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Best L^2-Extension of Algebraic Polynomials from the Unit Euclidean Sphere to a Concentric Sphere

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Abstract—We consider the problem of extending algebraic polynomials from the unit sphere of the Euclidean space of dimension $m \geq 2$ to a concentric sphere of radius $r \neq 1$ with the smallest value of the L^2-norm. An extension of an arbitrary polynomial is found. As a result, we obtain the best extension of a class of polynomials of given degree $n \geq 1$ whose norms in the space L^2 on the unit sphere do not exceed 1. We show that the best extension equals r^n for $r > 1$ and r^{n-1} for $0 < r < 1$. We describe the best extension method. A.V. Parfenenkov obtained in 2009 a similar result in the uniform norm on the plane ($m = 2$).

Keywords: polynomial, Euclidean sphere, L^2-norm, best extension.

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1. INTRODUCTION

1.1. Problem statement. Main result. Let $\mathbb{R}^m$, $m \geq 2$, be the Euclidean space of points $x = (x_1, \ldots, x_m)$ equipped with the norm $|x| = |x|_m = \big(\sum_{k=1}^{m} x_k^2\big)^{1/2}$; let $\mathbb{B}_r = \mathbb{B}_r^m = \{x \in \mathbb{R}^m\colon |x| \leq r\}$ and $\mathbb{S}_r = \mathbb{S}_r^{m-1} = \{x \in \mathbb{R}^m\colon |x| = r\}$ be a ball and a sphere of radius $r > 0$ centered at the origin of the space $\mathbb{R}^m$, respectively; and let $\mathbb{Z}_+^m$ be the set of points $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ with nonnegative integer coordinates called *multi-indices*. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m$ and a point $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$.

Denote by $L^2(\mathbb{S}_r)$ the space of complex-valued measurable square integrable on $\mathbb{S}_r$ functions with the L^2-norm

$$\|f\| = \|f\|_{L^2(\mathbb{S}_r)} = \left(\frac{1}{|\mathbb{S}_r|} \int\limits_{\mathbb{S}_r} |f(x)|^2\, dx\right)^{1/2};$$

here $|\mathbb{S}_r|$ is the area of the sphere $\mathbb{S}_r$; $L^2(\mathbb{S}_r)$ is a Hilbert space with respect to the inner product

$$(f, g) = \frac{1}{|\mathbb{S}_r|} \int\limits_{\mathbb{S}_r} f(x)\overline{g(x)}\, dx, \quad f,\, g \in L^2(\mathbb{S}_r). \tag{1.1}$$

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For a nonnegative integer n, denote by $\mathscr{P}_n = \mathscr{P}_n^m$ the set of algebraic polynomials

$$P_n(x) = P_n(x_1, \dots, x_m) = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m, \\ \alpha_1 + \dots + \alpha_m \le n}} c_\alpha x^\alpha \tag{1.2}$$

in m real variables $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ of degree (at most) n with complex coefficients $\{c_\alpha\}$. A term x^α in (1.2) is called a *monomial*, and the sum $\alpha_1 + \dots + \alpha_m$ is the *degree* of this monomial. The largest degree of monomials with nonzero coefficients is called the *exact degree* of a polynomial P_n. Denote by $\mathscr{B}_n$ the set of polynomials from $\mathscr{P}_n$ whose norms in the space $L^2(\mathbb{S})$ on the unit sphere $\mathbb{S} = \mathbb{S}_1$ are bounded by 1; i.e.,

$$\mathscr{B}_n = \{P_n \in \mathscr{P}_n \colon \|P_n\|_{L^2(\mathbb{S})} \le 1\}.$$

For a polynomial $P_n \in \mathscr{P}_n$, we consider an associated class

$$\mathscr{Q}_n(P_n) = \{Q_n \in \mathscr{P}_n \colon Q_n(x) = P_n(x) \text{ for } x \in \mathbb{S}\} \tag{1.3}$$

of polynomials $Q_n \in \mathscr{P}_n$ coinciding with P_n on the unit sphere $\mathbb{S}$. For a polynomial $P_n \in \mathscr{P}_n$, the smallest value

$$u_n(P_n; r) = \inf\{\|Q_n\|_{L^2(\mathbb{S}_r)} \colon Q_n \in \mathscr{Q}_n(P_n)\}$$

