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Disclinations in the Geometric Theory of Defects

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Abstract—In the geometric theory of defects, media with a spin structure (for example, ferromagnets) are regarded as manifolds with given Riemann–Cartan geometry. We consider the case with the Euclidean metric, which corresponds to the absence of elastic deformations, but with nontrivial $\mathbb{SO}(3)$ connection, which produces nontrivial curvature and torsion tensors. We show that the 't Hooft–Polyakov monopole has a physical interpretation; namely, in solid state physics it describes media with continuous distribution of dislocations and disclinations. To describe single disclinations, we use the Chern–Simons action. We give two examples of point disclinations: a spherically symmetric point "hedgehog" disclination and a point disclination for which the n-field takes a fixed value at infinity and has an essential singularity at the origin. We also construct an example of linear disclinations with Frank vector divisible by 2π.

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1. INTRODUCTION

Many physical properties of solids such as plasticity, melting, growth, etc., are defined by defects in the crystalline structure. Therefore, the study of defects is a topical scientific problem important for applications. Although dozens of monographs and thousands of papers have been published, no fundamental theory of defects is yet available.

One of the promising approaches to constructing a theory of defects is based on the Riemann–Cartan geometry defined by nontrivial metric and torsion. In this approach, a crystal is considered as an elastic continuous medium with a spin structure. If the displacement vector field is a smooth function, then the crystal possesses only elastic stresses corresponding to diffeomorphisms of the flat Euclidean space. If the displacement vector field has discontinuities, then we say that the medium has defects in the elastic structure, which are called dislocations and result in nontrivial geometry. Namely, they lead to a nonzero torsion tensor, which is equal to the surface density of the Burgers vector.

The idea of relating torsion to dislocations aroused in the 1950s [3, 39, 41, 49]. This approach is still being successfully developed (we note the reviews [15, 21, 36, 37, 42, 43, 55]) and is often called the gauge theory of dislocations. A similar approach is also being developed in gravity [18]. Interestingly, É. Cartan introduced the notion of torsion in geometry using the analogy with mechanics of elastic media [6].

Parallel to the study of dislocations, another type of defects has been intensively investigated. The point is that many solids (for example, ferromagnets, liquid crystals, spin glasses, etc.) not only have elastic properties but also possess a spin structure. In this case, there are defects in the spin structure, called disclinations [16]. They arise when the n-field describing a spin structure has discontinuities. The presence of disclinations is also related to nontrivial geometry. Namely, the curvature tensor for the $\mathbb{SO}(3)$ connection is equal to the surface density of the Frank vector. The gauge approach based on the rotational group $\mathbb{SO}(3)$ was used in this case [14]. The $\mathbb{SO}(3)$ gauge models of spin glasses with defects were considered in [19, 53].

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The geometric theory of static distribution of defects, which describes both types of defects from a unique point of view, was proposed in [34]. In contrast to other approaches, the only independent variables in this case are the vielbein and $\mathbb{SO}(3)$ connection. The torsion and curvature tensors have a straightforward physical interpretation as the surface densities of dislocations and disclinations, respectively. Covariant equations of equilibrium are postulated for the vielbein and $\mathbb{SO}(3)$ connection as in gravity models with torsion. Since any solution of the equilibrium equations is defined up to general coordinate transformations and local $\mathbb{SO}(3)$ rotations, we have to choose a coordinate system (to fix the gauge) in order to specify a unique solution. The elastic gauge for the vielbein [22] and the Lorentz gauge for the $\mathbb{SO}(3)$ connection [23] have been proposed recently. We stress that the displacement and rotational vector fields are not considered as independent variables in our approach. These notions can be introduced only in those domains of a medium where defects are absent. In this case the equilibrium equations for the vielbein and $\mathbb{SO}(3)$ connection are identically satisfied, the elastic gauge reduces to equations of nonlinear elasticity theory for the displacement vector field, and the Lorentz gauge transforms into equations of the principal chiral $\mathbb{SO}(3)$ field. In other words, to fix the coordinate system, one can choose two fundamental models: elasticity theory and the principal chiral field.

The presence of defects produces nontrivial Riemann–Cartan geometry. This means that to study the phenomena directly related to elastic media, we should modify the corresponding equations. For example, if the propagation of phonons in an ideal crystal is described by the wave equation, then it is easy to take into account the influence of dislocations. To this end we should replace the Euclidean metric by a nontrivial metric describing the distribution of defects. The scattering of phonons on straight parallel dislocations was analyzed in [12, 35, 47]. To describe quantum phenomena, one should make the same substitution of the metric in the Schrödinger equation. Wedge dislocations with conical singularities and their influence on the properties of solid bodies were considered in [1, 17, 31, 32]. Cylindrical dislocations were described in [10, 11, 24–26]. Other types of dislocations were considered, for example, in [2, 5, 8, 9, 45, 52].

Up to now, mainly dislocations and their influence on the physical properties of various media have been considered as applications of the geometric theory of defects. It has been assumed that the $\mathbb{SO}(3)$ connection is trivial but the metric differs from the Euclidean one and corresponds to a given dislocation. One started to consider disclinations in the framework of the geometric theory of defects only quite recently. Problems of this type imply that the metric is Euclidean (elastic deformations are absent) but the $\mathbb{SO}(3)$ connection is nontrivial. As far as we know, the first papers on this subject describe a straight disclination [27, 28]. The Chern–Simons action for the $\mathbb{SO}(3)$ connection is used there.

