

On Graded Semigroup C^* -Algebras and Hilbert Modules

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Abstract—Reduced semigroup C^* -algebras for arbitrary cancellative semigroups are studied. It is proved that if there exists a semigroup epimorphism from a semigroup to an arbitrary group G , then the corresponding semigroup C^* -algebra is topologically G -graded. It is also demonstrated that if the group is finite, then the graded semigroup C^* -algebra has the structure of a projective Hilbert C^* -module.

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INTRODUCTION

In this paper, we consider reduced semigroup C^* -algebras, namely, the algebras generated by left regular representations of cancellative semigroups. The study of such C^* -algebras was initiated by Coburn [2, 3], Douglas [4], and Murphy [21, 22]. The further development of the theory of semigroup C^* -algebras is due to many authors (see, for example, the references cited in [16]).

The present work is a continuation of the studies of reduced semigroup C^* -algebras started in [1, 8–13, 17–19]. Here we study the following problems: the construction of a grading for reduced semigroup C^* -algebras, and the existence of a Hilbert C^* -module structure on a graded semigroup C^* -algebra.

The grading of an object of a category makes it possible to better understand the structure of this object. In the category of C^* -algebras, one uses C^* -bundles, or Fell bundles, to construct a grading. These bundles were introduced by Fell [7], who employed them to extend the notions of harmonic analysis to the noncommutative case. Exel [5] introduced the notion of a topologically graded C^* -algebra. A refinement of the definition of such an algebra is contained in [24]. It is important that the topological grading of a C^* -algebra guarantees the existence of special operators which are analogs of the Fourier coefficients. For a detailed account of the theory of graded C^* -algebras, we refer the reader to the monograph [6]. In [1, 8, 9, 11, 17], the authors studied questions related to the construction of gradings for different semigroup C^* -algebras. In [11], the notion of the σ -index of a monomial was introduced and used to construct a G -grading of a semigroup C^* -algebra for a finite cyclic group G . In the present study, we employ this notion to construct a G -grading of a semigroup C^* -algebra for an arbitrary group G , including a nonabelian one.

It is well known that the theory of Hilbert C^* -modules is a convenient tool for studying C^* -algebras. A detailed introduction to the theory of Hilbert C^* -modules, as well as many of its applications, is contained in the book [20]. In [23], the authors used the theory of Hilbert C^* -modules to define a noncommutative analog of branched coverings. To define the structure of a Hilbert C^* -module on a semigroup C^* -algebra, we use the conditional expectation of algebraically finite index (see [25]) and thus define a noncommutative covering of a semigroup C^* -algebra [23, Definition 1.4].

The paper consists of the introduction and four sections. In Section 1, we give necessary information on semigroup C^* -algebras and the definitions of graded and topologically graded C^* -algebras. We present all necessary definitions related to Banach and Hilbert modules over C^* -algebras.

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In Sections 2 and 3, we construct a grading and show that this grading is topological. The main result of these two sections is formulated as follows (Theorem 2). If there exists a semigroup epimorphism $\sigma: S \rightarrow G$ from an arbitrary cancellative semigroup S to an arbitrary group G , then the semigroup C^* -algebra $C_r^*(S)$ is a topologically G -graded C^* -algebra. This theorem generalizes the result obtained in [11], where a similar statement was proved for an abelian semigroup and a finite cyclic group.

In Section 4, we study the existence of a Hilbert C^* -module structure on the G -graded C^* -algebra $C_r^*(S)$ under consideration. Namely, we prove that if G is a finite group, then the C^* -algebra $C_r^*(S)$ has the structure of a projective Hilbert C^* -module.

1. PRELIMINARIES

Throughout the paper, we denote by S an arbitrary cancellative semigroup with identity. Denote the identity in S by e .

The object of the present study is the reduced semigroup C^* -algebra $C_r^*(S)$. In this connection, we recall the definition of this C^* -algebra.

Consider the Hilbert space $l^2(S)$ of square summable complex-valued functions defined on S :

$$l^2(S) := \left\{ f: S \rightarrow \mathbb{C} \mid \sum_{a \in S} |f(a)|^2 < +\infty \right\}.$$

Denote the canonical orthonormal basis of $l^2(S)$ by $\{e_a \mid a \in S\}$, where

$$e_a(b) := \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

In the algebra of all bounded operators on $l^2(S)$, consider the C^* -subalgebra generated by the set of isometries $\{T_a \mid a \in S\}$, where the operator T_a is defined by the formula

$$T_a(e_b) = e_{ab}, \quad a, b \in S.$$

It is this C^* -subalgebra that is called the reduced semigroup C^* -algebra $C_r^*(S)$.

Next, for the convenience of the reader, we recall the definition of a G -graded C^* -algebra from [6, Definition 16.2].

Let \mathfrak{A} be an arbitrary C^* -algebra and G be a group. Then \mathfrak{A} is said to be a G -graded C^* -algebra if there exists a family of linearly independent closed subspaces $\{\mathfrak{A}_g \subset \mathfrak{A}\}_{g \in G}$ such that the following properties hold for any $g, h \in G$:

- (1) $\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{gh}$,
- (2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$, and
- (3) $\mathfrak{A} = \overline{\bigoplus_{g \in G} \mathfrak{A}_g}$.

