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A Trajectory Minimizing the Exposure of a Moving Object

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Abstract—A corridor Y for the motion of an object is given in the space $X = \mathbb{R}^N$ $(N = 2, 3)$. A finite number of emitters s_i with fixed convex radiation cones $K(s_i)$ are located outside the corridor. The intensity of radiation $F(y)$, $y > 0$, satisfies the condition $F(y) \geq \lambda F(\lambda y)$ for $y > 0$ and $\lambda > 1$. It is required to find a trajectory minimizing the value

$$J(\mathcal{T}) = \sum_i \int_0^1 F\big(\|s_i - t(\tau)\|\big)\, d\tau$$

in the class of uniform motion trajectories $\mathcal{T} = \big\{t(\tau)\colon 0 \leq \tau \leq 1,\ t(0) = t_*,\ t(1) = t^*\big\} \subset Y$, $t_*, t^* \in \partial Y$, $t_* \neq t^*$. We propose methods for the approximate construction of optimal trajectories in the case where the multiplicity of covering the corridor Y with the cones $K(s_i)$ is at most 2.

Keywords: navigation, optimal trajectory, irradiation, moving object.

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1. INTRODUCTION

We study the problem of constructing a route such that an object moving at a constant speed along this route would be exposed to the lowest total dose of radiation from several emitters. The problem is considered on the plane and in the space $\mathbb{R}^3$. The route must connect fixed start and end points and not go beyond a given corridor Y. The boundary ∂Y of the corridor is a closed continuous line in $\mathbb{R}^2$ or a closed continuous surface in $\mathbb{R}^3$; more exactly, we analyze the case where the boundary ∂Y is homeomorphic to a sphere. The emitters are located outside the corridor and are directional; i.e., each of them radiates into its own convex cone (*radiation cone*) with a vertex at the point where the emitter is located. We consider emitters whose radiation cones completely cover the corridor Y. We assume that the emitters are "single-type" in the sense that they have the same dependence of the decrease in radiation intensity on the distance to the emitter. Our study is limited to the case where the multiplicity of covering the corridor Y by the radiation cones of the emitters is at most 2.

The results of this paper can be used, for example, for planning optimal paths of autonomous mobile robots in an environment with obstacles [1].

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An object t moves at a constant speed in the corridor $Y \subset \mathbb{R}^N$ $(N = 2, 3)$ from an initial point t_* to a final point t^* $(t_*, t^* \in \partial Y,\ t_* \neq t^*)$, and $\mathbb{T}$ is the family of trajectories

$$\mathcal{T} = \big\{ t(\tau)\colon 0 \le \tau \le 1,\ t(0) = t_*,\ t(1) = t^* \big\} \subset Y.$$

The boundary ∂Y of the corridor Y is homeomorphic to a sphere. Fixed emitting sources s_i $(i = 1, \dots, n)$ are located outside the corridor. Radiation is emitted along rays l from a given fixed convex open cone $K_i = K(s_i) = \{l\}$ with vertex s_i such that $t_* \notin K_i$, $t^* \notin K_i$. Each cone $K(s_i)$ satisfies the condition

$$K(s_i) \cap \mathcal{T} \neq \varnothing \quad \forall\, \mathcal{T} \in \mathbb{T}.$$

By $K_Y(s_i)$, we denote the connected subset of $K(s_i) \cap Y$ nearest to s_i and satisfying the above condition.

Two options are possible: radiation is either transmitted or not transmitted through the boundary ∂Y of the corridor. For definiteness, we assume first that the boundary of the corridor prevents the propagation of radiation. In this case, we denote by $K_V(s)$ the set of points of the set $K_Y(s)$ that are irradiated (visible) from the source s. The radiation intensity of each source $s = s_i$ is the same along all rays from $K(s_i)$ and is characterized by a positive decreasing function $\widetilde{F}_i(y)$ of the real variable $y > 0$. Define

$$F_i(x) = \begin{cases} \widetilde{F}_i(\|s_i - x\|) & \text{for } \ x \in l \in K(s_i), \\ 0 & \text{for } \ x \notin K(s_i). \end{cases}$$

Let

$$J(\mathcal{T}) = \sum_{i=1}^{n} \int_0^1 F_i(t(\tau))\, d\tau$$

be the total radiation received by the object when it moves along the trajectory $\mathcal{T}$, where n is the number of emitters.

In this paper, we consider the problem of constructing a trajectory $\widehat{\mathcal{T}}$ that provides the infimum

$$J = \inf\{J(\mathcal{T})\colon \mathcal{T} \in \mathbb{T}\}. \tag{1}$$

It is assumed that the multiplicity of covering the corridor Y by the cones $K(s_i)$ is at most two. This is a problem in the calculus of variations with nonsmooth constraints (if the boundary ∂Y is not smooth) complicated by the presence of several radiation sources. Obviously, it is possible to formulate problem (1) as an optimal control problem of the motion of the object under state constraints [2]. We propose geometric methods for finding an optimal or close to optimal trajectory depending on the mutual arrangement of the cones $K(s_i)$. The problem of the motion of an object in a radioactive medium was considered in another formulation in [3, 4].

It is natural to assume that the function $\widetilde{F}$ not only decreases but also has the following property:

$$\int_{C_\rho} F(x)\ dx > \int_{C_{\lambda\rho}} F(x)\, dx \qquad \forall\, \lambda > 1 \tag{2}$$

for any $\rho > 0$, where $C_\rho = \{x \in K(s)\colon \|s - x\| = \rho\}$. This property, in particular, is possessed by a function satisfying the condition

$$\widetilde{F}(\rho) > \lambda \widetilde{F}(\lambda\rho) \ \text{ for } \ \rho > 0,\ \lambda > 1. \tag{3}$$

Indeed,

$$\int\limits_{C_\rho} F(x)\,dx = \widetilde{F}(\rho)|C_\rho| > \lambda\widetilde{F}(\lambda\rho)|C_\rho| = \widetilde{F}(\lambda\rho)|C_{\lambda\rho}| = \int\limits_{C_{\lambda\rho}} F(x)\,dx,$$

where $|C_\rho|$ is the area (arc length) of C_ρ.

