \ln\ln x \le c_5\sqrt{\ln x} - \ln\ln x - \ln(4c_4)$$

holds if $c_0(\varepsilon)$ is chosen large enough. Inequality (5.56) is proved. Thus, (5.38) is proved.

II. Let us show that condition (1) of Hypothesis 1 holds. We show that

$$S := \sum_{1\le r\le x^{1/3}} \max_{b\in\mathbb{Z}} \left| \#\mathcal{A}(x;r,b) - \frac{\#\mathcal{A}(x)}{r} \right| \le \frac{\#\mathcal{A}(x)}{(\ln x)^{100k^2}}. \tag{5.57}$$

Let $1 \le r \le x^{1/3}$ and $b \in \mathbb{Z}$. We have

$$\mathcal{A}(x) = \{x \le n < 2x\} \qquad \text{and} \qquad \mathcal{A}(x;r,b) = \big\{x \le n < 2x\colon\ n \equiv b \pmod r\big\}.$$

Hence,

$$\#\mathcal{A}(x) = x \qquad \text{and} \qquad \#\mathcal{A}(x;r,b) = \frac{x}{r} + \rho, \quad |\rho| \le 1. \tag{5.58}$$

We obtain

$$\left| \#\mathcal{A}(x;r,b) - \frac{\#\mathcal{A}(x)}{r} \right| = |\rho| \le 1. \tag{5.59}$$

Hence, $S \le x^{1/3}$. Thus, to prove (5.57), it suffices to show that

$$x^{1/3} \le \frac{x}{(\ln x)^{100k^2}}$$

or, which is equivalent, $(\ln x)^{100k^2} \le x^{2/3}$. Taking logarithms, we obtain

$$100k^2 \ln\ln x \le \frac{2}{3} \ln x.$$

Since $k \le (\ln x)^{1/5}$, we have

$$100k^2 \ln\ln x \le 100(\ln x)^{2/5} \ln\ln x.$$

The inequality

$$100(\ln x)^{2/5} \ln\ln x \le \frac{2}{3} \ln x$$

holds if $c_0(\varepsilon)$ is chosen large enough. Thus, (5.57) is proved.

III. Let us show that condition (3) of Hypothesis 1 holds. To this end we show that for any integer r with $1 \le r < x^{1/3}$ we have

$$\max_{b\in\mathbb{Z}} \#\mathcal{A}(x;r,b) \le 2\frac{\#\mathcal{A}(x)}{r}. \tag{5.60}$$

Let $1 \le r < x^{1/3}$ and $b \in \mathbb{Z}$. We may assume that $c_0(\varepsilon) \ge 2$. Hence, $r \le x^{1/3} \le x$. Applying (5.58), we obtain

$$\#\mathcal{A}(x;r,b) \le \frac{x}{r} + 1 \le 2\frac{x}{r} = 2\frac{\#\mathcal{A}(x)}{r},$$

and (5.60) is proved. Thus, the set $(\mathcal{A}, \mathcal{L}, \mathcal{P}, B, x, 1/3)$ satisfies Hypothesis 1.

We can assume that

$$\exp(c_1\sqrt{\ln x}) \le x^{1/5} \qquad \text{and} \qquad \ln x \le x^{1/5}$$

provided that $c_0(\varepsilon)$ is chosen large enough. Since $1 \le B \le \exp(c_1\sqrt{\ln x})$, we obtain $1 \le B \le x^{1/5}$. Let $L = l_1 n + l_2 \in \mathcal{L}$. Applying (5.37), we have $1 \le l_1 \le x^{1/5}$ and $1 \le l_2 \le x^{1/5}$. Thus, the hypothesis of Proposition 5.1 holds and there are nonnegative weights $w_n = w_n(\mathcal{L})$ with the properties stated in Proposition 5.1. In that proposition, the implied constants in (5.1)–(5.5) depend only on α, θ and on the implied constants from Hypothesis 1, and in our case these constants are absolute ($\alpha = 1/5$, $\theta = 1/3$, and estimates (5.38), (5.57), and (5.60) hold). Therefore, in our case the implied constants in (5.1)–(5.5) are positive and absolute. Finally, let us denote c_1 by ϑ. Lemma 5.3 is proved. □

Lemma 5.4. *There are positive absolute constants c and C such that the following holds. Let ε be a real number with $0<\varepsilon<1$. Then there is a number $c_0(\varepsilon)>0$, depending only on ε, such that if $x\in\mathbb{N}$, $y\in\mathbb{R}$, $m\in\mathbb{Z}$, $q\in\mathbb{Z}$, and $a\in\mathbb{Z}$ satisfy the conditions $c_0(\varepsilon)\le y\le \ln x$, $1\le m\le c\varepsilon\ln y$, $1\le q\le y^{1-\varepsilon}$, and $(a,q)=1$, then*

$$\#\bigl\{qx<p_n\le 2qx-5q\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a \pmod q),\ p_{n+m}-p_n\le y\bigr\}\ge \pi(2qx)\biggl(\frac{y}{2q\ln x}\biggr)^{\exp(Cm)}.$$

Proof. Let $\mathcal{A}=\mathbb{N}$, $\mathcal{P}=\mathbb{P}$, $\alpha=1/5$, and $\theta=1/3$, and let $C_0=C(1/5,1/3)>0$ be the absolute constant in Proposition 5.1. Let $c_0(\varepsilon)$ be the quantity in Lemma 5.3. We will choose $c(\varepsilon)$ later; this number is large enough. Let $c(\varepsilon)\ge c_0(\varepsilon)$. Let $x\in\mathbb{N}$, $y\in\mathbb{R}$, and $q\in\mathbb{Z}$ be such that

$$c(\varepsilon)\le y\le \ln x, \tag{5.61}$$

$$1\le q\le y^{1-\varepsilon}. \tag{5.62}$$

By Lemma 5.3, there is a positive integer B such that (5.31) holds. We assume that

$$\widetilde{C}_0\le k\le y^{\varepsilon/14}, \tag{5.63}$$

where $\widetilde{C}_0>0$ is an absolute constant. We will choose $\widetilde{C}_0$ later. For now, we assume that $\widetilde{C}_0$ is large enough; in particular, $\widetilde{C}_0\ge C_0$. It follows from (5.61) and (5.63) that $k\le(\ln x)^{1/5}$. Thus, (5.32) holds. Let (5.33)–(5.35) hold. Let $\mathcal{L}=\{L_1,\ldots,L_k\}$ be an admissible set of k linear functions $L_i(n)=qn+a+qb_i$, $i=1,\ldots,k$, where $b_1,\ldots,b_k$ are positive integers such that $b_1<\ldots<b_k$ and $qb_k\le\eta y$. Then (see Lemma 5.3) the hypothesis of Proposition 5.1 holds and there are non-negative weights $w_n=w_n(\mathcal{L})$ with the properties stated in Proposition 5.1; the implied constants in (5.1)–(5.5) are positive and absolute. We write $\mathcal{L}=\mathcal{L}(\mathbf{b})$ for such a set defined by $b_1,\ldots,b_k$. Denote the class of admissible sets by AS.