In this case, the family of Banach spaces $\{\mathfrak{A}_g\}_{g \in G}$ is called a C^* -algebraic bundle or a Fell bundle over the group G .

In [6, Definition 19.2], Exel also considered the notion of grading in a stronger sense. Namely, a G -graded C^* -algebra \mathfrak{A} is said to be *topologically graded* if there exists a contractive linear map

$$F: \mathfrak{A} \rightarrow \mathfrak{A}_e$$

that coincides with the identity map on the subspace \mathfrak{A}_e , where e is the identity element of the group, and vanishes on every subspace \mathfrak{A}_g , where $g \in G, g \neq e$.

An important property of a topologically graded C^* -algebra is the existence of Fourier coefficients (see [6, Corollary 19.6]). This means that for every $g \in G$ there exists a contractive linear map

$$F_g: \mathfrak{A} \rightarrow \mathfrak{A}_g$$

such that the equality $F_g(A) = A_g$ holds for any finite sum $A = \sum_{h \in G} A_h$ with $A_h \in \mathfrak{A}_h$. Moreover, the maps $F_g, g \in G$, have the following property: the equalities

$$F_g(BA) = BF_{h^{-1}g}(A) \quad \text{and} \quad F_g(AB) = F_{gh^{-1}}(A)B$$

hold for any $B \in \mathfrak{A}_h, h \in G$, and $A \in \mathfrak{A}$.

The notions related to Banach and C^* -Hilbert modules are contained in the books [14, 20]. Recall the necessary definitions. By a module we mean a left module.

A module \mathfrak{M} over a C^* -algebra \mathfrak{A} is called a *Banach \mathfrak{A} -module* if it is a Banach space with norm satisfying the inequality $\|A \cdot M\| \leq \|A\| \cdot \|M\|$, where $A \in \mathfrak{A}$ and $M \in \mathfrak{M}$. A subset X in a Banach \mathfrak{A} -module \mathfrak{M} is called a *set of generators* if the finite \mathfrak{A} -linear combinations of elements of X are dense in \mathfrak{M} . If X is a finite set, then the module \mathfrak{M} is said to be *finitely generated*.

An element M in an \mathfrak{A} -module \mathfrak{M} is said to be *cyclic* if the following equality holds:

$$\mathfrak{M} = \mathfrak{A} \cdot M := \{A \cdot M \mid A \in \mathfrak{A}\}.$$

A module with a cyclic element is called a *cyclic module*.

A module \mathfrak{M} over a C^* -algebra \mathfrak{A} is called a *pre-Hilbert \mathfrak{A} -module* if it is equipped with a sesquilinear form $\langle \cdot, \cdot \rangle: \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{A}$, called an \mathfrak{A} -valued scalar (or inner) product, that has the following properties for any $M, N \in \mathfrak{M}$ and $A \in \mathfrak{A}$:

- (1) $\langle M, M \rangle \geq 0$,
- (2) $\langle M, M \rangle = 0$ if and only if $M = 0$,
- (3) $\langle M, N \rangle = \langle N, M \rangle^*$, and
- (4) $\langle A \cdot M, N \rangle = A \langle M, N \rangle$.

If \mathfrak{M} is a pre-Hilbert \mathfrak{A} -module, then one can define a norm $\|\cdot\|_{\mathfrak{M}}$ on it as $\|M\|_{\mathfrak{M}} = \|\langle M, M \rangle\|^{1/2}$ for any $M \in \mathfrak{M}$ (see [20, Proposition 1.2.4]).

A pre-Hilbert \mathfrak{A} -module \mathfrak{M} that is complete in the norm $\|\cdot\|_{\mathfrak{M}}$ is called a *Hilbert C^* -module*.

For a C^* -algebra \mathfrak{A} and a C^* -subalgebra \mathfrak{B} in \mathfrak{A} , a linear map $E: \mathfrak{A} \rightarrow \mathfrak{B}$ such that $E(B) = B$ for any $B \in \mathfrak{B}$ and $\|E(A)\| \leq \|A\|$ for any $A \in \mathfrak{A}$ is called a *conditional expectation* (see [20, 25]). A conditional expectation is said to be *faithful* if for any positive $A \in \mathfrak{A}$ the equality $E(A) = 0$ implies $A = 0$. It is well known (see [20, Example 1.3.6]) that in this case one can introduce the structure of a left pre-Hilbert \mathfrak{B} -module on the C^* -algebra \mathfrak{A} by defining the inner product

$$\langle A, B \rangle = E(AB^*) \quad \text{for any } A, B \in \mathfrak{A}.$$

2. GRADING OF THE C^* -ALGEBRA $C_r^*(S)$

Let G be an arbitrary group. Denote the identity element of the group by e .