In the geometric theory of defects, the $\mathbb{SO}(3)$ connection is used instead of the n-field. To introduce it, we need to transform the n-field into the rotation angle field. This transformation is nontrivial, because an additional gauge degree of freedom appears [29]. In addition, an $\mathbb{SO}(2)$ gauge model without the $\mathbb{SO}(2)$ gauge field appears.

In the present review, we consider the case of Euclidean metric but nontrivial $\mathbb{SO}(3)$ connection, which corresponds to the presence of disclinations. We start with a short introduction into the geometric theory of defects.

Since the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, the static solutions of $\mathbb{SU}(2)$ gauge models can be viewed as those describing some distribution of disclinations and, possibly, dislocations. In particular, the 't Hooft–Polyakov monopole has a straightforward physical interpretation in the geometric theory of defects: it describes media with continuous distribution of disclinations and dislocations [30]. This is considered in Section 5.

The first examples of point disclinations in the geometric theory of defects are based on the Chern–Simons action. The most general form of the trivial spherically symmetric $\mathbb{SO}(3)$ connection containing one arbitrary function of radius is found for this case. In Section 6, we construct two examples of point disclinations for different boundary conditions. The first one describes a hedgehog disclination, and the second corresponds to a point disclination with n-field taking a fixed value at infinity and having an essential singularity at the origin [33]. In Section 7, we describe straight linear disclinations in the framework of the geometric theory of defects with the Chern–Simons action.

2. ELASTIC DEFORMATIONS

Elasticity theory (see, e.g., [44, 48]) is a classical area of mathematical physics with its own language developed for decades, which is different from the language of modern differential geometry in many respects. In this section, we present necessary notions of elasticity theory from the point of view of differential geometry (see, e.g., [13, 38]).

In an equilibrium state, a body occupies a bounded domain in the Euclidean space $\mathbb{R}^3$ of the observer. The equilibrium state is not defined uniquely: we can rotate or move the body as a whole. No deformation or elastic stresses arise in this case, since the Euclidean metric is invariant with respect to these transformations. We denote the Cartesian coordinates of a point of the body by letters with Latin indices y^i, $i = 1, 2, 3$. After a deformation or motion, every point of the body occupies a new position: $y \mapsto x$. This deformation corresponds to some diffeomorphism of domains in the Euclidean space. In addition, the body acquires the induced metric

$$\delta_{ij} \mapsto g_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\delta_{kl}. \tag{2.1}$$

The difference

$$\epsilon_{ij}(y) := \frac{1}{2}\big(\delta_{ij} - g_{ij}(y)\big) \tag{2.2}$$

is called the *deformation tensor* in the Cartesian coordinates. This difference is well defined, because the tensor components are subtracted pointwise. The definition implies that the deformation is identically zero in the equilibrium state. It is also zero after translations and rotations of the body as a whole.

Since the Euclidean space of the observer has a natural affine structure, the observer can add or subtract point coordinates before and after a deformation. After a deformation, every point has new coordinates in the same coordinate system:

$$y^i \mapsto x^i := y^i + u^i, \tag{2.3}$$

where $u^i(x)$ is the *displacement vector field.*

The following terminology is used in elasticity theory. If the components of the displacement vector field are considered as functions of the initial coordinates y, then this system is called *Lagrangian coordinates.* If the post-deformation coordinates x are chosen as independent ones, then we say that *Eulerian coordinates* are chosen. The Lagrangian and Eulerian coordinates are equivalent if the domains of definition of the point coordinates x^i and y^i of the body are diffeomorphic. However, in the geometric theory of defects, which is considered in the next sections, the situation is different. In general, only in the final state (after creation of dislocations) an elastic medium occupies the whole Euclidean space $\mathbb{R}^3$. In the presence of dislocations, the initial medium coordinates y^i do not usually cover the whole $\mathbb{R}^3$, because part of the medium can be removed or, conversely, added. Therefore, we use Eulerian coordinates related to the medium points after an elastic deformation and creation of defects.

In the absence of defects, the displacement vector field is assumed to be a sufficiently smooth vector field in the Euclidean space $\mathbb{R}^3$. The presence of discontinuities and (or) singularities in the displacement field is interpreted as the existence of defects in an elastic medium, which are called *dislocations.*

We will consider only static deformations in what follows. Then the basic equilibrium equations of an elastic medium for small deformations in the Cartesian coordinates have the form (see, e.g., [44, Ch. I, §§ 2, 4])

$$\partial_j \sigma^{ji} + f^i = 0, \tag{2.4}$$

$$\sigma^{ij} = \lambda \delta^{ij} \epsilon_k{}^k + 2\mu \epsilon^{ij}, \tag{2.5}$$

where σ^{ji} is the stress tensor (ith component of the elastic force acting on the unit area element with normal n^j), which is assumed to be symmetric. The *tensor of small deformations* ϵ_{ij} is given by symmetrized partial derivatives of the displacement vector field:

$$\epsilon_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{2.6}$$

where the Latin indices are raised and lowered using the Euclidean metric δ_{ij} and its inverse δ^{ij}. The letters λ and μ denote constants characterizing the elastic properties of media and are called the *Lame coefficients*. The functions $f^i(x)$ describe the total density of inelastic forces inside a medium, which are induced, for example, by gravity forces. We assume in what follows that such forces are absent: $f^i(x) = 0$. Equation (2.4) is Newton's second law for an equilibrium state, and equality (2.5) represents *Hooke's law*.

Let us look at elastic deformations from the point of view of differential geometry. From the mathematical standpoint, the map (2.3) is a diffeomorphism of the Euclidean space $\mathbb{R}^3$, with the Euclidean metric δ_{ij} induced by the pullback of the map $y^i \mapsto x^i$. This means that in the linear approximation the deformed metric is

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\epsilon_{ij}; \tag{2.7}$$

that is, it is defined by the tensor of small deformations (2.6).