Let m be a positive integer. We consider

$$\begin{aligned} S&=\sum_{\substack{1\le b_1<\ldots<b_k\\ qb_k\le\eta y\\ \mathcal{L}=\mathcal{L}(\mathbf{b})\in\mathrm{AS}}}\ \sum_{n\in\mathcal{A}(x)}\Biggl(\sum_{i=1}^k\mathbf{1}_{\mathcal{P}}(L_i(n))-m-k\sum_{i=1}^k\sum_{\substack{p\mid L_i(n)\\ p<x^\rho,\ p\nmid B}}1-k\sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}}\mathbf{1}_{\mathcal{S}(\rho;B)}(L(n))\Biggr)w_n(\mathcal{L})\\ &=\sum_{\substack{1\le b_1<\ldots<b_k\\ qb_k\le\eta y\\ \mathcal{L}=\mathcal{L}(\mathbf{b})\in\mathrm{AS}}}\ \sum_{n\in\mathcal{A}(x)}A_n(\mathcal{L})w_n(\mathcal{L}). \end{aligned} \tag{5.64}$$

Let $\mathcal{L}=\{L_1,\ldots,L_k\}$ and n be in the range of summation of S and $A_n(\mathcal{L})>0$. Then the following statements hold:

(1) The number of primes among $L_1(n),\ldots,L_k(n)$ is at least $m+1$.

(2) For any $1\le i\le k$, $L_i(n)$ has no prime factor p such that $p<x^\rho$ and $p\nmid B$.

(3) For any linear function $L=qt+b\notin\mathcal{L}$, where b is an integer with $1\le b\le 2\eta y$, $L(n)$ has a prime factor p such that $p<x^\rho$ and $p\nmid B$ (we choose ρ so that x^ρ is not an integer; therefore, the conditions $p\le x^\rho$ and $p<x^\rho$ are equivalent). Since $L(n)>n\ge x>x^\rho$, we see that $L(n)$ is not a prime number.

As a consequence we obtain the following statements:

(i) None of $n\in\mathcal{A}(x)$ can make a positive contribution to S from two different admissible sets (since if n makes a positive contribution for some admissible set $\mathcal{L}=\{L_1,\ldots,L_k\}$, then the numbers $L_1(n),\ldots,L_k(n)$ are uniquely determined as the integers in $[qn+1,qn+2\eta y]$ with no prime factors p such that $p<x^\rho$ and $p\nmid B$).

(ii) If $\mathcal{L} = \{L_1, \dots, L_k\}$ and n are in the range of summation of S and $A_n(\mathcal{L}) > 0$, then there can be no primes in the interval $[qn+1, qn+2\eta y]$ apart from possibly $L_1(n), \dots, L_k(n)$, and so the primes counted in this way must be consecutive.

Let $\mathcal{L} = \{L_1, \dots, L_k\}$ and n be in the range of summation of S and $A_n(\mathcal{L}) > 0$. Let $1 \le i \le k$. If $p \mid L_i(n)$ and $p \nmid B$, then $p \ge x^\rho$. Setting

$$\Omega = \bigl\{p\colon\ p \mid L_i(n) \ \text{ and } \ p \nmid B\bigr\},$$

we have

$$x^{\rho\,\#\Omega} \le \prod_{p\in\Omega} p \le L_i(n).$$

Since

$$q \le y^{1-\varepsilon} \le y \le \ln x \qquad \text{and} \qquad a + qb_i \le 2\eta y \le \ln x,$$

we obtain

$$L_i(n) = qn + a + qb_i \le n \ln x + \ln x \le 2x \ln x + \ln x \le x^2$$

provided that $c(\varepsilon)$ is chosen large enough. Hence, $\rho\,\#\Omega \le 2$, i.e., $\#\Omega \le 2/\rho$. We have

$$\prod_{\substack{p\mid L_i(n)\\ p\nmid B}} 4 = \prod_{p\in\Omega} 4 = 4^{\#\Omega} \le 4^{2/\rho} = e^{(2/\rho)\ln 4} \le e^{4/\rho} \qquad \text{and} \qquad \prod_{i=1}^{k} \prod_{\substack{p\mid L_i(n)\\ p\nmid B}} 4 \le e^{(4k)/\rho}.$$

Thus, if $\mathcal{L} = \{L_1, \dots, L_k\}$ and n are in the range of summation of S and $A_n(\mathcal{L}) > 0$, then (see (5.1))

$$w_n(\mathcal{L}) \le C(\ln R)^{2k} e^{(4k)/\rho}, \tag{5.65}$$

where $C > 0$ is an absolute constant.

Let $\mathcal{L} = \{L_1, \dots, L_k\}$ be in the range of summation of S. We consider

$$\begin{aligned} \widetilde{S}(\mathcal{L}) &= \sum_{n\in\mathcal{A}(x)} \Biggl(\sum_{i=1}^{k} \mathbf{1}_{\mathcal{P}}(L_i(n)) - m - k \sum_{i=1}^{k} \sum_{\substack{p\mid L_i(n)\\ p<x^\rho,\ p\nmid B}} 1 - k \sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}} \mathbf{1}_{\mathcal{S}(\rho;B)}(L(n)) \Biggr) w_n(\mathcal{L}) \\ &= S_1 - S_2 - S_3 - S_4. \end{aligned}$$

Our aim is to obtain a lower bound for $\widetilde{S}(\mathcal{L})$. We write w_n instead of $w_n(\mathcal{L})$ for brevity. Let $1 \le i \le k$. Since $\#\mathcal{A}(x) = x$, we have (see (5.3))

$$\begin{aligned} \sum_{n\in\mathcal{A}(x)} \mathbf{1}_{\mathcal{P}}(L_i(n)) w_n &\ge (1+o(1)) \frac{B^{k-1}}{\varphi(B)^{k-1}} \mathfrak{S}_B(\mathcal{L}) \frac{\varphi(q)}{q} \#\mathcal{P}_{L_i,\mathcal{A}}(x) (\ln R)^{k+1} J_k \\ &\quad + O\biggl(\frac{B^k}{\varphi(B)^k} \mathfrak{S}_B(\mathcal{L}) x (\ln R)^{k-1} I_k \biggr). \end{aligned}$$

Hence,

$$\begin{aligned} S_1 &= \sum_{n\in\mathcal{A}(x)} \sum_{i=1}^{k} \mathbf{1}_{\mathcal{P}}(L_i(n)) w_n = \sum_{i=1}^{k} \sum_{n\in\mathcal{A}(x)} \mathbf{1}_{\mathcal{P}}(L_i(n)) w_n \\ &\ge (1+o(1)) \frac{B^{k-1}}{\varphi(B)^{k-1}} \mathfrak{S}_B(\mathcal{L}) \frac{\varphi(q)}{q} (\ln R)^{k+1} J_k \sum_{i=1}^{k} \#\mathcal{P}_{L_i,\mathcal{A}}(x) + O\biggl(k \frac{B^k}{\varphi(B)^k} \mathfrak{S}_B(\mathcal{L}) x (\ln R)^{k-1} I_k \biggr) \end{aligned}$$

$$= (1+o(1))\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})(\ln R)^{k+1}J_k\frac{\varphi(B)}{B}\frac{\varphi(q)}{q}\sum_{i=1}^{k}\#\mathcal{P}_{L_i,\mathcal{A}}(x) + o\biggl(\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k\biggr)$$
$$= S_1' + S_1'',$$

since

$$0 < \frac{k}{\ln R} \le \frac{(\ln x)^{1/5}}{(1/9)\ln x} \to 0 \qquad \text{as} \quad x \to +\infty.$$

We have shown (see (5.55)) that if $x \ge c_0$, where $c_0 > 0$ is an absolute constant, then for any $L \in \mathcal{L}$

$$\#\mathcal{P}_{L,\mathcal{A}}(x) \ge \frac{qx}{4\varphi(q)\ln x}.$$

We may assume that $c(\varepsilon) \ge c_0$. Since $\varphi(B)/B \ge 1/2$ (see (5.31)), we obtain

$$\frac{\varphi(B)}{B}\frac{\varphi(q)}{q}\sum_{i=1}^{k}\#\mathcal{P}_{L_i,\mathcal{A}}(x) \ge \frac{kx}{8\ln x} = \frac{kx}{72\ln R}.$$

We have $|o(1)| \le 1/2$ in S_1' if $x \ge c'$, where $c' > 0$ is an absolute constant. We may assume that $c(\varepsilon) \ge c'$. Since (see (5.8))

$$J_k \ge c''\frac{\ln k}{k}I_k,$$

where $c'' > 0$ is an absolute constant, we get

$$S_1' \ge \frac{c''}{144}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \ln k.$$

We have

$$|S_1''| \le \frac{c''}{288}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \le \frac{c''}{288}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \ln k$$

provided that $c(\varepsilon)$ is chosen large enough. Therefore,

$$S_1 \ge \frac{c''}{288}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \ln k = c\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \ln k, \tag{5.66}$$

where $c > 0$ is an absolute constant.