Suppose that there exists a surjective semigroup homomorphism

$$\sigma: S \rightarrow G. \tag{2.1}$$

Then the semigroup S can be represented as a disjoint union of subsets S_g ,

$$S = \bigsqcup_{g \in G} S_g, \tag{2.2}$$

such that every S_g is the complete preimage of the element $g \in G$, i.e., $\sigma^{-1}(g) = S_g$.

To construct a grading of the semigroup C^* -algebra $C_r^*(S)$, we introduce the notion of a monomial as well as the notion of a σ -index for a monomial and for the corresponding operator in this C^* -algebra, where σ is the surjective homomorphism (2.1). We use the construction that was first introduced by the author in the joint paper [11].

Consider the free semigroup generated by the set $\{T_a^{-1}, T_a^1 \mid a \in S\}$. The elements of this semigroup are words of the form

$$V = T_{a_k}^{i_k} T_{a_{k-1}}^{i_{k-1}} \dots T_{a_1}^{i_1}, \tag{2.3}$$

where $a_1, \dots, a_k \in S$ and $i_1, \dots, i_k \in \{-1, 1\}$. We will call these words *monomials*. The number k in (2.3) is called the *length* of the monomial. The semigroup itself is called the *monomial semigroup* and is denoted by Mon .

The monomial semigroup is an involutive semigroup. The involution is defined on a monomial of the form (2.3) by the formula

$$V^* = T_{a_1}^{-i_1} T_{a_2}^{-i_2} \dots T_{a_k}^{-i_k}.$$

Now we define a map of semigroups $\text{ind}: \text{Mon} \rightarrow G$. For a monomial of the form (2.3), we set by definition

$$\text{ind } V = \sigma(a_k)^{i_k} \sigma(a_{k-1})^{i_{k-1}} \dots \sigma(a_1)^{i_1}.$$

It is easy to see that the equalities

$$\text{ind}(V \cdot W) = \text{ind } V \cdot \text{ind } W \quad \text{and} \quad \text{ind}(V^*) = (\text{ind } V)^{-1} \tag{2.4}$$

hold for any $V, W \in \text{Mon}$. Hence, the map ind is an involutive surjective semigroup homomorphism.

Every monomial V defines an operator \widehat{V} on the Hilbert space $l^2(S)$ as follows:

$$\widehat{T}_a^1 = T_a, \quad \widehat{T}_a^{-1} = T_a^*,$$

and if V is a monomial of the form (2.3), then

$$\widehat{V} = \widehat{T}_{a_k}^{i_k} \widehat{T}_{a_{k-1}}^{i_{k-1}} \dots \widehat{T}_{a_1}^{i_1}. \tag{2.5}$$

We will call operators of the form (2.5) *operator monomials*.

Lemma 1. *Let $V \in \text{Mon}$ and $\widehat{V}e_a \neq 0$ for some basis vector $e_a \in l^2(S)$. Then there exists an element $b \in S$ such that the following equalities hold:*

$$\widehat{V}e_a = e_b \quad \text{and} \quad \sigma(b) = \text{ind } V \cdot \sigma(a).$$

Proof. Let V be a monomial of the form (2.3). We will prove the lemma by induction on the length k of the monomial.

Let $k = 1$. Then two cases are possible: either $V = T_{a_1}^1$ or $V = T_{a_1}^{-1}$. Let $\widehat{V}e_a \neq 0$. Then, in the first case, we obtain $T_{a_1}e_a = e_{a_1a}$. Hence, $b = a_1a$ and

$$\sigma(b) = \sigma(a_1)\sigma(a) = \text{ind } V \cdot \sigma(a).$$

In the second case, we have $T_{a_1}^*e_a = e_b$, where $a = a_1b$. Therefore, $\sigma(a) = \sigma(a_1)\sigma(b)$, and so

$$\sigma(b) = (\sigma(a_1))^{-1}\sigma(a) = \text{ind } V \cdot \sigma(a).$$

Now, consider a monomial V of arbitrary length k . Obviously, we can write it as $V = T_{a_k}^{i_k} V'$, where V' satisfies the equalities $\widehat{V}'e_a = e_{b'}$ and $\sigma(b') = \text{ind } V' \cdot \sigma(a)$ by the induction hypothesis. Then we have $\widehat{V}e_a = \widehat{T}_{a_k}^{i_k} e_{b'} = e_b$. In the same way as in the case of $k = 1$, we obtain the equality $\sigma(b) = \text{ind } T_{a_k}^{i_k} \cdot \sigma(b')$. This implies the required assertion

$$\sigma(b) = \text{ind } T_{a_k}^{i_k} \cdot \text{ind } V' \cdot \sigma(a) = \text{ind } V \cdot \sigma(a). \quad \square$$

It follows from Lemma 1 that if $\widehat{V}_1 = \widehat{V}_2$, then $\text{ind } V_1 = \text{ind } V_2$.

For an arbitrary monomial $V \in \text{Mon}$, we refer to the value $\text{ind } V$ of the map ind on V both as the σ -index of the monomial V and the σ -index of the operator monomial \widehat{V} .

Finite linear combinations of operator monomials form an involutive subalgebra, which is dense in the C^* -algebra $C_r^*(S)$. Denote this subalgebra by $P(S)$.