In the Riemannian geometry, the metric defines the Levi–Civita connection $\widetilde{\Gamma}_{ij}{}^k(x)$ (Christoffel's symbols). The corresponding curvature tensor after an elastic deformation is identically zero, $\widetilde{R}_{ijk}{}^l(x) = 0$, because the curvature of the Euclidean space is zero and the map $y^i \mapsto x^i$ is a diffeomorphism. The torsion tensor is also zero for the same reason, since it is set to zero in the observer space. Thus, elastic deformations of media correspond to the trivial Riemann–Cartan geometry, because the curvature and torsion tensors vanish.

3. DISLOCATIONS

We start with describing linear dislocations in elastic media (see, e.g., [40, 44]). The simplest and most common examples of straight dislocations are shown in Fig. 1. Let us cut a medium along the half-plane $x^2 = 0$, $x^1 > 0$, then move the upper part of the medium $x^2 > 0$, $x^1 > 0$ (which is above the cut) by a vector $\boldsymbol{b}$ towards the dislocation axis x^3, and glue the two sides of the cut. The vector $\boldsymbol{b}$ is called the Burgers vector. In general, the Burgers vector may be nonconstant on the cut. For the edge dislocation, it varies from zero to some constant value $\boldsymbol{b}$ as the distance from the dislocation axis increases. After the gluing, the medium comes to an equilibrium state, which is called an edge dislocation and is shown in Fig. 1a. If the Burgers vector is parallel to the dislocation line, then the dislocation is called a screw dislocation (Fig. 1b).

One and the same dislocation can be formed in different ways. For example, if the Burgers vector is perpendicular to the cut plane and directed away from it, then the arising cavity should be filled with an extra medium. One can easily imagine that the resulting defect is an edge dislocation but rotated through the angle $\pi/2$ around the x^3 axis. This example shows that the dislocation is characterized by the dislocation line or axis (edge of the cut) and the Burgers vector rather than by the cutting surface.

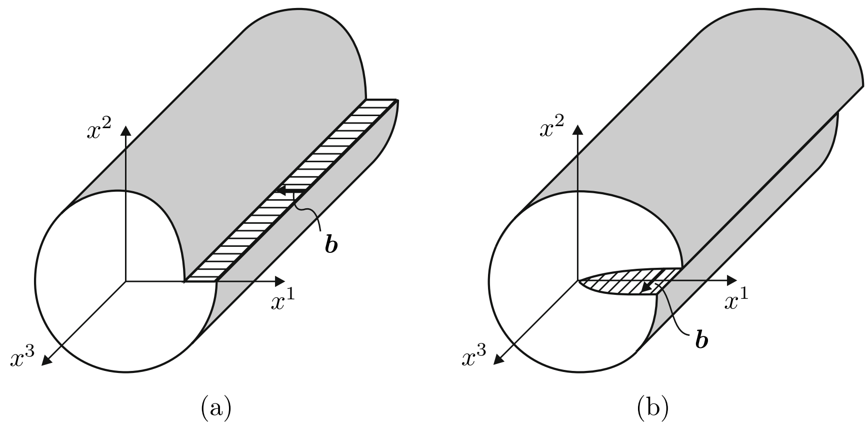

Fig. 1. Straight linear dislocations: (a) edge and (b) screw dislocations.

From the topological viewpoint, a medium containing several or even infinitely many dislocations represents the Euclidean space $\mathbb{R}^3$. In contrast to elastic deformations, the displacement vector field fails to be a smooth function because of cutting surfaces. At the same time we assume that the partial derivatives of the displacement vector $\partial_j u^i$ (distortion tensor) are smooth functions on the cutting surface. From the physical point of view, this assumption is reasonable because these derivatives define the deformation tensor (2.6). In turn, the partial derivatives of the deformation tensor should exist and be continuous functions everywhere in an equilibrium state except, possibly, the dislocation axis, because the elastic forces on the two sides of the cut must be equal in an equilibrium state. Since the deformation tensor defines the induced metric (2.7), we assume that the metric and vielbein in $\mathbb{R}^3$ are sufficiently smooth functions everywhere except, possibly, dislocation axes.

The main idea of the geometric approach is the following. To describe single dislocations in the framework of elasticity theory, one has to solve equations for the displacement vector with given boundary conditions on the cuts. This is possible for a small number of dislocations. However, the boundary conditions become too complicated when the number of dislocations increases, so that the problem is hardly solvable. Moreover, the same dislocation can be formed by different cuttings, which results in an ambiguous displacement vector field. Another disadvantage of this approach is that it cannot be used to describe a continuous distribution of defects, because the displacement vector field does not exist in this case since it has discontinuities at every point. The main variable in the geometric approach is a vielbein, which is a smooth function everywhere except, possibly, the dislocation cores. We postulate new equations for the vielbein. The transition from a finite number of dislocations to a continuous distribution of them is natural and simple in the geometric theory of defects. The singularities in dislocation cores are smoothed in the same way as the masses of point particles are smoothed after a transition to continuous media.