In the case of $\mathbb{R}^2$, we observe the following relations for the function $\widetilde{F}$ satisfying condition (3):

- let $[a,b]$ and $[\lambda a, \lambda b]$ $(\lambda > 1)$ be segments lying on a ray $l \in K(s)$ with vertex $s = 0$; then

$$\int\limits_a^b F(x)\,dx > \int\limits_a^b \lambda F(\lambda x)\,dx = \int\limits_a^b F(\lambda x)\,d\lambda x = \int\limits_{\lambda a}^{\lambda b} F(x)\,dx;$$

- suppose that Φ is the opening of the cone $K = K(s)$, $\Phi < \pi$, $s = 0$ is the origin of a polar coordinate system, $\mathcal{T} \in \mathbb{T}$, the part of this trajectory $\mathcal{T}_K = \mathcal{T} \cap K_Y$ is given by the function $\rho = \rho(\varphi)$, and $\lambda\mathcal{T}_K \subset K_Y$ for $\lambda > 1$. Then

$$\int\limits_0^\Phi \widetilde{F}(\rho(\varphi))\,d\varphi > \int\limits_0^\Phi \lambda\widetilde{F}(\lambda\rho(\varphi))\,d\varphi > \int\limits_0^\Phi \widetilde{F}(\lambda\rho(\varphi))\,d\varphi. \tag{4}$$

Thus, $\displaystyle\int_0^\Phi \widetilde{F}(\lambda\rho(\varphi))\,d\varphi$ as a function of λ also monotonically decreases with the growth of λ.

An example of a function satisfying condition (3) is the function

$$\widetilde{F}(\rho) = \rho^{-\alpha} \quad \text{for} \quad \alpha \geq 1.$$

2. THE CASE OF A SINGLE RADIATION SOURCE s IN $\mathbb{R}^2$

Consider the case of a single radiation source s on the plane. Then

$$J(\mathcal{T}) = \int\limits_0^1 F(t(\tau))\,d\tau.$$

Since $t_*, t^* \in \partial Y$, the boundary ∂Y is decomposed by the points t_* and t^* into two parts. Let Γ be the part of the boundary of Y opposite to s. Using the curve Γ, we define the function $\rho = \rho_\Gamma(\varphi)$, $0 \leq \varphi \leq \Phi$, in a polar coordinate system with the origin s so that the value $\rho_\Gamma(\varphi)$ is the distance from s to Γ along the ray $l_\varphi \subset K(s)$ outgoing from s at the angle φ. The function $\rho_\Gamma(\varphi)$ may have discontinuity points. Let us supplement the graph of the function $\rho_\Gamma(\varphi)$ at each discontinuity point φ with the segment $[\rho'_\varphi, \rho''_\varphi]$ $(\rho'_\varphi < \rho''_\varphi)$ of the ray l_φ so that we get a continuous curve, which is denoted by $\operatorname{gr}\rho(\cdot)$ (see Fig. 1). Note that, when the object moves along the half-open interval $(\rho'_\varphi, \rho''_\varphi] \subset l_\varphi$, it is not irradiated, since the point (ρ'_φ, φ) belongs to the boundary Γ. At first glance, it might be expected that $\operatorname{gr}\rho(\cdot)$ would be an optimal trajectory. The following examples show that the curve $\operatorname{gr}\rho(\cdot)$ is not always optimal.

Example 1. Let Γ be the graph of the function $\rho = \rho(\varphi) = 1 + \varepsilon \sin k\varphi$ $(\varepsilon < 1/2)$ (see Fig. 2). Then, for sufficiently large k, the function ρ has a large variation $\bigvee_0^\Phi \rho$ and, hence, a large value of $J(\operatorname{gr}\rho(\cdot))$. In this case, the trajectory coinciding on $K(s)$ with an arc of the circle of radius $1 - \varepsilon$ centered at s is more preferable.

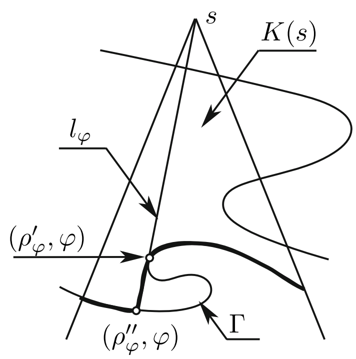

Fig. 1. Discontinuity of the function $\rho_\Gamma(\varphi)$).

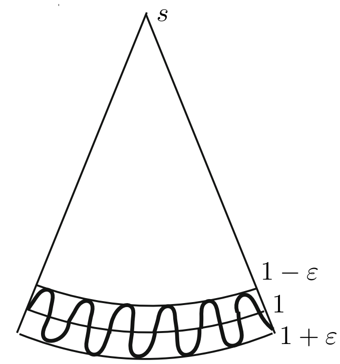

Fig. 2. Illustration of Example 1.

Example 2. For $\rho_1 < \rho_2$ and angles φ_1 and φ_2, $0 \le \varphi_1 < \varphi_2 \le \Phi < \pi$, we define a curve Γ consisting of the two arcs (ρ_1, φ), $0 \le \varphi \le \varphi_1$, and (ρ_1, φ), $\varphi_2 \le \varphi \le \Phi$, and of the two segments connecting the point $(\rho_2, (\varphi_1 + \varphi_2)/2)$ with the points (ρ_1, φ_1) and (ρ_1, φ_2). When $\rho_2 - \rho_1$ is large, the arc of the circle C_{ρ_1} is more preferable as a trajectory in comparison with Γ.

These examples and the monotonicity property of the function $\widetilde{F}$ show that the optimal trajectory must lie as far as possible from s and have a small length in small areas.