of the $L^2(\mathbb{S}_r)$-norms of polynomials from the associated class on a sphere of radius r can be interpreted as the value of the best L^2-extension of the polynomial P_n from the unit sphere $\mathbb{S}$ to the sphere $\mathbb{S}_r$. In this case, it is natural to consider the value

$$\theta_n^m(r) = \sup\{u_n(P_n; r) \colon P_n \in \mathscr{B}_n\} = \sup\{u_n(P_n; r) \colon P_n \in \mathscr{P}_n,\ \|P_n\|_{L^2(\mathbb{S})} \le 1\} \tag{1.4}$$

as the value of the best L^2-extension of the set of polynomials $\mathscr{B}_n$ from $\mathbb{S}$ to $\mathbb{S}_r$. In the present paper, we consider the problem of finding the exact value of the best extension $\theta_n^m(r)$ and the best extension method; we call it problem (1.4).

In 2009 Parfenenkov (see [1]) solved a problem similar to (1.4) in the uniform norm on the plane ($m = 2$) for all $r > 0$.

In what follows, we give a solution to problem (1.4) for arbitrary dimension $m \ge 2$ and all $r > 0$, $r \ne 1$; problem (1.4) in the case $r = 1$ is trivial: $\theta_n^m(1) = 1$.

Denote by $\mathscr{H}_n = \mathscr{H}_n^m$ the set of harmonic (in $\mathbb{R}^m$) polynomials $H_n \in \mathscr{P}_n^m$. Finally, let $\mathscr{G}_n = \mathscr{G}_n^m$ be the set of homogeneous harmonic polynomials $G_n \in \mathscr{H}_n^m$ whose exact degree (and order of homogeneity) is n.

The main result in the present paper is the following theorem.

Theorem. *The following statements hold for every $m \ge 2$ and $n \ge 1$.*

(1) *For $r > 1$, the value of the best extension* (1.4) *is*

$$\theta_n^m(r) = r^n, \tag{1.5}$$

and only homogeneous harmonic polynomials $G_n \in \mathscr{G}_n$ of degree n with unit norm $\|G_n\|_{L_2(\mathbb{S})} = 1$ and polynomials from the associated classes $\mathscr{Q}_n(G_n)$ are extremal in problem (1.4).

(2) *For $0 < r < 1$, the formula*

$$\theta_n^m(r) = r^{n-1}$$

holds, and only homogeneous harmonic polynomials $G_{n-1} \in \mathscr{G}_{n-1}$ *of degree* $n-1$ *with unit norm* $\|G_{n-1}\|_{L_2(\mathbb{S})} = 1$ *and polynomials from the associated classes* $\mathscr{Q}_n(G_{n-1})$ *are extremal in this case.*

We describe the best method for extending polynomials from the unit sphere to a sphere $\mathbb{S}_r$ of radius $r \neq 1$ in Section 4 after the proof of the theorem.

As will be seen from Lemma 1 and the proof of the theorem, the case $n = 1$ is degenerate, though formally the theorem holds for it.

Problem (1.4) is also meaningful for $n = 0$. This is a trivial case, and it is convenient to discuss it after the proof of Lemma 1.

Note two known facts.

(1) The following formula holds for the area of a sphere $\mathbb{S}_r$, $r > 0$ (see, for example, [2, Ch. XVIII, Sect. 676, Example 3]):

$$|\mathbb{S}_r| = \sigma_m \, r^{m-1}, \quad \sigma_m = |\mathbb{S}| = \frac{2\pi^{m/2}}{\Gamma\,(m/2)}.$$

(2) The following formula holds for functions f defined, measurable, and integrable on a sphere $\mathbb{S}_r$, $r > 0$ (details can be found, for example, in [3, (26)]):

$$\int\limits_{\mathbb{S}_r} f(x)\, dx = r^{m-1} \int\limits_{\mathbb{S}} f(rx)\, dx.$$

In what follows, the statements that a polynomial from $\mathscr{P}_n$ and, in particular, from $\mathscr{H}_n$ has degree n mean only that the exact degree of the polynomial is at most n. The cases when the exact degree of a polynomial is n will be discussed explicitly.