Now we proceed to constructing the formalism for the geometric theory of defects. In the presence of defects, in an equilibrium state, there is no symmetry, and therefore the notion of distinguished Cartesian coordinates is absent. Hence we consider an arbitrary curvilinear coordinate system x^μ, $\mu = 1, 2, 3$, in $\mathbb{R}^3$. Now we use Greek letters to number the coordinates because we admit arbitrary coordinate changes. Then the Burgers vector can be expressed by an integral of the displacement vector:

$$\oint_C dx^\mu\, \partial_\mu u^i(x) = -\oint_C dx^\mu\, \partial_\mu y^i(x) = -b^i, \tag{3.1}$$

where C is a closed contour surrounding the dislocation axis (Fig. 2). This integral is invariant with respect to arbitrary coordinate changes $x^\mu \mapsto x^{\mu'}(x)$ and covariant under $\mathbb{SO}(3)$ global rotations

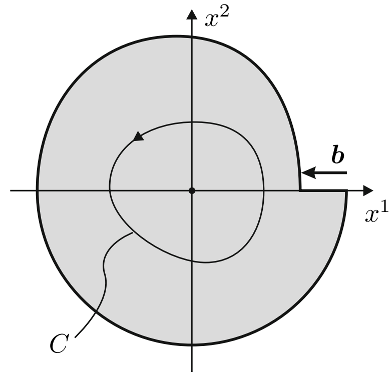

Fig. 2. Section of a medium with an edge dislocation. The dislocation axis is perpendicular to the figure plane; C is the integration contour for the Burgers vector $\boldsymbol{b}$.

of y^i. Here the components of the vector field $u^i(x)$ are considered with respect to an orthonormal basis in the tangent space, $u = u^i e_i$.

In the geometric theory of defects, we introduce a new independent variable (vielbein)

$$e_\mu{}^i(x) := \begin{cases} \partial_\mu y^i & \text{outside the cut,} \\ \lim \partial_\mu y^i & \text{on the cut,} \end{cases} \tag{3.2}$$

instead of the partial derivatives of the displacement vector field $\partial_\mu u^i$. By definition, the vielbein is a smooth function on the cut. Note that if the vielbein was defined just as the partial derivative $\partial_\mu y^i$, it would have a δ-function singularity on the cut because the functions $y^i(x)$ have a jump there.

The Burgers vector can be represented as an integral over a surface S with the contour C as the boundary:

$$\oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i) = b^i, \tag{3.3}$$

where $dx^\mu \wedge dx^\nu$ is the area element. The definition of vielbein (3.2) implies that the integrand vanishes everywhere except the dislocation axis. The integrand has a δ-function singularity at the origin for an edge dislocation with constant Burgers vector. The criterion for the presence of a dislocation is the violation of the integrability condition of the system of equations $\partial_\mu y^i = e_\mu{}^i$:

$$\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i \neq 0. \tag{3.4}$$

If dislocations are absent, then the functions $y^i(x)$ exist and define the transformation to Cartesian coordinates.

The field $e_\mu{}^i$ is identified with the vielbein in the geometric theory of defects. Next, we compare the integrand in (3.3) with the expression for torsion in Cartan variables:

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - e_\mu{}^j \omega_{\nu j}{}^i - (\mu \leftrightarrow \nu). \tag{3.5}$$

They differ only by the terms containing the $\mathbb{SO}(3)$ connection $\omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$. This allows us to introduce the following postulate. In the geometric theory of defects, the Burgers vector corresponding to a surface S is defined by the integral of the torsion tensor:

$$b^i := \iint_S dx^\mu \wedge dx^\nu \, T_{\mu\nu}{}^i.$$

This definition is invariant with respect to general coordinate transformations of x^μ and covariant under global rotations. Thus the torsion tensor in the geometric theory of defects has a straightforward physical meaning: it is equal to the surface density of the Burgers vector.

The physical meaning of the $\mathbb{SO}(3)$ connection will be given in Section 4, and now we show how this definition reduces to the expression for the Burgers vector (3.3) obtained within elasticity theory. If the curvature tensor for the $\mathbb{SO}(3)$ connection vanishes, then the connection is locally trivial and there exists an $\mathbb{SO}(3)$ rotation such that $\omega_{\mu i}{}^j = 0$. In this case, we return to the previous expression (3.3).

We have shown that the presence of linear dislocations results in nontrivial torsion. In the geometric theory of defects, the vanishing of torsion, $T_{\mu\nu}{}^i = 0$, is naturally considered as a criterion for the absence of dislocations. Then the term "dislocation" includes not only linear dislocations but also arbitrary defects in elastic media. In three dimensions, there are also point and surface defects along with linear dislocations. In the geometric approach, all of them are thought of as dislocations because they are related to nontrivial torsion.

4. DISCLINATIONS

In the previous section, we related dislocations in elastic media to the torsion tensor. To this end, we introduced the $\mathbb{SO}(3)$ connection. Now we show that the curvature of the $\mathbb{SO}(3)$ connection defines the surface density of the Frank vector, which characterizes other well-known defects: disclinations in the spin structure of media [44].

Let a unit vector field $n^i(x)$ ($n^i n_i = 1$) be given at every point (spin structure). For example, n^i has the meaning of magnetic moments at every point of a medium for ferromagnets (Fig. 3a). For nematic liquid crystals the unit vector field n^i with the equivalence relation $n^i \sim -n^i$ describes the director field (Fig. 3b).