The original problem is the problem of the calculus of variations and consists in finding a continuous function $\rho(\varphi)$:

$$\inf_{\rho(\varphi)} \left\{ \int_0^\Phi \widetilde{F}(\rho(\varphi))\, d\varphi \colon \rho(\varphi) \le \rho_\Gamma(\varphi) \right\}. \tag{5}$$

The difficulty of this problem is specified by the properties of the function $\rho_\Gamma(\varphi)$.

Suppose that the function $\rho_\Gamma(\varphi)$ is piecewise monotone. Let $x = (\overline{\rho}, \overline{\varphi})$ be a local maximum point of the function $\rho_\Gamma(\varphi)$ (see Fig. 3); possibly, it belongs to an interval consisting of maximum points; i.e., there exist $\varphi' \le \varphi''$ such that

$$\rho_\Gamma(\overline{\varphi}) \ge \rho_\Gamma(\varphi) \ \text{ for } \ \varphi' \le \varphi \le \varphi'' \quad \text{ and } \quad \rho_\Gamma(\overline{\varphi}) > \min\{\rho(\varphi'), \rho(\varphi'')\}.$$

For $\rho < \overline{\rho}$ close to $\overline{\rho}$, we denote by $a = a(\rho) = a(\rho, \varphi_\rho)$ and $b = b(\rho) = b(\rho, \varphi^\rho)$ with $\varphi_\rho < \varphi^\rho$ the points of $C_\rho \cap \operatorname{gr} \rho(\cdot)$ nearest to $l_{\overline{\varphi}}$ and by $\rho^* = \rho^*_x$ the largest of the numbers ρ for which one of the points a or b is a one-sided local minimum of the function $\rho_\Gamma(\varphi)$ over the cone K. Define

$$\Delta = \Delta_x = [\varphi_{\rho^*}, \varphi^{\rho^*}].$$

If the maximum point x is not a point of strict local maximum, then this interval Δ can also be generated by other points y for which $\Delta = \Delta_y$. Let C^{ab}_ρ be the arc of the circle C_ρ between the points a and b intersecting the segment $[s, x]$, and let G^{ab}_ρ be the part of the graph $\operatorname{gr} \rho(\cdot)$ between the points a and b containing x.

It is required to solve the following problem of finding a continuous function $\gamma = \gamma(\varphi)$ on Δ that provides the minimum

$$J_{\Delta_x} = \min_\gamma \left\{ \int_\Delta \widetilde{F}(\gamma(\varphi))\, d\varphi \colon \gamma(\varphi) \le \rho_\Gamma(\varphi) \text{ for } \varphi \in \Delta,\ \gamma(\varphi_{\rho^*}) = \gamma(\varphi^{\rho^*}) = \rho^* \right\}. \tag{6}$$

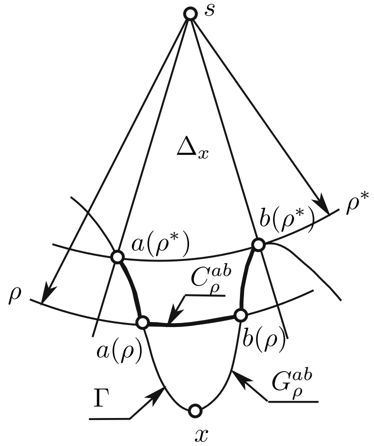

Fig. 3. Neighborhood of a local maximum of the function $\rho_\Gamma(\varphi)$).

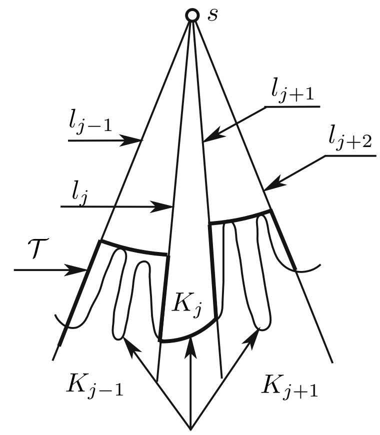

Fig. 4. Example of a trajectory close to optimal.

It is clear that the graph of the function $\widehat{\gamma} = \widehat{\gamma}(\varphi)$, which is a solution of problem (6), lies between the curves $C_{\rho^*}^{ab}$ and $G_{\rho^*}^{ab}$. Note that the function $\widehat{\gamma}$ is downward convex on those intervals $[\varphi_1, \varphi_2] \subset \Delta$, $\varphi_1 < \varphi_2$, where $\widehat{\gamma}(\varphi) < \rho_\Gamma(\varphi)$. Indeed, if this is not so, then there exist angles φ' and φ'', $\varphi_1 \le \varphi' < \varphi'' < \varphi_2$, such that the graph of the function $\widehat{\gamma}(\varphi)$, $\varphi' < \varphi < \varphi''$, lies between s and the straight line segment connecting the points $(\widehat{\gamma}(\varphi'), \varphi')$ and $(\widehat{\gamma}(\varphi''), \varphi'')$. By reflecting the graph with respect to this segment and using property (2), one can easily construct a trajectory with a lower irradiation rate in comparison with $(\widehat{\gamma}(\varphi), \varphi)$.

For each local maximum point x (specifically, for each interval consisting of local maximum points), we solve problem (6) on Δ_x.

Theorem. *Let the function $\rho_\Gamma(\varphi)$ be piecewise monotone. An extremal trajectory of problem* (5) *is given by a function $\widehat{\rho}(\varphi) = \widehat{\rho}(\varphi, s)$ $(0 \le \varphi \le \Phi)$ that coincides with the function $\widehat{\gamma}(\varphi)$ $(\varphi \in \Delta)$ on each interval $\Delta = \Delta_x$ and with $\rho_\Gamma(\varphi)$ on the set $\Phi \backslash \bigcup_x \Delta_x$.*

Proof. To prove the theorem, it suffices to observe that, due to the monotonicity of the functions F, the extremal trajectory coincides with $\rho_\Gamma(\varphi)$ on the set $\Phi \backslash \bigcup_x \Delta_x$, whereas outside this set it is constructed optimally and coincides with $\widehat{\gamma}(\varphi)$. □

Thus, the original problem (5) reduces to a series of extremal problems of form (6).