1.2. Gauss expansion of algebraic polynomials in several variables. In the study of extremal problems for algebraic polynomials in several variables, the well-known Gauss theorem on the representation of an arbitrary homogeneous polynomial in several variables in terms of homogeneous harmonic polynomials is of great importance; a proof of this theorem can be found in the monographs [4, Ch. XI, Sect. 2, Theorem XI.1] and [5, Ch. IV, Sect. 2, Theorem 2.1]. In the present paper, we use the Gauss representation of an arbitrary polynomial in several variables in terms of harmonic polynomials, which follows from this theorem. We formulate this statement in the form of the following theorem (see, for example, [4, Ch. XI, Sect. 5, (XII.5.1)]).

Theorem A (Gauss expansion for algebraic polynomials). *For every nonnegative integer* n, *every polynomial* $P_n \in \mathscr{P}_n$ *can be uniquely presented in the form*

$$P_n(x) = \sum_{s=0}^{[n/2]} |x|^{2s}\, H_{n-2s}(x), \quad x \in \mathbb{R}^m,$$

where $H_j \in \mathscr{H}_j$.

Corollary. *For every* $r > 0$ *and every polynomial* $P_n \in \mathscr{P}_n$, *there exists a unique harmonic polynomial* $H_n = H_n(P_n, r)$ *that coincides with* P_n *on the sphere* $\mathbb{S}_r$. *In particular, for every polynomial* $P_n \in \mathscr{P}_n$, *its associated class* (1.3) *contains a unique harmonic polynomial.*

2. THE STRUCTURE OF THE ASSOCIATED CLASS

The authors do not claim the novelty of the following statement about the representation of the associated class (1.3). However, we did not find a reference to such a result in the mathematical literature. Therefore, here it is presented with a proof.

Lemma 1 (The structure of the associated class)**.** *If $n \ge 2$, then, for every polynomial $P_n \in \mathscr{P}_n$, its associated class $\mathscr{Q}_n(P_n)$ consists exactly of polynomials of the form*

$$Q_n(x) = P_n(x) + \big(|x|^2 - 1\big)\, R_{n-2}(x), \quad R_{n-2} \in \mathscr{P}_{n-2}.$$

For $n = 0$ and 1, the class $\mathscr{Q}_n(P_n)$ consists only of the polynomial P_n.

Proof. Let us first discuss the assertions of the lemma for $n = 0$ and 1. For $n = 0$, a polynomial $P_0 \in \mathscr{P}_0$ is a constant, and its associated class $\mathscr{Q}_0(P_0)$, obviously, consists only of the constant P_0.

In the case $n = 1$, a polynomial $P_1 \in \mathscr{P}_1$ has the form

$$P_1(x_1, \dots, x_m) = a_1 x_1 + a_2 x_2 + \dots + a_m x_m + a_{m+1},$$

where $\{a_k\}$ are the coefficients of P_1. Let

$$Q_1(x_1, \dots, x_m) = b_1 x_1 + b_2 x_2 + \dots + b_m x_m + b_{m+1}$$

also be a polynomial of the first degree, which coincides with P_1 on the unit sphere. Consider their difference

$$F_1(x_1, \dots, x_m) = P_1(x_1, \dots, x_m) - Q_1(x_1, \dots, x_m)$$
$$= (a_1 - b_1)x_1 + \dots + (a_m - b_m)x_m + (a_{m+1} - b_{m+1}).$$

Define $c_k = a_k - b_k$. We take $2m$ points, one of whose coordinates is 1 or -1 while the others are 0. These points belong to the unit sphere; hence, the difference F_1 vanishes at them. As a result, we obtain a system of $2m$ linear equations with respect to the variables c_k:

$$c_{m+1} \pm c_k = 0, \quad k = 0, \dots, m.$$

This system has only the zero solution: $c_k = 0$, $1 \le k \le m+1$. Hence, the polynomial Q_1 coincides with the polynomial P_1 everywhere in $\mathbb{R}^m$.