It is easy to see that the monomials of σ -index e form an involutive subsemigroup in the monomial semigroup Mon . Denote by \mathfrak{A}_e the C^* -subalgebra in $C_r^*(S)$ generated by the operator monomials of σ -index e .

For any $g \in G$, the closure of the linear hull of the set of all operator monomials of σ -index g is a Banach subspace in the C^* -algebra $C_r^*(S)$, which we denote by \mathfrak{A}_g .

Next, using the expansion (2.2), we represent the Hilbert space $l^2(S)$ as an orthogonal sum of subspaces:

$$l^2(S) = \bigoplus_{g \in G} H_g, \tag{2.6}$$

where every subspace H_g , $g \in G$, has a Hilbert basis given by the family of functions $\{e_a \mid a \in S_g\}$.

The following lemma shows how the spaces H_g behave under the action of the elements of the space \mathfrak{A}_h , for $g, h \in G$.

Lemma 2. *For any $g, h \in G$ and any operator $A \in \mathfrak{A}_h$, the following inclusion holds:*

$$A(H_g) \subset H_{hg}.$$

In particular, for any $g \in G$, the subspace H_g is invariant under the action of any element of the C^ -algebra \mathfrak{A}_e .*

Proof. Since finite linear combinations of operator monomials of σ -index h form a dense subspace in the Banach space \mathfrak{A}_h , it suffices to prove the lemma for such operator monomials.

Fix elements $h, g \in G$. Let V be an arbitrary monomial of σ -index h . Take any basis vector $e_a \in H_g$ with $\widehat{V}e_a \neq 0$. If there is no such vector, then $\widehat{V}(H_g) = \{0\} \subset H_{hg}$. By Lemma 1, we obtain $\widehat{V}e_a = e_b$, where $b \in S$ is an element such that $\sigma(b) = \text{ind } V \cdot \sigma(a)$. Since $\sigma(a) = g$, we have $\sigma(b) = hg$. Thus, $b \in S_{hg}$ and $e_b \in H_{hg}$. \square

Next, we apply Lemma 2 in order to prove that the family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ is a C^* -algebraic bundle over the group G . This statement is a generalization of Lemma 3 from [11].

Lemma 3. *The following assertions hold for the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$:*

- (1) $\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{gh}$,
- (2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$,
- (3) *the family $\{\mathfrak{A}_g \mid g \in G\}$ is a linearly independent system of closed subspaces in the C^* -algebra $C_r^*(S)$, and*
- (4) $C_r^*(S) = \overline{\bigoplus_{g \in G} \mathfrak{A}_g}$.

Proof. For operator monomials, assertions (1) and (2) follow from equalities (2.4). In the general case, the assertions hold since the finite linear combinations of operator monomials of σ -index g are dense in the Banach space \mathfrak{A}_g .

Let us prove assertion (3). Let $A = \sum_{g \in G} A_g = 0$, where $A_g \in \mathfrak{A}_g$. Note that this sum contains a finite number of nonzero terms. We will show that then $A_g = 0$ for any g . Let $g_0 \in G$ be an element such that $A_{g_0} \neq 0$. Then we have the representation

$$A_{g_0} = - \sum_{g \in G, g \neq g_0} A_g.$$

By Lemma 2, the inclusion $A_{g_0}(H_h) \subset H_{g_0h}$ holds for any $h \in G$. This implies the relation

$$- \sum_{g \in G, g \neq g_0} A_g(H_h) \subset H_{g_0h}.$$

On the other hand, by the same Lemma 2, we have the inclusion

$$- \sum_{g \in G, g \neq g_0} A_g(H_h) \subset \bigoplus_{g \in G, g \neq g_0} H_{gh}.$$

Since $g_0h \neq gh$ and the subspaces $H_g, g \in G$, are orthogonal, we obtain the equality

$$H_{g_0h} \cap \bigoplus_{g \in G, g \neq g_0} H_{gh} = \{0\}.$$

This implies the relation

$$A_{g_0} = - \sum_{g \in G, g \neq g_0} A_g = 0.$$

Let us prove assertion (4). Note that the finite linear combinations of operator monomials are dense in the C^* -algebra $C_r^*(S)$. On the other hand, they are dense in $\bigoplus_{g \in G} \mathfrak{A}_g$, since every such finite linear combination can be represented as $\sum_{g \in G} A_g$, where A_g is a finite linear combination of operator monomials of σ -index g . From the inclusion $\bigoplus_{g \in G} \mathfrak{A}_g \subset C_r^*(S)$ we find that the subspace $\bigoplus_{g \in G} \mathfrak{A}_g$ is dense in $C_r^*(S)$. \square

Thus, Lemma 3 implies the following result on the G -grading of the semigroup C^* -algebra $C_r^*(S)$.