For a number ρ, $\rho^* \le \rho \le \overline{\rho}$, we define the function

$$\delta_\rho(\varphi) = \min\{\rho_\Gamma(\varphi), \rho\} \quad \text{for} \quad \varphi \in \Delta_x.$$

It is of interest to consider the simpler (in comparison with (6)) problem of finding the minimum on a narrower class of functions:

$$\overline{J}_\Delta = \min_{\rho^* \le \rho \le \overline{\rho}} \int\limits_\Delta \widetilde{F}(\delta_\rho(\varphi))\, d\varphi; \tag{7}$$

the solution of this problem approximates the function $\widehat{\gamma}(\cdot)$.

Let us mention one more method of constructing a trajectory $\mathcal{T}$ close to optimal. We divide the cone $K(s)$ into partial cones K_i bounded by rays l_j and l_{j+1}, where $l_j = l_{\varphi_j}$ $(0 = \varphi_0 < \varphi_1 < \dots < \varphi_m = \Phi)$ and $\Delta_j = \Delta_j\varphi = [\varphi_j, \varphi_{j+1})$. In the simplest case, the partition is either uniform

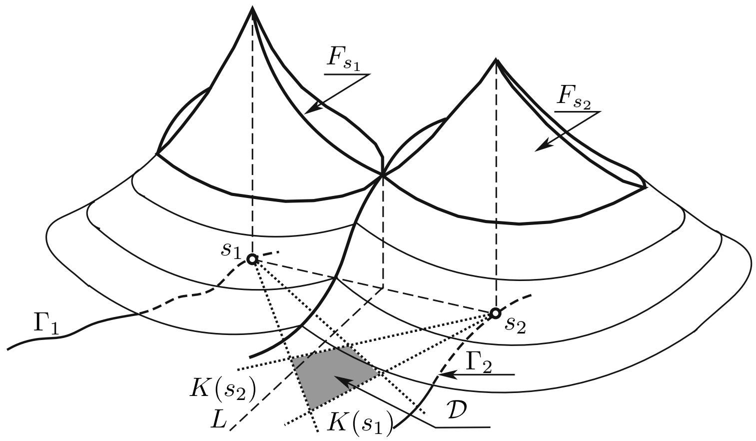

Fig. 5. The case of two sources at opposite parts of the boundary.

or is determined by the properties of Γ. We construct the trajectory $\mathcal{T}$ based on the function $t(\varphi)$, which is constant on some partial intervals Δ_j with a large variation of the function $\rho_\Gamma(\varphi)$. Without violating the constraint $\rho(\varphi) \le \rho_\Gamma(\varphi)$, due to the mentioned (see (4)) property of monotonicity of the quantity $\int \widetilde{F}(\lambda\rho(\varphi))\, d\varphi$ with respect to λ, it is natural to take a piecewise constant function on these intervals:

$$t(\varphi) = \rho_j = \min\{\rho_\Gamma(\alpha)\colon \alpha \in \Delta_j\varphi\} \quad \text{for} \quad \varphi \in \Delta_j\varphi.$$

The trajectory $\mathcal{T} \subset Y$ of the object on the chosen intervals is constructed by completing the graph of the function $t(\varphi)$ at each discontinuity point φ_j with the segment $[a_j, b_j]$ of the ray l_j, where $a_j = (\rho_j, \varphi_j)$ and $b_j = (\rho_{j-1}, \varphi_j)$. On the remaining partial intervals, the trajectory coincides with $\rho_\Gamma(\varphi)$ (Fig. 4).

3. THE CASE OF TWO SOURCES IN $\mathbb{R}^2$

The points t_* and t^* split the boundary ∂Y into two parts Γ_1 and Γ_2. In what follows, we assume that $s_i \in \Gamma_i$ $(i = 1, 2)$; i.e., the sources are located on opposite parts of ∂Y. Let

$$F_1 = F_2 \overset{\text{def}}{=} F, \qquad [s_1, s_2] \cap \big((\mathring{K}_Y(s_1) \cup \mathring{K}_Y(s_2)\big) = \varnothing$$

(see Fig. 5). Define $K_Y(s_1) \cap K_Y(s_2) = \mathcal{D}$. By l_i and l_i', we denote the "left" and "right" rays bounding the cone $K(s_i)$, $i = 1, 2$. We consider possible options for the location of the sources s_1 and s_2 and methods of constructing optimal trajectories. Let L be a straight line orthogonal to the segment $[s_1, s_2]$ containing the point $(s_1 + s_2)/2$.

If $\mathcal{D} \cap L = \varnothing$ and the sets $\mathcal{D}$ and $K_Y(s_1) \cap \Gamma_1$, for example, lie on the same side from L, then, as one can see, the optimal trajectory is $\mathcal{T}_{2,1}$, which traverses the cone $K_Y(s_2)$ along the segment $A_2 = l_1 \cap K_V(s_2)$ and the cone $K_Y(s_1)$ along the graph of the function $\rho_{\Gamma_2}\colon$ $G_1 \overset{\text{def}}{=} (\operatorname{gr}\rho_2(\cdot)) \cap K_Y(s_1)$ (in Fig. 6, these portions of the trajectory are marked with the bold dashed line).

We now assume that $\mathring{\mathcal{D}} \cap L \ne \varnothing$.

Among the trajectories not intersecting the set $\mathring{\mathcal{D}}$, there are two candidates for optimal ones; these are $\mathcal{T}_{1,2}$ and $\mathcal{T}_{2,1}$, which intersect the cones $K_Y(s_i)$ alternately. The latter trajectory is defined above, and the trajectory $\mathcal{T}_{1,2}$ intersects $K_Y(s_1)$ along the segment $A_1 = l_2 \cap K_Y(s_1)$ and

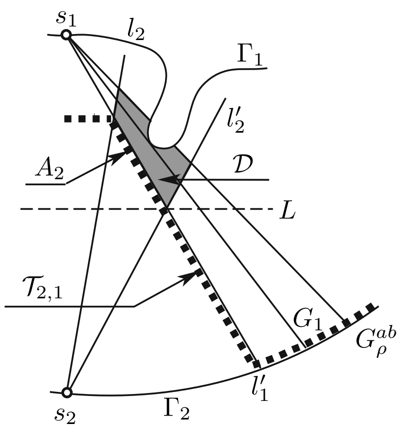

Fig. 6. The case $\mathcal{D} \cap L = \varnothing$.