Let us fix some direction n_0^i in the medium. Then the field $n^i(x)$ at a point x can be uniquely defined by the rotation angle field $\omega^{ij}(x) = -\omega^{ji}(x)$ taking values in the Lie algebra of rotations $\mathfrak{so}(3)$ (rotation angle): $n^i = n_0^j S_j{}^i(\omega)$, where $S_j{}^i \in \mathbb{SO}(3)$ is the rotational matrix corresponding to the algebra element ω^{ij}. We use the following parameterization of the rotational group by its algebra elements:

$$S_i{}^j = (\mathsf{e}^{(\omega\varepsilon)})_i{}^j = \delta_i^j \cos\omega + \frac{(\omega\varepsilon)_i{}^j}{\omega}\sin\omega + \frac{\omega_i\omega^j}{\omega^2}(1-\cos\omega) \in \mathbb{SO}(3), \tag{4.1}$$

where $(\omega\varepsilon)_i{}^j := \omega^k\varepsilon_{ki}{}^j$ and $\omega := \sqrt{\omega^i\omega_i}$ is the magnitude of the vector ω^i. The pseudovector $\omega^k = \omega_{ij}\varepsilon^{ijk}/2$, where ε^{ijk} is the totally antisymmetric third-rank tensor, $\varepsilon^{123} = 1$, is directed along the rotational axis, its length being equal to the rotation angle.

If the medium has a spin structure, then it may have defects, called disclinations. For linear disclinations parallel to the x^3 axis, the vector field n lies in the perpendicular plane x^1, x^2. The simplest examples of linear disclinations are shown in Fig. 4. Every linear disclination is characterized

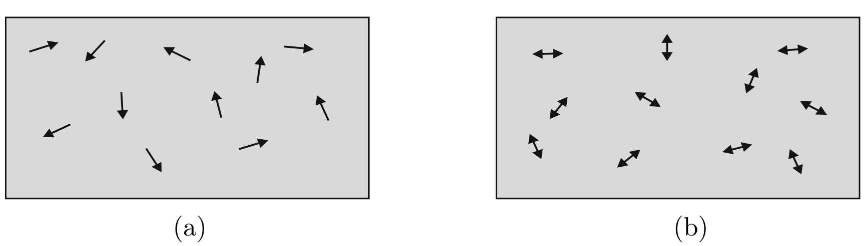

Fig. 3. Examples of media with a spin structure: (a) ferromagnets and (b) nematic liquid crystals.

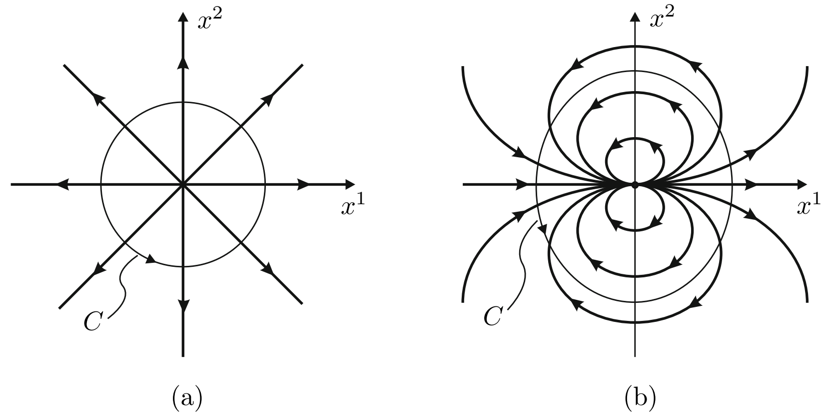

Fig. 4. Distribution of the unit vector field in the plane x^1, x^2 for linear disclinations parallel to the x^3 axis: (a) for $\Theta = 2\pi$ and (b) for $\Theta = 4\pi$.

by the Frank vector

$$\Theta_i = \varepsilon_{ijk}\Omega^{jk}, \qquad \text{where} \qquad \Omega^{ij} = \oint_C dx^\mu\, \partial_\mu \omega^{ij}, \tag{4.2}$$

with the integral taken along a closed contour C around the disclination axis. The length of the Frank vector is equal to the total rotation angle of field n^i around the disclination.

The vector field n^i defines a map of the Euclidean space into a sphere $n\colon \mathbb{R}^3 \to \mathbb{S}^2$. For linear disclinations parallel to the x^3 axis, this map is restricted to a map of the plane $\mathbb{R}^2$ into a circle $\mathbb{S}^1$. It is clear that the total rotation angle must be a multiple of 2π in this case.

Just as in the case of the displacement vector field for dislocations, the field $\omega^{ij}(x)$ taking values in the algebra $\mathfrak{so}(3)$ is not a continuous function on $\mathbb{R}^3$ in the presence of disclinations. Let us make a cut in $\mathbb{R}^3$ bounded by the disclination axis. Then we can consider the field $\omega^{ij}(x)$ as a smooth field on the whole space without the cut. Assume that all partial derivatives of $\omega^{ij}(x)$ have the same limit on different sides of the cut. Then we define a new field

$$\omega_\mu{}^{ij} := \begin{cases} \partial_\mu \omega^{ij} & \text{outside the cut,} \\ \lim \partial_\mu \omega^{ij} & \text{on the cut.} \end{cases} \tag{4.3}$$

By construction, the functions $\omega_\mu{}^{ij}$ are smooth everywhere except, possibly, the disclination axis. Then the Frank vector can be represented by the surface integral

$$\Omega^{ij} = \oint_C dx^\mu\, \omega_\mu{}^{ij} = \iint_S dx^\mu \wedge dx^\nu\, (\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij}), \tag{4.4}$$

where S is an arbitrary surface with boundary C. If the field $\omega_\mu{}^{ij}$ is fixed, then the integrability conditions for the system of equations $\partial_\mu \omega^{ij} = \omega_\mu{}^{ij}$ are given by the equalities

$$\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij} = 0. \tag{4.5}$$

This noncovariant equality yields a criterion of the absence of disclinations.