Theorem 1. *Let $\sigma: S \rightarrow G$ be a surjective semigroup homomorphism. Let \mathfrak{A}_g be a closed subspace in $C_r^*(S)$ generated by the operator monomials of σ -index g , where $g \in G$. Then the family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a Fell bundle of the semigroup C^* -algebra $C_r^*(S)$ over the group G .*

3. TOPOLOGICAL GRADING OF THE C^* -ALGEBRA $C_r^*(S)$

Here we prove that the semigroup C^* -algebra $C_r^*(S)$ is topologically graded (for the grading constructed in Section 2).

In the next lemma, we construct a linear bounded operator that allows us to talk about the topological G -grading of the semigroup C^* -algebra $C_r^*(S)$.

Lemma 4. *There exists a contractive linear operator*

$$F: C_r^*(S) \rightarrow \mathfrak{A}_e$$

that coincides with the identity operator on \mathfrak{A}_e and vanishes on every subspace $\mathfrak{A}_g, g \in G, g \neq e$.

Proof. Recall that the involutive subalgebra $P(S)$, which consists of all finite linear combinations of operator monomials, is dense in the C^* -algebra $C_r^*(S)$. Therefore, to prove the lemma, it suffices to construct a linear bounded operator

$$F: P(S) \rightarrow \mathfrak{A}_e$$

that leaves invariant the linear combinations of operator monomials of σ -index e and vanishes on the linear combinations of operator monomials of σ -index g for every $g \neq e$.

Since every element $A \in P(S)$ can be uniquely represented as a finite sum of nonzero terms of the form

$$A = \sum_{g \in G} A_g, \tag{3.1}$$

where $A_g \in \mathfrak{A}_g$, the formula

$$F(A) = A_e$$

obviously defines a linear operator on the normed space $P(S)$ that satisfies the required conditions.

Let us prove that F is a contractive operator. To this end, we fix an arbitrary element $A \in P(S)$ and consider its representation (3.1).

Let us show that the following estimate for the norms holds:

$$\|F(A)\| = \|A_e\| \leq \|A\|. \quad (3.2)$$

Recall that the Hilbert space $l^2(S)$ decomposes into the orthogonal sum (2.6). By Lemma 2, the subspaces H_g are invariant with respect to \mathfrak{A}_e . Therefore, for every element $A_e \in \mathfrak{A}_e$, we have the decomposition

$$A_e = \bigoplus_{g \in G} A_e^g,$$

where A_e^g denotes the restriction of the operator A_e to the subspace H_g . This implies the following equality for the operator norms:

$$\|A_e\| = \sup_{g \in G} \|A_e^g\|.$$

Fix an arbitrary number $\varepsilon > 0$. Let $g_0 \in G$ be an element such that the inequality

$$\|A_e^{g_0}\| \geq \|A_e\| - \varepsilon \quad (3.3)$$

is satisfied. Note that the following inequality holds:

$$\|A\| = \sup_{\|x\|=1, x \in l^2(S)} (Ax, Ax)^{1/2} \geq \sup_{\|x\|=1, x \in H_{g_0}} (Ax, Ax)^{1/2}.$$

For $x \in H_{g_0}$, consider the inner product

$$(Ax, Ax) = \left(\sum_{g \in G} A_g x, \sum_{h \in G} A_h x \right) = \sum_{g, h \in G} (A_g x, A_h x).$$

Let us show that $(A_g x, A_h x) = 0$ for $g \neq h$. Indeed, since $x \in H_{g_0}$, by Lemma 2 we have $A_g x \in H_{gg_0}$ and $A_h x \in H_{hg_0}$. If $g \neq h$, then $gg_0 \neq hg_0$, and since the subspaces H_{gg_0} and H_{hg_0} are orthogonal, we obtain $(A_g x, A_h x) = 0$. Thus, we have the estimate

$$(Ax, Ax) = \sum_{g \in G} (A_g x, A_g x) \geq (A_e x, A_e x),$$

which implies the inequality

$$\|A\| \geq \sup_{\|x\|=1, x \in H_{g_0}} (Ax, Ax)^{1/2} \geq \sup_{\|x\|=1, x \in H_{g_0}} (A_e x, A_e x)^{1/2} = \|A_e^{g_0}\|. \quad (3.4)$$

Since ε is arbitrary, from inequalities (3.3) and (3.4) we obtain the required inequality (3.2). \square

Note that the constructed linear operator

$$F: C_r^*(S) \rightarrow \mathfrak{A}_e$$

is a conditional expectation. We will call it the *conditional expectation associated with the grading* $\{\mathfrak{A}_g \mid g \in G\}$.

Lemmas 3 and 4 allow us to claim that the C^* -algebra $C_r^*(S)$ is topologically graded.

Theorem 2. *Let $\sigma: S \rightarrow G$ be a surjective semigroup homomorphism. Let \mathfrak{A}_g be a closed subspace in $C_r^*(S)$ generated by the operator monomials of σ -index g , where $g \in G$. Then the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a topological G -grading of the semigroup C^* -algebra $C_r^*(S)$.*

4. FINITELY GENERATED PROJECTIVE HILBERT \mathfrak{A}_e -MODULE

In this section, we show that the C^* -algebra $C_r^*(S)$ is a left Banach \mathfrak{A}_e -module. Moreover, if G is a finite group, then $C_r^*(S)$ is a finitely generated projective Hilbert \mathfrak{A}_e -module.