We have (see (5.2))

$$S_2 = m\sum_{n\in\mathcal{A}(x)} w_n = m(1+o(1))\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \ge \frac{m}{2}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \tag{5.67}$$

provided that $c(\varepsilon)$ is chosen large enough. Applying (5.5), we obtain

$$S_3 = k\sum_{n\in\mathcal{A}(x)}\sum_{i=1}^{k}\sum_{\substack{p\mid L_i(n)\\ p<x^\rho,\ p\nmid B}} w_n = k\sum_{i=1}^{k}\sum_{n\in\mathcal{A}(x)} w_n \sum_{\substack{p\mid L_i(n)\\ p<x^\rho,\ p\nmid B}} 1 \le c_2\rho^2k^6(\ln k)^2\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k,$$

where $c_2 > 0$ is an absolute constant. Let $c_3 > 0$ be an absolute constant such that

$$c_2c_3^2 \le \frac{1}{12} \qquad \text{and} \qquad \frac{c_3}{j^3\ln j} \le \frac{1}{30} \quad \text{for any } j \ge 2.$$

We choose an arbitrary number ρ in the interval

$$\Big[\frac{c_3}{2k^3\ln k},\frac{c_3}{k^3\ln k}\Big] \tag{5.68}$$

so that x^ρ is not an integer. It is clear that $\rho\le 1/30$. Let us show that the first inequality in (5.33) holds. It suffices to show that

$$\frac{k(\ln\ln x)^2}{\ln x}\le\frac{c_3/2}{k^3\ln k}.$$

This inequality is equivalent to

$$k^4\ln k(\ln\ln x)^2\le\frac{c_3}{2}\ln x.$$

Since $k\le(\ln x)^{1/5}$, we have

$$k^4\ln k(\ln\ln x)^2\le\frac{1}{5}(\ln x)^{4/5}(\ln\ln x)^3\le\frac{c_3}{2}\ln x$$

provided that $c(\varepsilon)$ is chosen large enough. Thus, the inequalities in (5.33) hold. We have (see (5.67))

$$\begin{aligned} S_3&\le c_2\frac{c_3^2}{k^6(\ln k)^2}k^6(\ln k)^2\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^kI_k\le\frac{1}{12}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^kI_k\\ &\le\frac{m}{12}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^kI_k\le\frac{1}{6}S_2. \end{aligned} \tag{5.69}$$

Now we estimate the quantity

$$S_4=k\sum_{n\in\mathcal{A}(x)}\sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}}\mathbf{1}_{\mathcal{S}(\rho;B)}(L(n))w_n=k\sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}}\sum_{n\in\mathcal{A}(x)}\mathbf{1}_{\mathcal{S}(\rho;B)}(L(n))w_n.$$

Let b be in the range of summation of S_4. Then $L=qt+b\notin\mathcal{L}$ and

$$\Delta_L=q^{k+1}\prod_{i=1}^{k}|(a+qb_i)-b|\ne 0.$$

Since $1\le B\le x^{1/5}$, we have (see (5.4))

$$\sum_{n\in\mathcal{A}(x)}\mathbf{1}_{\mathcal{S}(\rho;B)}(L(n))w_n\le\frac{c_4}{\rho}\frac{\Delta_L}{\varphi(\Delta_L)}\frac{B}{\varphi(B)}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^{k-1}I_k,$$

where $c_4>0$ is an absolute constant. Since $B/\varphi(B)\le 2$ and ρ lies in the interval (5.68), we obtain

$$\begin{aligned} \sum_{n\in\mathcal{A}(x)}\mathbf{1}_{\mathcal{S}(\rho;B)}(L(n))w_n&\le\frac{4c_4}{c_3}k^3\ln k\frac{\Delta_L}{\varphi(\Delta_L)}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^{k-1}I_k\\ &=c_5k^3\ln k\frac{\Delta_L}{\varphi(\Delta_L)}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^{k-1}I_k. \end{aligned}$$

Hence,

$$S_4\le c_5k^4\ln k\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^{k-1}I_k\sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}}\frac{\Delta_L}{\varphi(\Delta_L)}. \tag{5.70}$$

We put

$$c_6=36Cc_5, \tag{5.71}$$

where $C > 0$ is the absolute constant in Lemma 5.2, and

$$\eta = \frac{1}{12c_6k^4(\ln k)^2 \ln\ln(q+2)}. \tag{5.72}$$

Let us show that

$$(\ln x)^{-9/10} \le 2\eta \le 1. \tag{5.73}$$

The second inequality in (5.73) is equivalent to the inequality

$$6c_6k^4(\ln k)^2 \ln\ln(q+2) \ge 1.$$

We may assume that $\widetilde{C}_0 \ge 3$; therefore, $\ln k \ge 1$. We have

$$6c_6k^4(\ln k)^2 \ln\ln(q+2) \ge 6c_6(\ln\ln 3)k^4 \ge 6c_6(\ln\ln 3)\widetilde{C}_0^4 \ge 1$$

provided that $\widetilde{C}_0$ is chosen large enough. The first inequality in (5.73) is equivalent to the inequality

$$6c_6k^4(\ln k)^2 \ln\ln(q+2) \le (\ln x)^{9/10}.$$

Since $q \le \ln x$ and $k \le (\ln x)^{1/5}$, we have

$$6c_6k^4(\ln k)^2 \ln\ln(q+2) \le 6c_6(\ln x)^{4/5}\frac{1}{25}(\ln\ln x)^2 \ln\ln(\ln x+2) \le (\ln x)^{9/10}$$

provided that $c(\varepsilon)$ is chosen large enough. Thus, (5.73) holds. We can assume that $x \ge c$, where c is the absolute constant in Lemma 5.2, provided that $c(\varepsilon)$ is chosen large enough. Applying Lemma 5.2 and taking into account that $\ln(k+1) \le 2\ln k$, we have

$$\sum_{\substack{1\le b\le 2\eta y\\ L=qt+b\notin\mathcal{L}}} \frac{\Delta_L}{\varphi(\Delta_L)} \le \sum_{\substack{1\le b\le 2\eta \ln x\\ L=qt+b\notin\mathcal{L}}} \frac{\Delta_L}{\varphi(\Delta_L)} \le 4C\ln\ln(q+2)(\ln k)\eta\ln x = 36C\ln\ln(q+2)(\ln k)\eta\ln R.$$