We identify the field $\omega_\mu{}^{ij}$ with the $\mathbb{SO}(3)$ connection in the geometric theory of defects. The terms with derivatives in the expression for the curvature

$$R_{\mu\nu j}{}^i = \partial_\mu \omega_{\nu j}{}^i - \omega_{\mu j}{}^k \omega_{\nu k}{}^i - (\mu \leftrightarrow \nu) \tag{4.6}$$

coincide with those in (4.5). Therefore, we postulate the covariant criterion of the absence of disclinations as the vanishing of the curvature tensor for the $\mathbb{SO}(3)$ connection: $R_{\mu\nu}{}^{ij} = 0$. At the same time we assign a physical meaning to the curvature in Cartan variables as the surface density of the Frank vector:

$$\Omega^{ij} := \iint dx^\mu \wedge dx^\nu R_{\mu\nu}{}^{ij}. \tag{4.7}$$

This definition reduces to the previous expression for the Frank vector (4.4) in the case when rotations of the vector n are confined to a fixed plane. Then rotations are restricted to the subgroup $\mathbb{SO}(2) \subset \mathbb{SO}(3)$. In this case the quadratic terms in the expression for the curvature (4.6) vanish, because the rotation group of the plane $\mathbb{SO}(2)$ is abelian, and we get the previous expression (4.4) for the Frank vector.

5. THE 'T HOOFT–POLYAKOV MONOPOLE

The 't Hooft–Polyakov monopole is a static spherically symmetric solution with finite energy of the field equations of the $\mathbb{SU}(2)$ gauge Yang–Mills model with the triplet of scalar fields φ in the adjoint representation and $\lambda\varphi^4$ interaction [20, 50] (see also [46, 54, 56]). Many other static solutions are related to this one, but they do not have spherical symmetry and satisfy some boundary conditions at infinity, where the triplet of scalar fields takes values on a two-dimensional sphere and the components of the $\mathbb{SU}(2)$ connection tend to zero. These solutions are divided into homotopically inequivalent classes and characterized by the topological charge (index of the map $\mathbb{S}^2 \to \mathbb{S}^2$ of the boundary of the three-dimensional Euclidean space, represented by a two-dimensional sphere, into the range of values of the triplet of scalar fields). These classes of field configurations have some properties of particles (finiteness of energy, stability, and localization in space) and are highly interesting from the theoretical point of view.

We will show below that solutions of the 't Hooft–Polyakov type have a straightforward physical interpretation in the geometric theory of defects and describe a continuous distribution of dislocations and disclinations, because the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic [30].

5.1. The action and vacuum solutions. Recall that the Lie algebra $\mathfrak{su}(2)$ is compact, simple, coincides with the Lie algebra of three-dimensional rotations $\mathfrak{so}(3)$, and is defined by the commutation relations

$$[J_i, J_j] = -\varepsilon_{ij}{}^k J_k, \qquad i,j,k = 1,2,3, \tag{5.1}$$

where J_i is a basis of the Lie algebra, ε_{ijk} is the totally antisymmetric third-rank tensor, and the indices are raised and lowered by using the Euclidean metric δ_{ij}, which is proportional to the Killing–Cartan form in this case.

Consider the $\mathbb{SU}(2)$ gauge model in the Minkowskian space $\mathbb{R}^{1,3}$ with Cartesian coordinates x^α which is described by the Lagrangian

$$L = -\frac{1}{4}F^{\alpha\beta i}F_{\alpha\beta i} + \frac{1}{2}\nabla^\alpha\varphi^i\nabla_\alpha\varphi_i - \frac{1}{4}\lambda(\varphi^2 - a^2)^2, \tag{5.2}$$

where $A_\alpha{}^i$ are the components of the local form of the $\mathbb{SU}(2)$ connection (Yang–Mills fields),

$$F_{\alpha\beta}{}^i := \partial_\alpha A_\beta{}^i - \partial_\beta A_\alpha{}^i + eA_\alpha{}^j A_\beta{}^k \varepsilon_{jk}{}^i$$

is the Yang–Mills field strength (components of the local curvature form of the $\mathbb{SU}(2)$ connection), $e \in \mathbb{R}$, $\lambda > 0$, and $a > 0$ are coupling constants, $\varphi := (\varphi^i) \in \mathbb{R}^3$ is the triplet of real scalar fields transforming under the adjoint representation of the group $\mathbb{SU}(2)$, $\varphi^2 := \varphi^i\varphi_i$, and $\nabla_\alpha\varphi^i := \partial_\alpha\varphi^i + eA_\alpha{}^j\varphi^k\varepsilon_{jk}{}^i$ is the covariant derivative of scalar fields.

Since the gauge fields in (5.2) transform under the adjoint representation of $\mathbb{SU}(2)$ and it coincides with the fundamental representation of $\mathbb{SO}(3)$, everything is reduced to the orthogonal rotational group $\mathbb{SO}(3)$ from the point of view of equations of motion.

The Lagrangian (5.2) yields the following equations of motion:

$$\frac{\delta S}{\delta A_\alpha{}^i} = \nabla_\beta F^{\beta\alpha}{}_i + e(\nabla^\alpha \varphi^j)\varphi^k \varepsilon_{ikj} = 0, \tag{5.3}$$

$$\frac{\delta S}{\delta \varphi^i} = -\nabla^\alpha \nabla_\alpha \varphi_i - \lambda(\varphi^2 - a^2)\varphi_i = 0. \tag{5.4}$$