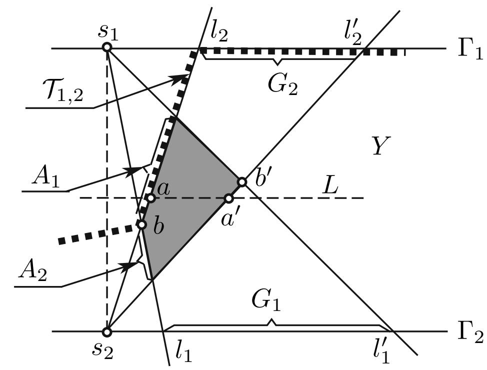

Fig. 7. The case $\mathring{\mathcal{D}} \cap L \neq \varnothing$).

the cone $K_Y(s_2)$ along the graph of the function $\rho_{\Gamma_1}\colon\ G_2 \stackrel{\text{def}}{=} (\operatorname{gr}\rho(\cdot)) \cap K_Y(s_2)$ (Fig. 7). These trajectories correspond to the irradiation values

$$J_{1,2} = \int\limits_{A_1} F(s_1 - x)\,dx + \int\limits_{G_2} F(s_2 - x)\,dx, \quad J_{2,1} = \int\limits_{A_2} F(s_2 - x)\,dx + \int\limits_{G_1} F(s_1 - x)\,dx.$$

Let us consider trajectories intersecting the set $\mathring{\mathcal{D}}$. All of them contain the segment $[a, a'] = L \cap \mathcal{D}$ designated with regard to the direction of motion from t_* to t^*. Indeed, differentiating the function $F(s_1 - x - \alpha(s_1 - s_2)) + F(s_2 - x - \alpha(s_1 - s_2))$ with respect to α, we see that the minimum over α of this function is attained at $\alpha = 0$ for all $x \in [a, a']$. The search for an optimal trajectory outside the set $\mathcal{D}$ can be implemented by solving the following *problem*.

Suppose that l is a ray from $K(s)$, η is a segment from l, c is a point from $K_Y(s)\backslash l$, and $\mathbb{T}(x, c)$ is the set of trajectories

$$\mathcal{T}_x = \{t(\tau)\colon 0 \le \tau \le 1,\ t(0) = x,\ t(1) = c\} \subset K_Y(s)$$

for $x \in \eta$. It is required to find the value

$$J(\eta, c) = \inf_{x \in \eta} \inf \left\{ \int\limits_0^1 F_s(t(\tau))\,d\tau : \mathcal{T} \in \mathbb{T}(x, c) \right\}. \tag{8}$$

Define $[q, q'] = L \cap (K_Y(s_1) \cup K_Y(s_2))$, $b = l_1 \cap l_2$, $q'_{1,2} = l'_1 \cap \Gamma_2$, and $q'_{2,1} = l'_2 \cap \Gamma_1$. It is easily seen that, to approach optimally the point a from the point t_*, the object must intersect the segment $[q, b]$ using the solution of the problem $J([q, b], a)$, and, to escape from the point a' towards the point t^*, it must intersect the segment $[q', q'_{1,2}]$ if $q' \in l'_1$ (Fig. 8a) and the interval $[q', q'_{2,1}]$ if $q' \in l'_2$ (Fig. 8b).

Lemma. *Suppose that* $s = 0$, *the points* v *and* w *belong to* $K_Y(s)$, $w \notin l = \{\alpha v,\ \alpha > 0\}$, *and*

$$J(\alpha) = \int\limits_0^1 F_s(\lambda\alpha v + (1 - \lambda)w)\,d\lambda, \quad \alpha > 0.$$

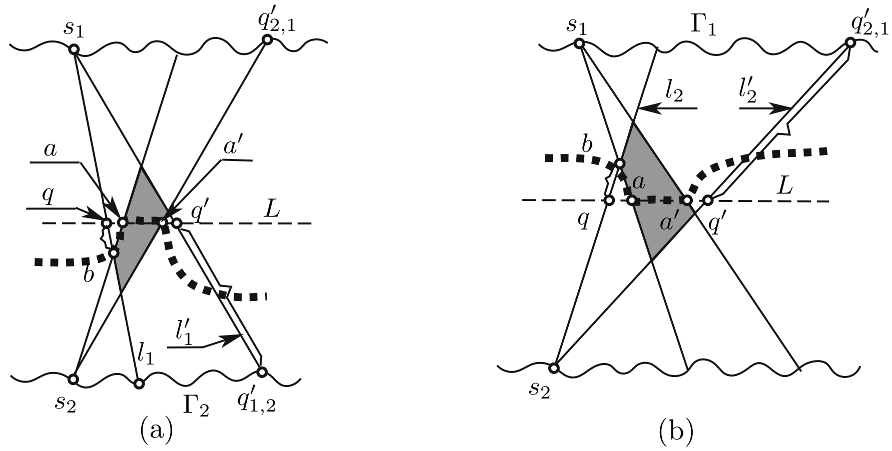

Fig. 8. (a) The case $q' \in l'_1$. (b) The case $q' \in l'_2$.