Now, let $n \ge 2$. Assume that F is a polynomial equal to the difference of algebraic polynomials taking identical values on the unit sphere $\mathbb{S}$, i.e., F is a polynomial vanishing on the unit sphere:

$$F(x) \equiv 0, \quad x \in \mathbb{S}. \tag{2.1}$$

We represent the points $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ in the form $x = (\xi, y)$, where $\xi = x_1 \in \mathbb{R}$ and $y = (x_2, \dots, x_m) \in \mathbb{R}^{m-1}$. We write the polynomial $F(x) = F(\xi, y)$ as a polynomial in the variable ξ whose coefficients are polynomials in y:

$$F(\xi, y) = \sum_{k=0}^{n} c_k(y)\xi^k.$$

Divide $F(\xi, y)$ by a second-degree polynomial $\xi^2 + |y|^2_{m-1} - 1 = \xi^2 + x_2^2 + \dots + x_m^2 - 1$ in the variable ξ regarding the coordinates of the point y as parameters. We obtain the relation

$$F(\xi, y) = R(\xi, y)(\xi^2 + |y|^2 - 1) + \xi\, U(y) + V(y), \tag{2.2}$$

where $R(\xi, y)$, $U(y)$, and $V(y)$ are polynomials in the variables $x = (\xi, y)$ and y, respectively.

For an arbitrary point y from the open unit ball $\overset{\circ}{\mathbb{B}} = \overset{\circ}{\mathbb{B}}{}^{m-1}$ of the space $\mathbb{R}^{m-1}$, the two (distinct) values

$$\xi^{\pm} = \pm\sqrt{1 - |y|^2_{m-1}}$$

of the parameter ξ are such that the points $x^{\pm} = (\xi^{\pm}, y)$ lie on the unit sphere of the space $\mathbb{R}^m$; hence, according to (2.1), we have $F(\xi^{\pm}, y) = 0$. Substituting these points into representation (2.2), we obtain

$$F(\xi^{\pm}, y) = \xi^{\pm} U(y) + V(y) = 0, \quad y \in \overset{\circ}{\mathbb{B}}\,. \tag{2.3}$$

Since $\xi^+ \neq \xi^-$, relations (2.3) imply that $U(y) = 0$ and $V(y) = 0$ for $y \in \overset{\circ}{\mathbb{B}}$. Since U and V are polynomials, this implies that U and V are identically equal to zero on $\mathbb{R}^{m-1}$.

As a result, we obtain the following representation for the polynomial F:

$$F(x) = R(x)\left(|x|^2_m - 1\right), \quad x \in \mathbb{R}^m.$$

It is easy to see that the polynomial R has degree at most $n - 2$, i.e., belongs to $\mathscr{P}_{n-2}$.

Lemma 1 is proved completely.

Remark 1. The assertion of Lemma 1 for $n = 0$ shows that problem (1.4) is trivial in this case. Specifically, $\theta^m_0(r) = 1$, $r > 0$, and only the two polynomials ± 1 are extremal.

3. THE BEST EXTENSION OF HARMONIC POLYNOMIALS

In this section, we study the best extension

$$u_n(H_n; r) = \inf\{\|Q_n\|_{L^2(\mathbb{S}_r)} \colon Q_n \in \mathscr{Q}_n(H_n)\}$$

of harmonic polynomials H_n of degree $n \geq 1$ from the unit sphere to a sphere of radius $r > 0$, $r \neq 1$. We will proceed from the quite obvious fact that a harmonic polynomial $H_n \in \mathscr{H}^m_n$ is the sum

$$H_n = \sum_{k=0}^{n} G_k \tag{3.1}$$

of homogeneous harmonic polynomials $G_k \in \mathscr{G}^m_k$ of degree k, $0 \leq k \leq n$.

Lemma 2 (Extension of harmonic polynomials)**.** *The following formula holds for a harmonic polynomial H_n written in the form (3.1) for $n \geq 2$ and $r > 0$, $r \neq 1$:*

$$u_n(H_n; r) = \left(r^{2n}\|G_n\|^2_{L^2(\mathbb{S})} + r^{2(n-1)}\|G_{n-1}\|^2_{L^2(\mathbb{S})}\right)^{1/2}. \tag{3.2}$$

The best extension is implemented only by the polynomials

$$Q^*_n(x) = H_n(x) + (|x|^2 - 1)R^*_{n-2}(x), \tag{3.3}$$

where R^*_{n-2} are polynomials of degree (at most) $n - 2$ for which the following representation necessarily holds on the sphere $\mathbb{S}_r$:

$$R^*_{n-2}(x) = -\frac{1}{r^2 - 1}\sum_{k=0}^{n-2} G_k(x), \quad x \in \mathbb{S}_r. \tag{3.4}$$