Substituting this estimate into (5.70), we get (see also (5.71), (5.72), and (5.67))

$$\begin{aligned} S_4 &\le 36Cc_5k^4(\ln k)^2\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k\eta\ln\ln(q+2) \\ &= c_6k^4(\ln k)^2\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k\ln\ln(q+2)\frac{1}{12c_6k^4(\ln k)^2\ln\ln(q+2)} \\ &= \frac{1}{12}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \le \frac{m}{12}\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \le \frac{1}{6}S_2. \end{aligned} \tag{5.74}$$

From (5.69) and (5.74) we obtain

$$\widetilde{S}(\mathcal{L}) = S_1 - S_2 - S_3 - S_4 \ge S_1 - \frac{4}{3}S_2.$$

We have (see (5.2))

$$S_2 = m\sum_{n\in\mathcal{A}(x)} w_n = m(1+o(1))\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k \le \frac{3}{2}m\frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k$$

provided that $c(\varepsilon)$ is chosen large enough. Applying (5.66) with c replaced by $3c_1$, we obtain

$$\widetilde{S}(\mathcal{L}) \ge \frac{B^k}{\varphi(B)^k}\mathfrak{S}_B(\mathcal{L})x(\ln R)^k I_k(3c_1\ln k - 2m),$$

where $c_1 > 0$ is an absolute constant. We put

$$\widetilde{c} = \widetilde{C}_0 + \frac{1}{c_1}, \tag{5.75}$$

$$k = \lceil \exp(\widetilde{c}m) \rceil. \tag{5.76}$$

It is not hard to see that

$$k \geq \widetilde{C}_0 \qquad \text{and} \qquad 3c_1 \ln k - 2m \geq m.$$

Since m is a positive integer, we see that $3c_1 \ln k - 2m \geq 1$. Hence,

$$\widetilde{S}(\mathcal{L}) \geq \frac{B^k}{\varphi(B)^k} \mathfrak{S}_B(\mathcal{L}) x (\ln R)^k I_k.$$

Since $B^k/\varphi(B)^k \geq 1$, $\ln R = (1/9)\ln x$, $\mathfrak{S}_B(\mathcal{L}) \geq \exp(-c_2 k)$, and $I_k \geq c_3(2k\ln k)^{-k}$, where c_2 and c_3 are positive absolute constants (see (5.6) and (5.7)), it follows that

$$\widetilde{S}(\mathcal{L}) \geq \frac{1}{9^k} c_3 (2k\ln k)^{-k} \exp(-c_2 k) x (\ln x)^k \geq \exp(-k^2) x (\ln x)^k$$

provided that $\widetilde{C}_0$ is chosen large enough. We obtain

$$S = \sum_{\substack{1 \leq b_1 < \ldots < b_k \\ qb_k \leq \eta y \\ \mathcal{L} = \mathcal{L}(\mathbf{b}) \in \mathrm{AS}}} \widetilde{S}(\mathcal{L}) \geq \exp(-k^2) x (\ln x)^k \sum_{\substack{1 \leq b_1 < \ldots < b_k \\ qb_k \leq \eta y \\ \mathcal{L} = \mathcal{L}(\mathbf{b}) \in \mathrm{AS}}} 1 = \exp(-k^2) x (\ln x)^k S'. \tag{5.77}$$

Now we derive a lower bound for S'. First let us show that

$$2 \leq k \leq \frac{1}{2}\left[\frac{\eta y}{q}\right]. \tag{5.78}$$

The first inequality obviously holds, since we may assume that $\widetilde{C}_0 \geq 2$. To prove the second inequality, it suffices to show that

$$2k \leq \frac{\eta y}{q}. \tag{5.79}$$

We have (see (5.62) and (5.72))

$$\frac{\eta y}{q} \geq \eta y^{\varepsilon} = \frac{c_4 y^{\varepsilon}}{k^4 (\ln k)^2 \ln\ln(q+2)},$$

where $c_4 > 0$ is an absolute constant. Thus, to prove (5.79), it suffices to show that

$$2k^5 (\ln k)^2 \ln\ln(q+2) \leq c_4 y^{\varepsilon}.$$

In particular, from (5.62) it follows that $q \leq y$. Applying (5.63), we have

$$2k^5 (\ln k)^2 \ln\ln(q+2) \leq 2y^{5\varepsilon/14} \frac{\varepsilon^2}{196} (\ln y)^2 \ln\ln(y+2) \leq c_4 y^{\varepsilon}$$

provided that $c(\varepsilon)$ is chosen large enough. Thus, (5.78) is proved.

We put

$$\Omega = \left\{ 1 \leq n \leq \left[\frac{\eta y}{q}\right] \colon\ (n,p) = 1\ \ \forall p \leq k \right\}.$$

Applying Lemma 2.13, we have

$$\#\Omega = \Phi\left(\left[\frac{\eta y}{q}\right], k\right) \geq c_0 \frac{[\eta y/q]}{\ln k},$$

where $c_0 > 0$ is an absolute constant. In particular, from (5.78) it follows that $\eta y/q \ge 4$, and so

$$\left[\frac{\eta y}{q}\right] \ge \frac{\eta y}{q} - 1 \ge \frac{\eta y}{2q}.$$

We obtain

$$\#\Omega \ge c_5 \frac{\eta y}{q \ln k}, \tag{5.80}$$

where $c_5 > 0$ is an absolute constant. Let us show that

$$c_5 \frac{\eta y}{q \ln k} \ge 2k. \tag{5.81}$$

Applying (5.62) and (5.72), we have

$$c_5 \frac{\eta y}{q \ln k} \ge \frac{c_6 y^{\varepsilon}}{k^4 (\ln k)^3 \ln\ln(q+2)},$$

where $c_6 > 0$ is an absolute constant. Therefore, it suffices to show that

$$2k^5 (\ln k)^3 \ln\ln(q+2) \le c_6 y^{\varepsilon}.$$

Applying (5.63) and taking into account that $q \le y$, we have

$$2k^5 (\ln k)^3 \ln\ln(q+2) \le 2y^{5\varepsilon/14} \left(\frac{\varepsilon}{14}\right)^3 (\ln y)^3 \ln\ln(y+2) \le c_6 y^{\varepsilon}$$

provided that $c(\varepsilon)$ is chosen large enough. Thus, (5.81) is proved.

Let $b_1 < \ldots < b_k$ be positive integers from the set Ω. Let us show that for any prime p with $p \nmid q$ there is an integer m_p such that $m_p \not\equiv b_i \pmod{p}$ for all $1 \le i \le k$. Let p be a prime with $p \nmid q$. If $p > k$, then the statement is obvious. If $p \le k$, then we may put $m_p = 0$; from the definition of the set Ω it follows that $b_i \not\equiv 0 \pmod{p}$ for all $1 \le i \le k$. Thus, the statement is proved. By Lemma 5.1, $\mathcal{L}(\mathbf{b})$ is an admissible set. Hence (see also Lemma 2.12, (5.80), (5.81), and (5.72)),

$$\begin{aligned} S' &\ge \binom{\#\Omega}{k} \ge k^{-k} (\#\Omega - k)^k \ge k^{-k} \left(c_5 \frac{\eta y}{q \ln k} - k\right)^k \ge k^{-k} \left(\frac{c_5}{2} \frac{\eta y}{q \ln k}\right)^k \\ &= k^{-k} \left(c_6 \frac{y}{q \ln\ln(q+2) k^4 (\ln k)^3}\right)^k = \left(\frac{y}{q \ln\ln(q+2)}\right)^k \left(\frac{c_6}{k^5 (\ln k)^3}\right)^k, \end{aligned}$$

where $c_6 > 0$ is an absolute constant. We have

$$\left(\frac{c_6}{k^5 (\ln k)^3}\right)^k \ge \exp(-k^2)$$

provided that $\widetilde{C}_0$ is chosen large enough. Hence,

$$S' \ge \left(\frac{y}{q \ln\ln(q+2)}\right)^k \exp(-k^2).$$

Substituting this estimate into (5.77), we obtain

$$S \ge \exp(-2k^2) x (\ln x)^k \left(\frac{y}{q \ln\ln(q+2)}\right)^k \ge \exp(-2k^5) x (\ln x)^k \left(\frac{y}{q \ln\ln(q+2)}\right)^k. \tag{5.82}$$