Recall that the semigroup S can be represented as the disjoint union (2.2). Let us fix an arbitrary element x_g in each set S_g . Denote the set of all such x_g by X . Thus, $X \subset S$ and $X \cap S_g = \{x_g\}$ for any $g \in G$.

By a *set of representatives of the classes* $\{S_g \mid g \in G\} = \{\sigma^{-1}(g) \mid g \in G\}$ we will mean an arbitrary subset $X \subset S$ with the following property: for any $g \in G$, there exists a unique $x \in X$ such that $X \cap S_g = \{x\}$.

Lemma 5. *For every $g \in G$, the equality*

$$\mathfrak{A}_g = \mathfrak{A}_e \cdot T_{x_g}$$

holds; i.e., the space \mathfrak{A}_g is a cyclic Banach \mathfrak{A}_e -module, and the element T_{x_g} is a cyclic element of the module \mathfrak{A}_g .

Proof. First, let us show that the inclusion $\mathfrak{A}_e \cdot T_{x_g} \subset \mathfrak{A}_g$ holds. If $A^{(e)} = \sum_i \alpha_i \widehat{V}_i^{(e)}$ is a finite linear combination of operator monomials of σ -index e , where $\alpha_i \in \mathbb{C}$, then $A^{(e)}T_{x_g} = \sum_i \alpha_i \widehat{V}_i^{(e)}T_{x_g}$ is obviously a finite linear combination of operator monomials of σ -index g . Let $\{A_n^{(e)}\}$ be a sequence of such linear combinations, and let $\lim_{n \rightarrow \infty} A_n^{(e)} = B_e \in \mathfrak{A}_e$. Then, in view of the inequality

$$\|A_n^{(e)}T_{x_g} - B_eT_{x_g}\| \leq \|A_n^{(e)} - B_e\| \cdot \|T_{x_g}\| = \|A_n^{(e)} - B_e\|,$$

we obtain $B_eT_{x_g} = \lim_{n \rightarrow \infty} A_n^{(e)}T_{x_g} \in \mathfrak{A}_g$.

Let us establish the reverse inclusion $\mathfrak{A}_g \subset \mathfrak{A}_e \cdot T_{x_g}$. Let $B_g \in \mathfrak{A}_g$. Set $B_e := B_gT_{x_g}^*$. Then we obviously have $B_g = B_eT_{x_g}$. Just as above, taking into account that the σ -index of $T_{x_g}^*$ is equal to g^{-1} , we can show that $B_e \in \mathfrak{A}_e$. This completes the proof of the lemma. \square

Assertion (4) of Lemma 3 and Lemma 5 imply the following theorem.

Theorem 3. *Let G be an arbitrary group with identity e , $\sigma: S \rightarrow G$ be a surjective semigroup homomorphism, and X be a set of representatives of the classes $\{\sigma^{-1}(g) \mid g \in G\}$. Let \mathfrak{A}_e be the C^* -subalgebra in $C_r^*(S)$ generated by the operator monomials of σ -index e . Then the C^* -algebra $C_r^*(S)$ is a Banach \mathfrak{A}_e -module with generating set $\{T_x \mid x \in X\}$.*

As pointed out in Section 1, since the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a topological G -grading of the C^* -algebra $C_r^*(S)$, for every g there exists a contractive linear map

$$F_g: C_r^*(S) \rightarrow \mathfrak{A}_g$$

such that for any finite sum $A = \sum_{g \in G} A_g$ with $A_g \in \mathfrak{A}_g$ we have the equality $F_g(A) = A_g$. Moreover, the maps F_g satisfy the equalities

$$F_g(AB) = F_{gh^{-1}}(A)B \quad \text{and} \quad F_g(BA) = BF_{h^{-1}g}(A) \tag{4.1}$$

for any $A \in C_r^*(S)$ and $B \in \mathfrak{A}_h$.

Note that the construction of the operator F in Lemma 4 and the principle of extension by continuity (see, for example, [15, Ch. 2, § 1, Theorem 2]) immediately imply the equality $F_e = F$.

Next, let G be a finite group.

First, we examine the geometry of the underlying Banach space of the reduced semigroup C^* -algebra $C_r^*(S)$.

Theorem 4. *Let G be a finite group with identity e , $\sigma: S \rightarrow G$ be a surjective semigroup homomorphism, and X be a set of representatives of the classes $\{\sigma^{-1}(g) \mid g \in G\}$. Let \mathfrak{A}_e be the*