Proof. By Lemma 1, the following representation can be written for an arbitrary polynomial $Q_n \in \mathscr{Q}_n(H_n)$:

$$Q_n(x) = H_n(x) + (|x|^2 - 1)R_{n-2}(x),$$

where R_{n-2} is an arbitrary polynomial from $\mathscr{P}_{n-2}$. By the corollary to Theorem A, the polynomial R_{n-2} coincides on the sphere $\mathbb{S}_r$ with some harmonic polynomial H_{n-2} of degree (at most) $n-2$, which, according to formula (3.1), has the form

$$H_{n-2}(x) = \sum_{k=0}^{n-2} G_k^0(x), \quad x \in \mathbb{R}^m,$$

where $\{G_k^0\}_{k=0}^{n-2}$ are homogeneous harmonic polynomials. Consequently, the following representation holds for the polynomial R_{n-2} on $\mathbb{S}_r$:

$$R_{n-2}(x) = \sum_{k=0}^{n-2} G_k^0(x), \quad x \in \mathbb{S}_r.$$

Thus, we have the following formula for the polynomial Q_n on the sphere $\mathbb{S}_r$:

$$Q_n(x) = G_n(x) + G_{n-1}(x) + \sum_{k=0}^{n-2} \left(G_k(x) + (r^2-1)G_k^0(x)\right), \quad x \in \mathbb{S}_r.$$

All $n+1$ terms on the right-hand side of this formula are homogeneous harmonic polynomials of the corresponding degree. Such polynomials are orthogonal on the sphere $\mathbb{S}_r$ with respect to the inner product (1.1) (see, for example, [5, Ch. IV, Sect. 2, Corollary 2.4] or [4, Ch. XI, Sect. 3]). Therefore,

$$\|Q_n\|^2_{L_2(\mathbb{S}_r)} = \|G_n\|^2_{L_2(\mathbb{S}_r)} + \|G_{n-1}\|^2_{L_2(\mathbb{S}_r)} + \sum_{k=0}^{n-2} \|G_k + (r^2-1)G_k^0\|^2_{L_2(\mathbb{S}_r)}.$$

The smallest value of the latter quantity is attained at polynomials $\{G_k^0\}_{k=0}^{n-2}$ defined by the formulas

$$G_k^0 = -G_k/(r^2-1), \quad 0 \le k \le n-2,$$

at least on the sphere $\mathbb{S}_r$. Thus, it is proved that

$$u_n(H_n; r) = \inf\{\|Q_n\|_{L^2(\mathbb{S}_r)} \colon Q_n \in \mathscr{Q}_n(H_n)\} = \|G_n + G_{n-1}\|_{L_2(\mathbb{S}_r)}$$

$$= \left(\|G_n\|^2_{L_2(\mathbb{S}_r)} + \|G_{n-1}\|^2_{L_2(\mathbb{S}_r)}\right)^{1/2} = \left(r^{2n}\|G_n\|^2_{L_2(\mathbb{S})} + r^{2(n-1)}\|G_{n-1}\|^2_{L_2(\mathbb{S})}\right)^{1/2}.$$

The minimum here is attained at every polynomial of the form (3.3) in which R^*_{n-2} is a polynomial of degree $n-2$ having the form (3.4) on the sphere $\mathbb{S}_r$.

Lemma 2 is proved.

Remark 2. In our opinion, it is of interest that polynomials (3.3) on the sphere $\mathbb{S}_r$, regardless of the value of $r > 0$, $r \ne 1$, are defined by the same formula

$$Q_n^*(x) = G_n(x) + G_{n-1}(x), \quad x \in \mathbb{S}_r.$$

4. COMPLETION OF STUDYING PROBLEM (1.4)

4.1. Proof of the main theorem. First, we justify the assertions of the theorem in the case $n \geq 2$. According to the corollary to Theorem A, for any polynomial $P_n \in \mathscr{P}_n$, the associated class contains a (unique) harmonic polynomial $H_n \in \mathscr{P}_n$. The associated classes of these two polynomials are the same; hence,

$$u_n(P_n; r) = u_n(H_n; r).$$

Using representation (3.1) of the polynomial H_n and formula (3.2), we obtain the relation

$$u_n(P_n; r) = \big(r^{2n}\|G_n\|^2_{L^2(\mathbb{S})} + r^{2(n-1)}\|G_{n-1}\|^2_{L^2(\mathbb{S})}\big)^{1/2}. \tag{4.1}$$