Now we obtain an upper bound for S. Applying (5.64) and (5.65), we get

$$S \le \sum_{\substack{1\le b_1<\ldots<b_k\\ qb_k\le \eta y\\ \mathcal{L}=\mathcal{L}(\mathbf{b})\in \mathrm{AS}}} \sum_{n\in\mathcal{A}(x):\ A_n(\mathcal{L})>0} A_n(\mathcal{L})w_n(\mathcal{L}) \le Ck(\ln R)^{2k}e^{(4k)/\rho} \sum_{\substack{1\le b_1<\ldots<b_k\\ qb_k\le \eta y\\ \mathcal{L}=\mathcal{L}(\mathbf{b})\in \mathrm{AS}}} \sum_{n\in\mathcal{A}(x):\ A_n(\mathcal{L})>0} 1.$$

We have (see assertions (1)–(3), (i), and (ii) at the beginning of the proof)

$$\sum_{\substack{1\le b_1<\ldots<b_k\\ qb_k\le \eta y\\ \mathcal{L}=\mathcal{L}(\mathbf{b})\in \mathrm{AS}}} \sum_{n\in\mathcal{A}(x):\ A_n(\mathcal{L})>0} 1$$

$$\le \#\big\{x\le n<2x\colon\ \exists p_j,p_{j+1},\ldots,p_{j+m}\in[qn+1,qn+2\eta y],\ p_j,p_{j+1},\ldots,p_{j+m}\equiv a \ (\mathrm{mod}\ q)\big\}$$
$$\le \#\big\{x\le n<2x\colon\ \exists p_j,p_{j+1},\ldots,p_{j+m}\in[qn+1,qn+y],\ p_j,p_{j+1},\ldots,p_{j+m}\equiv a \ (\mathrm{mod}\ q)\big\} := N_1.$$

Hence,

$$S\le Ck(\ln R)^{2k}e^{(4k)/\rho}N_1.$$

Since ρ lies in the interval (5.68), we have

$$\frac{4k}{\rho}\le \frac{8k^4\ln k}{c_3}=c_4k^4\ln k,$$

where $c_4>0$ is an absolute constant. Since $\ln R=(1/9)\ln x$, it follows that

$$Ck(\ln R)^{2k}e^{(4k)/\rho}\le C\frac{k}{9^{2k}}\exp(c_4k^4\ln k)(\ln x)^{2k}\le \exp(k^5)(\ln x)^{2k}$$

provided that $\widetilde{C}_0$ is chosen large enough. Hence,

$$S\le \exp(k^5)(\ln x)^{2k}N_1. \tag{5.83}$$

From (5.82) and (5.83) we obtain

$$N_1\ge x\Big(\frac{y}{\ln x}\Big)^k\Big(\frac{1}{q\ln\ln(q+2)}\Big)^k\exp(-3k^5). \tag{5.84}$$

We define

$$\Omega_1=\big\{x\le n\le 2x-1\colon\ \exists p_j,p_{j+1},\ldots,p_{j+m}\in[qn+1,qn+y],\ p_j,p_{j+1},\ldots,p_{j+m}\equiv a\ (\mathrm{mod}\ q)\big\},$$
$$\Omega_2=\big\{qx+1\le p_n\le q(2x-1)+y\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\big\}$$

and put $N_2=\#\Omega_2$. Since x is a positive integer, we have $N_1=\#\Omega_1$. Let us show that

$$N_1\le(\lceil y\rceil+1)N_2. \tag{5.85}$$

Let $n\in\Omega_1$. Then there are at least $m+1$ consecutive primes all congruent to a (mod q) in the interval $[qn+1,qn+y]$. Let p be the first of them. Then $p\in\Omega_2$. We put

$$\Lambda=\big\{j\in\mathbb{Z}\colon\ qj+1\le p\le qj+y\big\}$$

and claim that

$$\#\Lambda\le\lceil y\rceil+1. \tag{5.86}$$

Let $I_j = [qj+1, qj+y]$, $j\in\mathbb{Z}$. Since $p\in I_n$, we have $\Lambda\ne\varnothing$. Let l be the minimal element in Λ. We put $t=\lceil y\rceil+1$. Then $t>y$ and

$$q(l+t)+1>q(l+t)=ql+qt\ge ql+t>ql+y\ge p.$$

Hence, $p\notin I_j$ for $j\ge l+t$ and $j\le l-1$. We obtain $\#\Lambda\le t$. Thus, (5.86) is proved; (5.85) follows from (5.86). We have $\lceil y\rceil+1\le y+2\le 2y$ provided that $c(\varepsilon)$ is chosen large enough. Since

$$N_2\le\#\bigl\{qx+1\le p_n\le 2qx+y\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\bigr\}=:N_3,$$

we obtain (see (5.84))

$$N_3\ge\frac{1}{2}\frac{x}{y}\Bigl(\frac{y}{\ln x}\Bigr)^k\biggl(\frac{1}{q\ln\ln(q+2)}\biggr)^k\exp(-3k^5). \tag{5.87}$$

We put

$$N_4=\#\bigl\{qx<p_n\le 2qx-5q\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\bigr\}, \tag{5.88}$$
$$N_5=\#\bigl\{2qx-5q<p_n\le 2qx+y\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\bigr\}.$$

Then

$$N_3=N_4+N_5. \tag{5.89}$$

Since $q\le y$, we have

$$N_5\le 5q+[y]\le 5q+y\le 6y. \tag{5.90}$$

Let us show that

$$y\le\frac{1}{24}\frac{x}{y}\Bigl(\frac{y}{\ln x}\Bigr)^k\biggl(\frac{1}{q\ln\ln(q+2)}\biggr)^k\exp(-3k^5):=T_1. \tag{5.91}$$

Since $q\le y^{1-\varepsilon}\le y$ and $k\le y^{\varepsilon/14}$, we have

$$T_1\ge\frac{1}{24}\frac{x}{y}\biggl(\frac{y^\varepsilon}{\ln x\ln\ln(y+2)}\biggr)^k\exp\bigl(-3y^{5\varepsilon/14}\bigr).$$

Therefore, to prove (5.91), it suffices to show that

$$y\le\frac{1}{24}\frac{x}{y}\biggl(\frac{y^\varepsilon}{\ln x\ln\ln(y+2)}\biggr)^k\exp\bigl(-3y^{5\varepsilon/14}\bigr).$$

Taking logarithms, we obtain

$$\ln y\le-\ln 24+\ln x-\ln y+k\bigl(\varepsilon\ln y-\ln\ln x-\ln\ln\ln(y+2)\bigr)-3y^{5\varepsilon/14}$$

or, which is equivalent,

$$T_2:=2\ln y+\ln 24-\varepsilon k\ln y+k\ln\ln x+k\ln\ln\ln(y+2)+3y^{5\varepsilon/14}\le\ln x.$$