Then the function $J(\alpha)$ *is decreasing.*

Proof. Indeed,

$$\frac{1}{\Delta\alpha}\big[J(\alpha+\Delta\alpha) - J(\alpha)\big] = \frac{1}{\Delta\alpha}\int\limits_0^1 \big[F(\lambda(\alpha+\Delta\alpha)v + (1-\lambda)w) - F(\lambda\alpha v + (1-\lambda)w)\big]\, d\lambda$$

$$\xrightarrow[\Delta\alpha\to 0]{} \int\limits_0^1 F'_\alpha(\lambda\alpha v + (1-\lambda)w)\lambda\, d\lambda < 0. \qquad \square$$

Using the lemma, we conclude that one unknown part of the trajectory to the point a coincides with the segment $[a,b]$ (see Figs. 8a, 8b). To construct another part of the trajectory from the point a' to t^*, we use the following *algorithm* of finding an approximate solution of the problem $J([q,p_0],a)$ (8), where $a \in \overset{\circ}{K}(s)$, l' and l are the boundary rays with vertex s of the cone $K(s)$, $p_o \in l$, and $q \in (s,p)$. We use the notation $\int_a^b = \int_a^b F(x)\,dx$, where $[a,b]$ is a straight line segment.

Fix a finite family of rays $l_i \in K(s)$ with vertex s that intersect the interval (a,q) in order from the point q to the point a. Let $p_1 \in l_1$ be the point farthest from s such that the interval $[p_1,p_0]$ is disjoint with the interior of the set $X\backslash Y$. Define $c_i = l_i \cap [a,p_{i-1}]$. Applying the lemma (see Fig. 9), we get

$$\int\limits_a^q > \int\limits_a^{p_0} \quad \text{for} \quad w = a, \tag{9}$$

$$\left.\begin{array}{l} \displaystyle\int\limits_{c_1}^{p_0} > \int\limits_{p_0}^{p_1} \quad \text{for} \quad w = p_0, \\ \displaystyle\int\limits_a^{c_1} > \int\limits_a^{p_1} \quad \text{for} \quad w = a \end{array}\right\} \Rightarrow \int\limits_a^{p_0} > \int\limits_a^{p_1} + \int\limits_{p_1}^{p_0}. \tag{10}$$

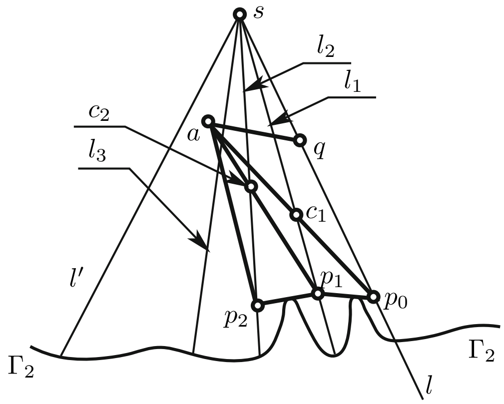

Fig. 9. Construction of an approximate solution of problem (8).

Let $p_2 \in l_2$ be the point farthest from s such that the segment $[p_1, p_2]$ is disjoint with the interior of the set $X\backslash Y$. Applying the lemma, we get

$$\left.\begin{aligned} &\int\limits_{c_2}^{p_1} > \int\limits_{p_2}^{p_1} \quad \text{for} \quad w = p_1, \\ &\int\limits_{a}^{c_2} > \int\limits_{a}^{p_2} \quad \text{for} \quad w = a \end{aligned}\right\} \Rightarrow \int\limits_{a}^{p_1} > \int\limits_{a}^{p_2} + \int\limits_{p_2}^{p_1}. \tag{11}$$

It follows from (9)–(11) that

$$\int\limits_{a}^{q} > \int\limits_{a}^{p_0} > \int\limits_{a}^{p_1} + \int\limits_{p_1}^{p_0} > \int\limits_{a}^{p_2} + \int\limits_{p_2}^{p_1} + \int\limits_{p_1}^{p_0}.$$

Thus, four trajectories are constructed: $[a, q]$, $[a, p_0]$, $[a, p_1] \cup [p_1, p_0]$, and $[a_0, p_2] \cup [p_2, p_1] \cup [p_1, p_0]$ with decreasing values of the total exposure. These trajectories connect the point a with the segment $[q, p_0]$. The trajectory construction process can be continued by using all rays from the given family $\{l_i\}$. Expanding the family $\{l_i\}$, we construct a trajectory for which the segment $[p_0, p_1] \cup [p_1, p_2] \cup \ldots \cup [p_k, p_{k+1}]$ better approximates a part of the boundary Γ_2. Moreover, we can define an infinite set of rays in $\{l_i\}$ approaching a point and construct a "converging" sequence of trajectories.

Thus, if there is a pair of sources s_1 and s_2 with nonempty intersection $\overset{\circ}{K}_Y(s_1) \cap \overset{\circ}{K}_Y(s_2)$, in order to find an optimal trajectory, one should determine to which of the considered cases the relative position of the cones $K_Y(s_i)$ belongs.

If the sources lie on the same part of the boundary ∂Y, say, on Γ_1 and $\overset{\circ}{K}_Y(s_1) \cap \overset{\circ}{K}_Y(s_2) \neq \varnothing$, then the above reasoning in the case of a single irradiator remains valid.

4. APPROXIMATE CONSTRUCTION OF OPTIMAL TRAJECTORIES IN $\mathbb{R}^3$

Consider the case of a single emitter s. Assume that the boundary ∂Y does not block radiation, i.e., does not form shadows. Assume also that $s \notin Y$, any trajectory $\mathcal{T} \in \mathbb{T}$ intersects the interior of the truncated cone $K_Y(s) = K(s) \cap Y$, and, moreover, K_Y is the connected part of the corridor Y nearest to s that has the indicated property. We will proceed from the fact that the opening of the cone $K(s)$ is less than π. For the chosen value of m, we construct two one-parameter families of planes

$$\{P^i\},\ \{Q^j\},\quad i,j \in \{1,2,\ldots,m\}.$$

Suppose that $\mathcal{L}$ and $\mathcal{R}$ are the left and right sides of the surface $Y \cap \partial K(s)$, respectively (in the sense that any trajectory from $\mathbb{T}$ enters the truncated cone $K_Y(s)$ through $\mathcal{L}$ and leaves it through $\mathcal{R}$), p_l and p_r are points from the intersection of the corridor Y and the surfaces $\mathcal{L}$ and $\mathcal{R}$, respectively, and P_l and P_r are the tangent planes to $\mathcal{L}$ and $\mathcal{R}$ at the points p_l and p_r. Clearly, $s \in P_l \cap P_r$. Let Q_I and Q_{II} be the planes containing the point s and supporting $K_Y(s)$ from different sides of the corridor Y. We define