C^* -subalgebra in $C_r^*(S)$ generated by the operator monomials of σ -index e . Then the C^* -algebra $C_r^*(S)$ is a finitely generated Banach \mathfrak{A}_e -module with a set of generators $\{T_x \mid x \in X\}$. Moreover, it can be represented as a direct sum of a finite number of cyclic \mathfrak{A}_e -modules:

$$C_r^*(S) = \bigoplus_{x \in X} \mathfrak{A}_e \cdot T_x.$$

Proof. First, we prove the equality of spaces

$$C_r^*(S) = \bigoplus_{g \in G} \mathfrak{A}_g. \tag{4.2}$$

To this end, we show that the equality

$$A = \sum_{g \in G} F_g(A) \tag{4.3}$$

holds for any $A \in C_r^*(S)$. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence in $P(S)$ that converges to A . Note that equality (4.3) holds for any finite sum of the form $A = \sum_{g \in G} A_g$; i.e., for any $A_n \in P(S)$, we have

$$A_n = \sum_{g \in G} (A_n)_g = \sum_{g \in G} F_g(A_n).$$

Then we obtain the chain of inequalities

$$\begin{aligned} \left\| A - \sum_{g \in G} F_g(A) \right\| &\leq \|A - A_n\| + \left\| \sum_{g \in G} F_g(A_n) - \sum_{g \in G} F_g(A) \right\| \\ &\leq \|A - A_n\| + \sum_{g \in G} \|F_g\| \cdot \|A_n - A\| = (k + 1) \cdot \|A - A_n\|, \end{aligned}$$

where k is the order of the group G . This estimate implies equality (4.3).

Now, employing Lemma 5, we obtain the assertion of the theorem. \square

In fact, as we will see below, if the group G is finite, then the C^* -algebra $C_r^*(S)$ is a finitely generated projective C^* -Hilbert \mathfrak{A}_e -module.

Recall the definition of a finitely generated projective Hilbert module [20]. Let \mathfrak{M} be a Hilbert \mathfrak{A} -module such that there exists a Hilbert \mathfrak{A} -module \mathfrak{N} for which the direct sum $\mathfrak{M} \oplus \mathfrak{N}$ is isomorphic as a module to the direct sum of a finite number of copies of the Hilbert \mathfrak{A} -module \mathfrak{A} . Then \mathfrak{M} is called a *finitely generated projective \mathfrak{A} -module*.

In [25, Corollary 3.1.4], conditions on the conditional expectation $E: \mathfrak{A} \rightarrow \mathfrak{B} \subset \mathfrak{A}$ are presented under which \mathfrak{A} is a C^* -Hilbert module over the C^* -algebra \mathfrak{B} .

The conditional expectation $E: \mathfrak{A} \rightarrow \mathfrak{B} \subset \mathfrak{A}$ is called a conditional expectation of *algebraically finite index* (see [25, Definition 1.2.2] and [20, Sect. 4.5]) if there exists a finite set of elements $X_1, \dots, X_n \in \mathfrak{A}$ such that every element $A \in \mathfrak{A}$ can be represented as

$$A = \sum_{k=1}^n E(AX_k^*)X_k.$$

If the conditional expectation is a conditional expectation of algebraically finite index, then it is faithful [25]. In [25, Corollary 3.1.4], it is demonstrated that the algebraic finiteness of the index is equivalent to the fact that \mathfrak{A} is a finitely generated projective Hilbert C^* -module over the C^* -algebra \mathfrak{B} (see also [23, Theorem 5.7]).

In the following lemma and theorem, we prove that in the case of a finite group G the conditional expectation $F: C_r^*(S) \rightarrow \mathfrak{A}_e$ constructed in Lemma 4 is a conditional expectation of algebraically finite index and that the C^* -algebra $C_r^*(S)$ is a finitely generated projective Hilbert \mathfrak{A}_e -module.

Lemma 6. *The conditional expectation $F: C_r^*(S) \rightarrow \mathfrak{A}_e$ associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ is a conditional expectation of algebraically finite index.*

Proof. By Theorem 4, any element $A \in C_r^*(S)$ can be represented as the finite sum (4.3). Then, using the properties (4.1) and taking into account that $F_e = F$, we obtain for any $A \in C_r^*(G)$ the equalities

$$A = \sum_{g \in G} F_g(A) = \sum_{g \in G} F_g(AT_{x_g}^* T_{x_g}) = \sum_{g \in G} F(AT_{x_g}^*) T_{x_g}.$$

This means that the conditional expectation F constructed in Lemma 4 is a conditional expectation of algebraically finite index. \square

Theorem 5. *Let G be a finite group with identity e and $\sigma: S \rightarrow G$ be a surjective semigroup homomorphism. Let \mathfrak{A}_e be the C^* -subalgebra in $C_r^*(S)$ generated by the operator monomials of σ -index e . Then the C^* -algebra $C_r^*(S)$ is a finitely generated projective Hilbert \mathfrak{A}_e -module.*

Proof. It follows from Lemma 6 that the conditional expectation $F: C_r^*(S) \rightarrow \mathfrak{A}_e$ associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ is faithful. Then we can define the following \mathfrak{A}_e -valued inner product on the C^* -algebra $C_r^*(S)$:

$$\langle A, B \rangle = F(AB^*),$$

where $A, B \in C_r^*(S)$. On the other hand, as already mentioned above, the algebraic finiteness of the index of the conditional expectation F associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ implies that the C^* -algebra $C_r^*(S)$ is a finitely generated projective Hilbert C^* -module over the C^* -algebra \mathfrak{A}_e . \square