Since $y\le\ln x$ and $0<\varepsilon<1$, we have $k\le(\ln x)^{\varepsilon/14}\le(\ln x)^{1/14}$. Then

$$T_2\le 2\ln\ln x+\ln 24+(\ln x)^{1/14}\ln\ln x+(\ln x)^{1/14}\ln\ln\ln(\ln x+2)+3(\ln x)^{5/14}\le\ln x$$

provided that $c(\varepsilon)$ is chosen large enough. Thus, (5.91) is proved. From (5.90) and (5.91) it follows that

$$N_5\le\frac{1}{4}\frac{x}{y}\Bigl(\frac{y}{\ln x}\Bigr)^k\biggl(\frac{1}{q\ln\ln(q+2)}\biggr)^k\exp(-3k^5). \tag{5.92}$$

Applying (5.87), (5.89), and (5.92), we obtain

$$N_4 \geq \frac{1}{4}\frac{x}{y}\Big(\frac{y}{\ln x}\Big)^k \left(\frac{1}{q\ln\ln(q+2)}\right)^k \exp(-3k^5) =: T_3. \tag{5.93}$$

We have (see (2.2))

$$\pi(2qx) \leq c_1\frac{2qx}{\ln(2qx)} \leq c_1\frac{2qx}{\ln x} = c_2\frac{qx}{\ln x},$$

where $c_1 > 0$ and $c_2 = 2c_1 > 0$ are absolute constants. Therefore,

$$\begin{aligned} T_3 &= \frac{qx}{\ln x}\frac{\ln x}{qx}\frac{1}{4}\frac{x}{y}\Big(\frac{y}{\ln x}\Big)^k \left(\frac{1}{q\ln\ln(q+2)}\right)^k \exp(-3k^5) \\ &\geq \frac{1}{4c_2}\,\pi(2qx)\Big(\frac{y}{\ln x}\Big)^{k-1}\frac{1}{q^{k+1}(\ln\ln(q+2))^k}\exp(-3k^5). \end{aligned}$$

Using the inequality $\ln(1+x) \leq x$, $x > 0$, we obtain $\ln\ln(q+2) \leq \ln(1+q) \leq q$. Hence,

$$\frac{1}{q^{k+1}(\ln\ln(q+2))^k} \geq \frac{1}{q^{2k+1}} \geq \frac{1}{q^{3k^5}}.$$

We can assume that $4c_2 \leq 2^{3k^5}$ if $\widetilde{C}_0$ is chosen large enough. We have

$$T_3 \geq \pi(2qx)\Big(\frac{y}{\ln x}\Big)^{k-1}\frac{1}{(2eq)^{3k^5}}.$$

We can also assume that $3k^5 \leq k^6$ if $\widetilde{C}_0$ is chosen large enough. Hence,

$$\frac{1}{(2eq)^{3k^5}} \geq \frac{1}{(2eq)^{k^6}}.$$

We have $(2e)^{k^6} \leq 2^{k^7}$ if $\widetilde{C}_0$ is chosen large enough. It is clear that $q^{k^6} \leq q^{k^7}$. Then

$$\frac{1}{(2eq)^{k^6}} \geq \frac{1}{(2q)^{k^7}}.$$

Further (see (5.61)),

$$0 < \frac{y}{\ln x} \leq 1 \qquad \Rightarrow \qquad \Big(\frac{y}{\ln x}\Big)^{k-1} \geq \Big(\frac{y}{\ln x}\Big)^{k^7}.$$

We obtain

$$T_3 \geq \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{k^7}.$$

From (5.75) and (5.76) we find

$$k = \lceil\exp(\widetilde{c}m)\rceil \leq \exp(\widetilde{c}m) + 1 \leq \exp(2\widetilde{c}m) \tag{5.94}$$

provided that $\widetilde{C}_0$ is chosen large enough. Therefore, $k^7 \leq \exp(14\widetilde{c}m)$. Since $\widetilde{C}_0$ is a positive absolute constant, we see from (5.75) that $\widetilde{c}$ is a positive absolute constant. We have

$$T_3 \geq \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(14\widetilde{c}m)} = \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(Cm)}, \tag{5.95}$$

where $C = 14\widetilde{c} > 0$ is an absolute constant. Combining (5.88), (5.93), and (5.95) we obtain

$$\#\{qx < p_n \le 2qx - 5q\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\}$$
$$\ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(Cm)}.$$

Applying (5.94), we see that the inequality $k \le y^{\varepsilon/14}$ holds if $\exp(2\widetilde{c}m) \le y^{\varepsilon/14}$. This inequality is equivalent to

$$m \le \frac{\varepsilon}{28\widetilde{c}}\ln y = c\varepsilon\ln y,$$

where $c = 1/(28\widetilde{c}) > 0$ is an absolute constant. Let us redenote $c(\varepsilon)$ by $c_0(\varepsilon)$. Lemma 5.4 is proved. □

Lemma 5.5. *There are positive absolute constants c and C such that the following holds. Let ε be a real number with $0 < \varepsilon < 1$. Then there is a number $c_0(\varepsilon) > 0$, depending only on ε, such that if $x \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ satisfy the conditions $c_0(\varepsilon) \le y \le \ln x$, $1 \le m \le c\varepsilon\ln y$, $1 \le q \le y^{1-\varepsilon}$, and $(a,q) = 1$, then*

$$\#\{qx < p_n \le 2qx\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\} \ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(Cm)}.$$

Proof. Let c, C, and $c_0(\varepsilon)$ be the quantities mentioned in Lemma 5.4. We will choose a quantity $\widetilde{c}_0(\varepsilon)$ and an absolute constant $\widetilde{C}$ later; they will be large enough. In particular, let $\widetilde{c}_0(\varepsilon) \ge c_0(\varepsilon)$ and $\widetilde{C} \ge C$. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ be such that $\widetilde{c}_0(\varepsilon) \le y \le \ln x$, $1 \le m \le c\varepsilon\ln y$, $1 \le q \le y^{1-\varepsilon}$, and $(a,q) = 1$. We put $l = \lceil x\rceil$. Then, by Lemma 5.4, we have

$$N_1 = \#\{ql < p_n \le 2ql - 5q\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\}$$
$$\ge \pi(2ql)\left(\frac{y}{2q\ln l}\right)^{\exp(Cm)} =: T_1. \tag{5.96}$$

Since $x \le l < x + 1$, we have

$$ql \ge qx \qquad \text{and} \qquad 2ql - 5q \le 2q(x+1) - 5q = 2qx - 3q < 2qx.$$

Therefore,

$$N_1 \le \#\{qx < p_n \le 2qx\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\} =: N_2. \tag{5.97}$$

We have $x + 1 \le x^2$ provided that $\widetilde{c}_0(\varepsilon)$ is chosen large enough. Hence,

$$\ln l \le \ln(x+1) \le 2\ln x.$$

Since $\pi(2ql) \ge \pi(2qx)$, we have

$$T_1 \ge \pi(2qx)\left(\frac{y}{4q\ln x}\right)^{\exp(Cm)} = \pi(2qx)\left(\frac{y}{q\ln x}\right)^{\exp(Cm)}\left(\frac{1}{4}\right)^{\exp(Cm)}.$$

Then

$$2\exp(\widetilde{C}m) \le \exp(2\widetilde{C}m)$$

provided that $\widetilde{C}$ is chosen large enough. Since $\widetilde{C} \ge C$, we have

$$\left(\frac{1}{4}\right)^{\exp(Cm)} \ge \left(\frac{1}{4}\right)^{\exp(\widetilde{C}m)} = \left(\frac{1}{2}\right)^{2\exp(\widetilde{C}m)} \ge \left(\frac{1}{2}\right)^{\exp(2\widetilde{C}m)}.$$