Hence, for $r > 1$,

$$u_n(P_n; r) \leq r^n\big(\|G_n\|^2_{L^2(\mathbb{S})} + \|G_{n-1}\|^2_{L^2(\mathbb{S})}\big)^{1/2}$$

$$\leq r^n\Big(\sum_{k=0}^{n}\|G_k\|^2_{L^2(\mathbb{S})}\Big)^{1/2} = r^n\|H_n\|_{L^2(\mathbb{S})} = r^n\|P_n\|_{L^2(\mathbb{S})}.$$

Thus, the following estimate holds for every polynomial $P_n \in \mathscr{P}_n$ for $r > 1$:

$$u_n(P_n; r) \leq r^n\|P_n\|_{L^2(\mathbb{S})}.$$

It is easy to conclude from the proof that this inequality turns into an equality if and only if $H_n = G_n$. This implies the equality (1.5) and the characterization of extremal polynomials given in the statement of the theorem.

The statements of the main theorem for $r > 1$ are proved.

In the case $0 < r < 1$, formula (4.1) implies the inequality

$$u_n(P_n; r) \leq r^{n-1}\big(\|G_n\|^2_{L^2(\mathbb{S})} + \|G_{n-1}\|^2_{L^2(\mathbb{S})}\big)^{1/2}.$$

Hence, as for $r > 1$, the following estimate is valid for any polynomial $P_n \in \mathscr{P}_n$ in the case $0 < r < 1$:

$$u_n(P_n; r) \leq r^{n-1}\|P_n\|_{L^2(\mathbb{S})}.$$

This estimate implies all the assertions of the theorem for $0 < r < 1$.

The main theorem is proved for $n \geq 2$.

Consider the case $n = 1$. A polynomial P_1 of the first degree is harmonic and has the form $P_1 = G_0 + G_1$, where G_0 is a constant and G_1 is a homogeneous polynomial of the first degree. According to Lemma 1, the class associated with P_1 consists only of this polynomial. Therefore, in this case,

$$u_1(P_1; r) = \|P_1\|_{L^2(\mathbb{S}_r)} = \big(r^2\|G_1\|^2_{L^2(\mathbb{S})} + \|G_0\|^2_{L^2(\mathbb{S})}\big)^{1/2}. \tag{4.2}$$

Formula (4.2) is analogous to formula (4.1). The further argument in the proof of the theorem for $n = 1$ is carried out in the same way as for $n \geq 2$.

The theorem is proved completely.

4.2. The best extension method. In the above proof of the theorem, a method is constructed for extending a polynomial $P_n \in \mathscr{P}_n$ from the unit sphere of the space $\mathbb{R}^m$ to a polynomial of degree at most n with the smallest value of the L^2-norm on a sphere $\mathbb{S}_r$ of radius $r \neq 1$; we call it here the *best extension method*. In the case $n = 1$, for any $r > 0$, this method is the identity operator, which assigns to any polynomial $P_1 \in \mathscr{P}_1$ the same polynomial. In the case $n \geq 2$, the method consists of two steps.

(1) For a polynomial $P_n \in \mathscr{P}_n$, we consider its associated harmonic polynomial $H_n \in \mathscr{P}_n$.

(2) Using representation (3.1), we associate to the polynomial H_n a polynomial $Q_n^* \in \mathscr{P}_n$ by formula (3.3).

The constructed mapping $P_n \to Q_n^*$ of the set $\mathscr{P}_n$ into itself will be the best extension method. This mapping is many-valued for $n \geq 4$, since the polynomial R_{n-2}^* in formula (3.3) is defined uniquely only on the sphere $\mathbb{S}_r$.

However, a mapping A_r that extends a polynomial $P_n \in \mathscr{P}_n$ from the unit sphere $\mathbb{S}$ to a sphere $\mathbb{S}_r$, $r \neq 1$, in the best way (i.e., with the smallest value of the L^2-norm) is defined uniquely by the formula

$$(A_r P_n)(x) = G_n(x) + G_{n-1}(x), \quad x \in \mathbb{S}_r;$$

this mapping is a linear operator, and the following formula holds for the norm of this operator from $L^2(\mathbb{S})$ to $L^2(\mathbb{S}_r)$:

$$\|A_r\|_{L^2(\mathbb{S}) \to L^2(\mathbb{S}_r)} = \max\{r^n, r^{n-1}\}.$$

In this sense, the best extension method is single-valued, unique, and linear.

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