REFERENCES

1. M. A. Aukhadiev, S. A. Grigoryan, and E. V. Lipacheva, “An operator approach to quantization of semigroups,” *Sb. Math.* **205** (3), 319–342 (2014) [transl. from *Mat. Sb.* **205** (3), 15–40 (2014)].
2. L. A. Coburn, “The C^* -algebra generated by an isometry,” *Bull. Am. Math. Soc.* **73** (5), 722–726 (1967).
3. L. A. Coburn, “The C^* -algebra generated by an isometry. II,” *Trans. Am. Math. Soc.* **137**, 211–217 (1969).
4. R. G. Douglas, “On the C^* -algebra of a one-parameter semigroup of isometries,” *Acta Math.* **128**, 143–151 (1972).
5. R. Exel, “Amenability for Fell bundles,” *J. Reine Angew. Math.* **492**, 41–73 (1997).
6. R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications* (Am. Math. Soc., Providence, RI, 2017), *Math. Surv. Monogr.* **224**.
7. J. M. G. Fell, *An Extension of Mackey’s Method to Banach *-Algebraic Bundles* (Am. Math. Soc., Providence, RI, 1969), *Mem. AMS*, No. 90.
8. S. A. Grigoryan, R. N. Gumerov, and E. V. Lipacheva, “On extensions of semigroups and their applications to Toeplitz algebras,” *Lobachevskii J. Math.* **40** (12), 2052–2061 (2019).
9. S. A. Grigoryan, E. V. Lipacheva, and A. S. Sitdikov, “Nets of graded C^* -algebras over partially ordered sets,” *St. Petersburg Math. J.* **30** (6), 901–915 (2019) [transl. from *Algebra Anal.* **30** (6), 1–19 (2018)].
10. R. N. Gumerov and E. V. Lipacheva, “Inductive systems of C^* -algebras over posets: A survey,” *Lobachevskii J. Math.* **41** (4), 644–654 (2020).
11. R. N. Gumerov and E. V. Lipacheva, “Topological grading of semigroup C^* -algebras,” *Herald Bauman Moscow State Tech. Univ., Ser. Nat. Sci.*, No. 3, 44–55 (2020).
12. R. N. Gumerov, E. V. Lipacheva, and T. A. Grigoryan, “On inductive limits for systems of C^* -algebras,” *Russ. Math.* **62** (7), 68–73 (2018) [transl. from *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 7, 79–85 (2018)].
13. R. N. Gumerov, E. V. Lipacheva, and T. A. Grigoryan, “On a topology and limits for inductive systems of C^* -algebras over partially ordered sets,” *Int. J. Theor. Phys.* **60** (2), 499–511 (2021).
14. A. Ya. Helemskii, *Banach and Polynormed Algebras: General Theory, Representations, Homology* (Nauka, Moscow, 1989). Engl. transl.: *Banach and Locally Convex Algebras* (Clarendon Press, Oxford, 1993).

15. A. Ya. Helemskii, *Lectures on Functional Analysis* (MTsNMO, Moscow, 2004). Engl. transl.: *Lectures and Exercises on Functional Analysis* (Am. Math. Soc., Providence, RI, 2006), Transl. Math. Monogr. **233**.
16. X. Li, “Semigroup C^* -algebras,” in *Operator Algebras and Applications* (Springer, Cham, 2016), Abel Symposia **12**, pp. 185–196.
17. E. V. Lipacheva, “On a class of graded ideals of semigroup C^* -algebras,” *Russ. Math.* **62** (10), 37–46 (2018) [transl. from *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 10, 43–54 (2018)].
18. E. V. Lipacheva, “Embedding semigroup C^* -algebras into inductive limits,” *Lobachevskii J. Math.* **40** (5), 667–675 (2019).
19. E. V. Lipacheva and K. H. Hovsepyan, “Automorphisms of some subalgebras of the Toeplitz algebra,” *Sib. Math. J.* **57** (3), 525–531 (2016) [transl. from *Sib. Mat. Zh.* **57** (3), 666–674 (2016)].
20. V. M. Manuilov and E. V. Troitsky, *C^* -Hilbert Modules* (Faktorial Press, Moscow, 2001). Engl. transl.: *Hilbert C^* -Modules* (Am. Math. Soc., Providence, RI, 2005), Transl. Math. Monogr. **226**.
21. G. J. Murphy, “Ordered groups and Toeplitz algebras,” *J. Oper. Theory* **18** (2), 303–326 (1987).
22. G. J. Murphy, “Toeplitz operators and algebras,” *Math. Z.* **208** (3), 355–362 (1991).
23. A. A. Pavlov and E. V. Troitskii, “Quantization of branched coverings,” *Russ. J. Math. Phys.* **18** (3), 338–352 (2011).
24. I. Raeburn, “On graded C^* -algebras,” *Bull. Aust. Math. Soc.* **97** (1), 127–132 (2018).
25. Y. Watatani, *Index for C^* -Subalgebras* (Am. Math. Soc., Providence, RI, 1990), Mem. AMS **83** (424).

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