Further,

$$0 < \frac{y}{q\ln x} \le 1 \qquad \Rightarrow \qquad \left(\frac{y}{q\ln x}\right)^{\exp(Cm)} \ge \left(\frac{y}{q\ln x}\right)^{\exp(2\widetilde{C}m)}.$$

Hence,

$$T_1 \ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(2\widetilde{C}m)}. \tag{5.98}$$

From (5.96)–(5.98) we obtain

$$\#\{qx < p_n \le 2qx\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le y\} \ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(2\widetilde{C}m)}.$$

Let us denote $\widetilde{c}_0(\varepsilon)$ by $c_0(\varepsilon)$ and $2\widetilde{C}$ by C. Lemma 5.5 is proved. □

Let us complete the proof of Theorem 1.1. Let $c_0(\varepsilon)$, c, C be the quantities in Lemma 5.5. We will choose a quantity $\widetilde{c}_0(\varepsilon)$ and an absolute constant $\widetilde{C}$ later; they will be large enough. Let $\widetilde{c}_0(\varepsilon) \ge c_0(\varepsilon)$ and $\widetilde{C} \ge C$.

Let us prove the following statement.

Proposition 5.2. *Let ε be a real number with $0 < \varepsilon < 1$. Let $t \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ be such that*

$$t \ge 100, \qquad \widetilde{c}_0(\varepsilon) \le y \le \ln\frac{t}{2\ln t}, \qquad 1 \le m \le c\varepsilon\ln y, \qquad 1 \le q \le y^{1-\varepsilon}, \qquad (a,q) = 1.$$

Then

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le y\right\} \ge \pi(t)\left(\frac{y}{2q\ln t}\right)^{\exp(\widetilde{C}m)}. \tag{5.99}$$

Proof. Indeed, since $t \ge 100$, we have $2\ln t \ge 1$. Hence,

$$y \le \ln\frac{t}{2\ln t} \le \ln t.$$

We have $q \le y^{1-\varepsilon} \le y \le \ln t$. Therefore,

$$y \le \ln\frac{t}{2\ln t} \le \ln\frac{t}{2q}.$$

We put $x = t/2q$. Then $x \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ are such that

$$c_0(\varepsilon) \le y \le \ln x, \qquad 1 \le m \le c\varepsilon\ln y, \qquad 1 \le q \le y^{1-\varepsilon}, \qquad (a,q) = 1.$$

By Lemma 5.5, we have

$$\#\{qx < p_n \le 2qx\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le y\}$$
$$\ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(Cm)} \ge \pi(2qx)\left(\frac{y}{2q\ln x}\right)^{\exp(\widetilde{C}m)} \ge \pi(2qx)\left(\frac{y}{2q\ln(2qx)}\right)^{\exp(\widetilde{C}m)}.$$

Returning to the variable t, we obtain (5.99). □

Let us prove the following statement.

Proposition 5.3. *Let ε be a real number with $0 < \varepsilon < 1$. Let $t \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ be such that*

$$t \ge 100, \qquad \widetilde{c}_0\left(\frac{\varepsilon}{2}\right) \le \ln\frac{t}{2\ln t}, \qquad 1 \le m \le \frac{c}{4}\varepsilon\ln\ln t, \qquad 1 \le q \le (\ln t)^{1-\varepsilon}, \qquad (a,q) = 1.$$

Then

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le \ln\frac{t}{2\ln t}\right\} \ge \pi(t)\left(\frac{1}{4q}\right)^{\exp(\widetilde{C}m)}.$$

Proof. We need the following

Lemma 5.6. *Let t be a real number with $t \ge 100$. Then*

$$2\ln t \le \sqrt{t}, \qquad \ln\frac{t}{2\ln t} \ge \frac{1}{2}\ln t, \qquad \ln\ln\frac{t}{2\ln t} \ge \frac{1}{2}\ln\ln t, \qquad 1 - \frac{\ln(2\ln t)}{\ln t} \ge \frac{1}{2}.$$

The proof of lemma 5.6 is a simple exercise in calculus, and we omit it.

We put

$$y = \ln\frac{t}{2\ln t}.$$

Since $t \ge 100$, we have (see Lemma 5.6)

$$\ln y = \ln\ln\frac{t}{2\ln t} \ge \frac{1}{2}\ln\ln t.$$

Therefore,

$$1 \le m \le c\frac{\varepsilon}{2}\ln y.$$

We may assume that $\widetilde{c}_0(\varepsilon) \ge 2^{1/\varepsilon}$. Since $t \ge 100$, we have $t/(2\ln t) \le t$ and

$$\widetilde{c}_0\left(\frac{\varepsilon}{2}\right) \le \ln\frac{t}{2\ln t} \le \ln t.$$

Hence,

$$t \ge \exp\left(\widetilde{c}_0\left(\frac{\varepsilon}{2}\right)\right) \ge \exp\left(2^{2/\varepsilon}\right).$$

Therefore,

$$\frac{1}{2}(\ln t)^{1-\varepsilon/2} \ge (\ln t)^{1-\varepsilon}. \tag{5.100}$$

From (5.100) and the last inequality in Lemma 5.6 we find

$$y^{1-\varepsilon/2} = \left(\ln\frac{t}{2\ln t}\right)^{1-\varepsilon/2} = (\ln t)^{1-\varepsilon/2}\left(1 - \frac{\ln(2\ln t)}{\ln t}\right)^{1-\varepsilon/2} \ge (\ln t)^{1-\varepsilon/2}\left(\frac{1}{2}\right)^{1-\varepsilon/2}$$
$$\ge \frac{1}{2}(\ln t)^{1-\varepsilon/2} \ge (\ln t)^{1-\varepsilon}.$$

Since $1 \le q \le (\ln t)^{1-\varepsilon}$, we have $1 \le q \le y^{1-\varepsilon/2}$. Applying Proposition 5.2 with $\varepsilon/2$ and the second inequality of Lemma 5.6, we have

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le \ln\frac{t}{2\ln t}\right\}$$
$$\ge \pi(t)\left(\frac{\ln(t/(2\ln t))}{2q\ln t}\right)^{\exp(\widetilde{C}m)} \ge \pi(t)\left(\frac{1}{4q}\right)^{\exp(\widetilde{C}m)}.$$

The statement is proved. □

Proposition 5.4. *Let ε be a real number with $0<\varepsilon<1$. Let $t\in\mathbb{R}$, $y\in\mathbb{R}$, $m\in\mathbb{Z}$, $q\in\mathbb{Z}$, and $a\in\mathbb{Z}$ be such that*

$$t\ge 100,\qquad \widetilde{c}_0\Big(\frac{\varepsilon}{2}\Big)\le \ln\frac{t}{2\ln t}\le y\le \ln t,\qquad 1\le m\le \frac{c}{4}\varepsilon\ln y,\qquad 1\le q\le y^{1-\varepsilon},\qquad (a,q)=1.$$

Then

$$\#\Big\{\frac{t}{2}<p_n\le t\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\Big\}\ge \pi(t)\Big(\frac{y}{4q\ln t}\Big)^{\exp(\widetilde{C}m)}.$$

Proof. Since $y\le\ln t$, we have

$$1\le m\le \frac{c}{4}\varepsilon\ln\ln t\qquad\text{and}\qquad 1\le q\le(\ln t)^{1-\varepsilon}.$$

Applying Proposition 5.3, we obtain

$$\begin{aligned}\#\Big\{\frac{t}{2}<p_n\le t\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\Big\}&\\ \ge\#\Big\{\frac{t}{2}<p_n\le t\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le \ln\frac{t}{2\ln t}\Big\}&\\ \ge\pi(t)\Big(\frac{1}{4q}\Big)^{\exp(\widetilde{C}m)}\ge\pi(t)\Big(\frac{y}{4q\ln t}\Big)^{\exp(\widetilde{C}m)}.&\qquad\square\end{aligned}$$

For $0<\varepsilon<1$ we define the quantity $t_0(\varepsilon)$ as follows:

$$t_0(\varepsilon)\ge 100,\qquad \ln\frac{t}{2\ln t}\ge\max\Big\{\widetilde{c}_0\Big(\frac{\varepsilon}{2}\Big),\widetilde{c}_0(\varepsilon)\Big\}\quad\text{for any}\quad t\ge t_0(\varepsilon).$$

Let us prove the following statement.