It implies the Hamiltonian density

$$H = -\frac{1}{2}P^{\mu i}P_{\mu i} + \frac{1}{4}F^{\mu\nu i}F_{\mu\nu i} + \frac{1}{2}p^i p_i - \frac{1}{2}\nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4}\lambda(\varphi^2 - a^2)^2 + \mu^i \nabla_\mu P^\mu{}_i + \lambda^i P^0{}_i, \tag{5.5}$$

where $(P^\alpha{}_i) = (P^0{}_i, P^\mu{}_i)$ and p_i are momenta conjugate to the potentials $(A_\alpha{}^i) = (A_0{}^i, A_\mu{}^i)$ and scalar fields φ_i, and μ^i and λ^i are Lagrange multipliers standing in front of the first-class constraints $P^0{}_i = 0$ and $\nabla_\mu P^\mu{}_i = 0$. We recall that the Greek letters from the middle of the alphabet take only space values, $\mu, \nu, \ldots = 1, 2, 3$. The energy is, by definition, the numerical value of the Hamiltonian for physical degrees of freedom, i.e., the Hamiltonian after solving all constraints and gauge conditions. In the case under consideration, the energy density for a given field configuration is obtained from the Hamiltonian (5.5) after discarding the last two terms proportional to constraints:

$$E = -\frac{1}{2}P^{\mu i}P_{\mu i} + \frac{1}{4}F^{\mu\nu i}F_{\mu\nu i} + \frac{1}{2}p^i p_i - \frac{1}{2}\nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4}\lambda(\varphi^2 - a^2)^2. \tag{5.6}$$

It is explicitly positive definite. Recall that the space Greek indices μ and ν are raised and lowered by the negative definite metric $\eta_{\mu\nu} = -\delta_{\mu\nu}$ in our case.

The expression for the energy density (5.6) does not depend on $A_0{}^i$. For simplicity, we choose the time gauge $A_0{}^i = 0$. The solutions of equations of motion (5.3), (5.4) with minimal energy correspond to vacuum. In the case under study, the minimal value is zero and is achieved if and only if the following conditions hold:

$$P^\mu{}_i = 0, \qquad p_i = 0, \qquad F_{\mu\nu}{}^i = 0, \qquad \nabla_\alpha \varphi^i = 0, \qquad \varphi^2 = a^2. \tag{5.7}$$

The first two conditions mean that the vacuum solution in the time gauge must be static. The third condition implies that the components of the gauge fields must be pure gauge, and, without loss of generality, we put $A_\mu{}^i = 0$. Then the last two equations imply the equalities $\partial_\mu \varphi^i = 0$ and $\varphi^2 = a^2$.

5.2. Static spherically symmetric solutions. We consider the following ansatz: $A_0{}^i = 0$, $A_\mu{}^i = A_\mu{}^i(\boldsymbol{x})$, and $\varphi^i = \varphi^i(\boldsymbol{x})$, where $\boldsymbol{x} := (x^\mu) \in \mathbb{R}^3$ is a point in Euclidean space. In this case, the equations of motion (5.3), (5.4) are

$$\begin{aligned} &\nabla_\nu F^{\nu\mu}{}_i + e(\nabla^\mu \varphi^j)\varphi^k \varepsilon_{ikj} = 0, \\ &-\nabla^\mu \nabla_\mu \varphi_i - \lambda(\varphi^2 - a^2)\varphi_i = 0. \end{aligned} \tag{5.8}$$

These are exactly the Euler–Lagrange equations for the Euclidean three-dimensional action with the Lagrangian

$$L = -\frac{1}{4}F^{\mu\nu i}F_{\mu\nu i} + \frac{1}{2}\nabla^\mu \varphi^i \nabla_\mu \varphi_i - \frac{1}{4}\lambda(\varphi^2 - a^2)^2, \tag{5.9}$$

which depends only on the space components $A_\mu{}^i(\boldsymbol{x})$ and $\varphi^i(\boldsymbol{x})$.

Let us refine the definition of spherical symmetry. The rotational group $\mathbb{SO}(3)$ acts naturally in the coordinate space $(x^\mu) \in \mathbb{R}^3$, on which all fields are defined. Moreover, there is another three-dimensional Euclidean space $(\varphi^i) \in \mathbb{R}^3$, the target space. Therefore, the action of the rotational group should be extended. There is an alternative: we can say that the $\mathbb{SO}(3)$ group either does not act in the target space at all or acts in the same way as in the coordinate space x^μ. The

’t Hooft–Polyakov monopole corresponds to the second definition. In this case, the group of global $\mathbb{SO}(3)$ rotations acts on Greek and Latin indices in the same way, and they can be identified.

Since the Greek and Latin indices are identified in what follows, we change the sign of the space metric, $\eta_{\mu\nu}\mapsto\delta_{\mu\nu}$. In other words, we minimize the energy

$$\mathcal{E}:=\int d\boldsymbol{x}\left(\frac{1}{4}F^{\mu\nu i}F_{\mu\nu i}+\frac{1}{2}\nabla^\mu\varphi^i\nabla_\mu\varphi_i+\frac{1}{4}\lambda(\varphi^2-a^2)^2\right), \tag{5.10}$$

where the Greek indices $\mu,\nu=1,2,3$ are raised and lowered by using the Euclidean metric $\delta_{\mu\nu}$, and

$$\begin{aligned} F_{\mu\nu}{}^i&=\partial_\mu A_\nu{}^i-\partial_\nu A_\mu{}^i+eA_\mu{}^jA_\nu{}^k\varepsilon_{jk}{}^i,\\ \nabla_\mu\varphi^i&=\partial_\mu\varphi^i+eA_\mu{}^j\varphi^k\varepsilon_{jk}{}^i. \end{aligned} \tag{5.11}$$