$$P^i = \frac{1}{i}P_l + \Big(1 - \frac{1}{i}\Big)P_r, \qquad Q^j = \frac{1}{j}Q_I + \Big(1 - \frac{1}{j}\Big)Q_{II}, \qquad i,j \in \{1,2,\ldots,m\} \tag{12}$$

for given m. The planes P^i and Q^j decompose the cone $K(s)$ into partial cones K_{ij} bounded by the planes P^i, P^{i+1}, Q^j, and Q^{j+1}. Let, for a ray $l \subset K_{ij}$, the point $z(l) \in l \cap K_Y(s)$ be the most distant from s. Among these rays, we find a ray $l = l_{ij}$ for which $\|s - z(l_{ij})\|$ takes the greatest value. Define $z_{ij} = z(l_{ij})$. We construct a graph whose vertices are the points z_{ij} and edges are the segments connecting points z_{ij} and $z_{i'j'}$ from neighboring partial cones, i.e., cones having a common edge or a common face containing the point s. To determine the weight of an edge $[z_{ij}, z_{i'j'}]$, we find the distance ρ_z from s to the boundary ∂Y along the ray $\{s + \alpha(z - s),$ $\alpha > 0\}$, where $z = \lambda z_{ij} + (1-\lambda) z_{i'j'}$. The edge weight is defined as

$$\int_0^1 \widetilde{F}(\rho_z(\lambda z_{ij} + (1-\lambda) z_{i'j'}))\, d\lambda.$$

Let Z_l be the set of the vertices z_{ij} of the graph that correspond to the partial cones K_{ij} intersecting the left side $\mathcal{L}$ of the set $Y \cap \partial K(s)$. The set of vertices Z_r can be defined similarly. The problem of constructing a trajectory that approximates the optimal one is reduced to finding a path connecting the sets Z_l and Z_r and having the least sum of the weights of its edges.

Consider the case of two emitters s_1 and s_2 in $\mathbb{R}^3$. Assume that $F_{s_1} = F_{s_2}$ and the set $\mathcal{D} = \overset{\circ}{K}_Y(s_1) \cap \overset{\circ}{K}_Y(s_2)$ is nonempty. We can also assume that $[s_1, s_2] \cap \overset{\circ}{\mathcal{D}} = \varnothing$, since the emitters s_1 and s_2 should not irradiate each other. Here, as earlier, we use the notation $\mathcal{L}_i = \mathcal{L}K_Y(s_i)$ and $\mathcal{R}_i = \mathcal{R}K_Y(s_i)$ for the left and right sides of the truncated cone $K_Y(s_i)$. The part of the corridor Y located, say, to the right from $\mathcal{R}_2$ and to the left from $\mathcal{L}_1$ is called a box and is denoted by $\mathcal{R}_2\mathcal{L}_1$. Possible positions of the cones $K(s_1)$ and $K(s_2)$ are shown in Fig. 10. The boundary edges of the set $\mathcal{D}$ are marked with the bold solid and dashed lines. We have $s_1 \notin K(s_2)$ and $s_2 \notin K(s_1)$, and the set $\mathcal{D}$ is the intersection of Y with a tetrahedron with convex conical faces in the case (a); in the case (b), $s_2 \in K(s_1)$ and $\mathcal{D} = Y \cap K(s_1) \cap K(s_2)$.

For definiteness, we assume that the segment $[s_1, s_2]$ lies "to the left" from the set $\mathcal{D}$. Denote the plane orthogonal to this segment and containing the point $(s_1 + s_2)/2$ by L.

First, we assume that $L \cap \mathcal{D} = \varnothing$. In this case, the optimal trajectory $\mathcal{T}$ is disjoint with $\mathcal{D}$, since, if $t \in \mathcal{T} \cap \mathcal{D}$ and $\rho(s_1, t) \neq \rho(s_2, t)$, the point t could be shifted to a point $t' \in \mathcal{D}$ such that

$$\rho(s_1, t) + \rho(s_2, t) < \rho(s_1, t') + \rho(s_2, t').$$

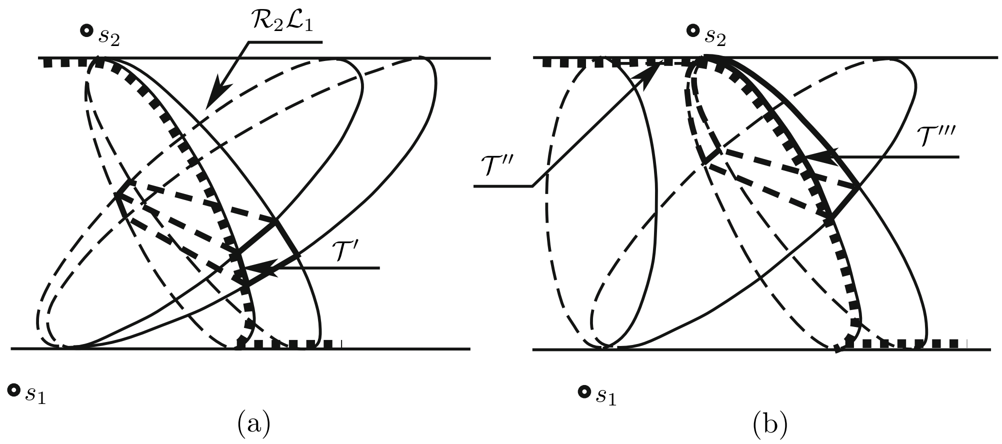

Fig. 10. (a) The case $s_1 \notin K(s_2)$, $s_2 \notin K(s_1)$. (b) The case $s_2 \in K(s_1)$, $\mathcal{D} = Y \cap K(s_1) \cap K(s_2)$.