Proposition 5.5. *Let ε be a real number with $0<\varepsilon<1$. Let $t\in\mathbb{R}$, $y\in\mathbb{R}$, $m\in\mathbb{Z}$, $q\in\mathbb{Z}$, and $a\in\mathbb{Z}$ be such that*

$$t\ge t_0(\varepsilon),\qquad \max\Big\{\widetilde{c}_0\Big(\frac{\varepsilon}{2}\Big),\widetilde{c}_0(\varepsilon)\Big\}\le y\le\ln t,\qquad 1\le m\le\frac{c}{4}\varepsilon\ln y,\qquad 1\le q\le y^{1-\varepsilon},\qquad (a,q)=1.$$

Then

$$\#\Big\{\frac{t}{2}<p_n\le t\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\Big\}\ge \pi(t)\Big(\frac{y}{4q\ln t}\Big)^{\exp(\widetilde{C}m)}.$$

Proof. Consider two cases. If

$$\ln\frac{t}{2\ln t}<y\le\ln t,$$

then t, y, m, q, and a satisfy the hypothesis of Proposition 5.4, which yields

$$\#\Big\{\frac{t}{2}<p_n\le t\colon\ p_n\equiv\ldots\equiv p_{n+m}\equiv a\ (\mathrm{mod}\ q),\ p_{n+m}-p_n\le y\Big\}\ge \pi(t)\Big(\frac{y}{4q\ln t}\Big)^{\exp(\widetilde{C}m)}.$$

If

$$y\le\ln\frac{t}{2\ln t},$$

then t, y, m, q, and a satisfy the hypothesis of Proposition 5.2, which yields

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\right\} \ge \pi(t)\left(\frac{y}{2q\ln t}\right)^{\exp(\widetilde{C}m)} \ge \pi(t)\left(\frac{y}{4q\ln t}\right)^{\exp(\widetilde{C}m)}. \qquad \square$$

For $0 < \varepsilon < 1$ we put

$$\rho(\varepsilon) = \max\left\{\widetilde{c}_0\left(\frac{\varepsilon}{2}\right), \widetilde{c}_0(\varepsilon)\right\} + t_0(\varepsilon).$$

Let us prove the following statement.

Proposition 5.6. *Let ε be a real number with $0 < \varepsilon < 1$. Let $t \in \mathbb{R}$, $y \in \mathbb{R}$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, and $a \in \mathbb{Z}$ be such that*

$$\rho(\varepsilon) \le y \le \ln t, \qquad 1 \le m \le \frac{c}{4}\varepsilon \ln y, \qquad 1 \le q \le y^{1-\varepsilon}, \qquad (a, q) = 1.$$

Then

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ p_{n+m} - p_n \le y\right\} \ge \pi(t)\left(\frac{y}{2q\ln t}\right)^{\exp(2\widetilde{C}m)}.$$

Proof. We have

$$\max\left\{\widetilde{c}_0\left(\frac{\varepsilon}{2}\right), \widetilde{c}_0(\varepsilon)\right\} \le y \le \ln t \qquad \text{and} \qquad t \ge \exp(\rho(\varepsilon)) \ge \rho(\varepsilon) \ge t_0(\varepsilon).$$

Applying Proposition 5.5, we obtain

$$\#\left\{\frac{t}{2} < p_n \le t\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \ (\mathrm{mod}\ q),\ \ p_{n+m} - p_n \le y\right\} \ge \pi(t)\left(\frac{y}{4q\ln t}\right)^{\exp(\widetilde{C}m)}. \tag{5.101}$$

We may assume that $\widetilde{C} \ge 2$. Therefore, $\exp(\widetilde{C}m) \ge \widetilde{C}m \ge \widetilde{C} \ge 2$. Hence, $2\exp(\widetilde{C}m) \le \exp(2\widetilde{C}m)$. We have

$$\left(\frac{1}{4}\right)^{\exp(\widetilde{C}m)} = \left(\frac{1}{2}\right)^{2\exp(\widetilde{C}m)} \ge \left(\frac{1}{2}\right)^{\exp(2\widetilde{C}m)}.$$

Further,

$$0 < \frac{y}{q\ln t} \le 1 \qquad \Rightarrow \qquad \left(\frac{y}{q\ln t}\right)^{\exp(\widetilde{C}m)} \ge \left(\frac{y}{q\ln t}\right)^{\exp(2\widetilde{C}m)}.$$

We obtain

$$\left(\frac{y}{4q\ln t}\right)^{\exp(\widetilde{C}m)} \ge \left(\frac{y}{2q\ln t}\right)^{\exp(2\widetilde{C}m)}. \tag{5.102}$$

Relations (5.101) and (5.102) imply the required assertion. $\square$

Let us denote $\rho(\varepsilon)$ by $c_0(\varepsilon)$, $c/4$ by c, and $2\widetilde{C}$ by C. Theorem 1.1 is proved. $\square$

Proof of Corollary 1.1. Let $c_0(\varepsilon)$, c, and C be the quantities in Theorem 1.1. We put

$$C_1 = \max\left\{\frac{2}{c}, c_0\left(\frac{1}{2}\right), C\right\}.$$

Let m be a positive integer. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be such that $\exp(C_1 m) \le y \le \ln x$. Then

$$y \ge \exp(C_1 m) \ge C_1 m \ge C_1 \ge c_0\left(\frac{1}{2}\right) \qquad \text{and} \qquad y \ge \exp(C_1 m) \ge \exp\left(\frac{2}{c} m\right).$$

The last inequality implies

$$m \le c\,\frac{1}{2}\ln y.$$

Putting $q = 1$ and $a = 1$, we have

$$c_0\left(\frac{1}{2}\right) \le y \le \ln x, \qquad 1 \le m \le c\,\frac{1}{2}\ln y, \qquad 1 \le q \le y^{1/2}, \qquad (a,q) = 1.$$

Applying Theorem 1.1 with $\varepsilon = 1/2$, we see that

$$\begin{aligned}
\#\Big\{\frac{x}{2} < p_n \le x\colon\ p_{n+m} - p_n \le y\Big\} \\
&= \#\Big\{\frac{x}{2} < p_n \le x\colon\ p_n \equiv \ldots \equiv p_{n+m} \equiv a \pmod q,\ p_{n+m} - p_n \le y\Big\} \\
&\ge \pi(x)\left(\frac{y}{2q\ln x}\right)^{\exp(Cm)} = \pi(x)\left(\frac{y}{2\ln x}\right)^{\exp(Cm)} \ge \pi(x)\left(\frac{y}{2\ln x}\right)^{\exp(C_1 m)}.
\end{aligned}$$

Let us redenote C_1 by C. Corollary 1.1 is proved. □

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