Indeed, when $\rho(s_1, t) > \rho(s_2, t)$, we can take $t' = s_2 + \varepsilon(t - s_2)$ for small $\varepsilon > 0$. Hence, $\mathcal{T}$ intersects the cones $K(s_i)$ successively, say, first $K(s_1)$ and then $K(s_2)$ (the case of the other order is considered similarly). Thus, the problem is to find a trajectory $\mathcal{T}$ that optimally intersects $K(s_1)$ when passing from the box $\mathcal{L}_1\mathcal{L}_2$ to the box $\mathcal{R}_1\mathcal{L}_2$; further, we must find a trajectory that optimally intersects $K(s_2)$ when passing from the box $\mathcal{R}_1\mathcal{L}_2$ to the box $\mathcal{R}_1\mathcal{R}_2$. The optimality means minimizing the total exposure of the object over each segment $\delta\mathcal{T}_i = \mathcal{T} \cap K(s_i)$, $i = 1, 2$. The behavior of the optimal trajectory on a short interval can be tracked using the intersection points with the planes of a specially chosen one-parameter family similar to the family $P_\lambda = \lambda P_l + (1 - \lambda)P_r$ (see (12)). This family must cut a part of the space and the trajectory that obviously contains the segment $\delta\mathcal{T}_i$ of the trajectory. Taking into account the convexity of the cones $K(s_i)$ $(i = 1, 2)$ and choosing an appropriate one-parameter family of planes, we see that

- in the case (a), the farthest from s_1 trajectory of traversing the cone $K(s_1)$, denoted as $\mathcal{T}'$, belongs to the set $Y \cap K(s_1) \cap \mathcal{L}_2$,
- in the case (b), the trajectory most distant from s_1 is composed of two parts $\mathcal{T}''$ and $\mathcal{T}'''$, the first of which, $\mathcal{T}''$, belongs to ∂Y and connects $\mathcal{L}_1$ and $\mathcal{L}_2$, while the second, $\mathcal{T}'''$, belongs to the set $Y \cap K(s_1) \cap \mathcal{L}_2$.

In both cases, it remains to construct the trajectory $\mathcal{T}_2$ intersecting optimally the cone $K(s_2)$. As noted above, the farthest from s_i curve is not always optimal; it is necessary to take into account its length. To find a trajectory close to optimal, the methods proposed in the previous sections can be used: namely, the methods presented in the section where the problem is studied in $\mathbb{R}^2$ in order to construct $\mathcal{T}'$ and $\mathcal{T}'''$ and the algorithm for finding a trajectory in $\mathbb{R}^3$ in the case of a single emitter in order to find $\mathcal{T}''$. In Fig. 10, one of the optimal trajectories is shown by the dotted line.

Let $L_\mathcal{D} \stackrel{\mathrm{def}}{=} L \cap \mathcal{D}$. The points $x \in \mathcal{D}$ are exposed to irradiation of magnitude $2F(\|s_1 - x\|)$ from the virtual source $(s_1 + s_2)/2$. The set $\mathcal{D}$ is the intersection of Y with a convex set whose boundary in the case (a) consists of fragments of four (three in the case (b)) conical surfaces:

$$\mathcal{L}_1 \cap K_2, \quad \mathcal{R}_1 \cap K_2, \quad \mathcal{L}_2 \cap K_1, \quad \mathcal{R}_2 \cap K_1,$$

where $K_i = K(s_i)$. In the simplest case where the plane L intersects the edges $\widetilde{l} = (\mathcal{L}_1 \cap \mathcal{L}_2) \cap Y$ and $\widetilde{r} = (\mathcal{R}_1 \cap \mathcal{R}_2) \cap Y$, the problem reduces to searching for an optimal planar curve from $L_\mathcal{D}$ connecting the points of intersection of L with the specified edges. This curve can be constructed by using the above lemma. The optimal trajectory outside $\mathcal{D}$ belongs to the boxes $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{R}_1\mathcal{R}_2$, which lie outside the irradiation zone.

In the general case, we find the distances

$$\min\left\{\|x-d\|\colon x\in\widetilde{l},\ d\in L_{\mathcal{D}}\right\},\quad \min\left\{\|y-d\|\colon y\in\widetilde{r},\ d\in L_{\mathcal{D}}\right\}$$

and pairs of points $\{x_l\in\widetilde{l}, d_l\in L_{\mathcal{D}}\}$ and $\{y_r\in\widetilde{r}, d_r\in L_{\mathcal{D}}\}$ implementing these lower bounds. To solve the problem, it suffices to construct an optimal trajectory in $L_{\mathcal{D}}$ connecting the points d_l and d_r and two trajectories in $\overline{\mathcal{D}}$: one of them connects the points x_l and d_l and the other connects the points y_r and d_r.

The above methods make it possible to construct either an optimal or close to optimal trajectory in different cases of the location of the cones $K(s)$.

REFERENCES

1. W. Liu, "Path planning methods in an environment with obstacles (A review)," Mat. Mat. Model., No. 1, 15–58 (2018). doi 10.24108/mathm.0118.0000098
2. A. V. Arutyunov, G. G. Magaril-Il'yaev, and V. M. Tikhomirov, *Pontryagin Maximum Principle*: *Proofs and Applications* (Faktorial, Moscow, 2006) [in Russian].
3. V. V. Korobkin, A. N. Sesekin, O. L. Tashlykov, and A. G. Chentsov, *Routing Methods and Their Applications in Problems of the Enhancement of Safety and Efficiency of Nuclear Plant Operation* (Novye Tekhnologii, Moscow, 2012) [in Russian].
4. A. G. Chentsov, A. M. Grigoryev, and A. A. Chentsov, "Optimization "In windows" for routing problems with constraints," in *Mathematical Optimization Theory and Operations Research*: *MOTOR 2019*, Ed. by I. Bykadorov, V. Strusevich, and T. Tchemisova (Springer, Cham, 2019), Ser. Communications in Computer and Information Science, Vol. 1090, pp. 470–485. doi 10.1007/978-3-030-33394-2_36

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