The Euler–Lagrange equations for the functional (5.10) are

$$\begin{aligned} \frac{\delta\mathcal{E}}{\delta A_\mu{}^i}&=-\nabla_\nu F^{\nu\mu}{}_i+e(\nabla^\mu\varphi^j)\varphi^k\varepsilon_{ikj}=0,\\ \frac{\delta\mathcal{E}}{\delta\varphi^i}&=-\nabla^\mu\nabla_\mu\varphi^i+\lambda(\varphi^2-a^2)\varphi^i=0. \end{aligned} \tag{5.12}$$

This system of equations is solved with the spherically symmetric boundary conditions:

$$\lim_{r\to\infty}A_\mu{}^i\to 0,\qquad \lim_{r\to\infty}\varphi^i\to\frac{x^i}{r}a. \tag{5.13}$$

Now we make the spherically symmetric ansatz

$$\varphi^i=\frac{x^i}{r}\frac{H}{er},\qquad A_\mu{}^i=\frac{\varepsilon_\mu{}^{ij}x_j}{r}\frac{K-1}{er}, \tag{5.14}$$

where $H(r)$ and $K(r)$ are some unknown functions of radius only. After simple calculations, the Euler–Lagrange equations (5.12) become

$$r^2K''=K(K^2+H^2-1),\qquad r^2H''=2HK^2+\lambda\left(\frac{H^2}{e^2}-a^2r^2\right)H. \tag{5.15}$$

At present, an analytic solution is known only for $\lambda=0$. It is [4, 51]

$$K=\frac{ear}{\sinh(ear)},\qquad H=\frac{ear}{\tanh(ear)}-1 \tag{5.16}$$

and is called the *Bogomol’nyi–Prasad–Sommerfield solution*. It is easy to verify that this solution has finite energy. Numerical analysis of the system of equations (5.15) shows that there exist other spherically symmetric solutions with finite energy.

5.3. The ’t Hooft–Polyakov monopole in the geometric theory of defects. It was shown in Subsection 5.2 that static monopole solutions minimize the energy (5.10). This is a three-dimensional functional depending on the $\mathbb{SO}(3)$ connection, in which the metric is supposed to be Euclidean. Consider it as an expression for the free energy in the geometric theory of defects, the triplet of scalar fields φ^i being considered as the source of defects.

The Euclidean metric means that elastic stresses in the medium are absent. The Cartan variables for the monopole solutions are

$$e_\mu{}^i=\delta_\mu^i,\qquad \omega_\mu{}^{ij}=A_\mu{}^k\varepsilon_k{}^{ij}=\left(\delta_\mu^jx^i-\delta_\mu^ix^j\right)\frac{K-1}{er^2}, \tag{5.17}$$

where the spherically symmetric $\mathbb{SO}(3)$ connection (5.14) is used. Note that the vielbein is also chosen in a spherically symmetric form. Simple calculations yield the following expressions for the curvature and torsion:

$$R_{\mu\nu}{}^k=\frac{1}{2}R_{\mu\nu}{}^{ij}\varepsilon_{ij}{}^k=F_{\mu\nu}{}^k=\varepsilon_{\mu\nu}{}^k\frac{K'}{er}-\frac{\varepsilon_{\mu\nu}{}^j x_j x^k}{er^3}\left(K'-\frac{K^2-1}{r}\right), \tag{5.18}$$

$$T_{\mu\nu}{}^k=\left(\delta^k_\mu x_\nu-\delta^k_\nu x_\mu\right)\frac{K-1}{er^2}. \tag{5.19}$$

In the geometric theory of defects, the curvature (5.18) and torsion (5.19) are physically interpreted as the surface densities of the Frank and Burgers vectors. That is, they are equal to the kth components of the corresponding vectors on the unit surface area element $dx^\mu\wedge dx^\nu$. If s^μ is the normal to the area element, then the corresponding densities of the Frank and Burgers vectors are

$$f_\mu{}^i:=\frac{1}{2}\varepsilon_\mu{}^{\nu\rho}R_{\nu\rho}{}^i=\frac{1}{3er}\delta^i_\mu\left(2K'+\frac{K^2-1}{r}\right)-\frac{1}{er}\left(\widehat{x}_\mu\widehat{x}^i-\frac{1}{3}\delta^i_\mu\right)\left(K'-\frac{K^2-1}{r}\right), \tag{5.20}$$

$$b_\mu{}^i:=\frac{1}{2}\varepsilon_\mu{}^{\nu\rho}T_{\nu\rho}{}^i=\varepsilon_\mu{}^{ij}\widehat{x}_j\frac{K-1}{er}, \tag{5.21}$$

where $\widehat{x}^\mu:=x^\mu/r$ and the tensor $f_\mu{}^i$ is decomposed into irreducible parts.

The functions $K(r)$ and $H(r)$ for the Bogomol'nyi–Prasad–Sommerfield solution are given by (5.16). They have the following asymptotics:

$$\begin{aligned}K\big|_{r\to0}&\approx1-\frac{(ear)^2}{6}-\frac{(ear)^4}{120}, &\qquad K\big|_{r\to\infty}&\approx2ear\mathrm{e}^{-ear}\to0,\\ H\big|_{r\to0}&\approx1+\frac{(ear)^2}{3}-\frac{2(ear)^4}{15}, &\qquad H\big|_{r\to\infty}&\approx ear-1\to\infty.\end{aligned} \tag{5.22}$$

The corresponding asymptotics of the densities of the Frank and Burgers vectors are

$$\begin{aligned}f_\mu{}^i\big|_{r\to0}&\approx-\frac{1}{3}\delta^i_\mu\left(ea^2+\frac{7}{90}e^3a^4r^2\right)+\frac{2}{45}x_\mu x^ie^3a^4\to-\frac{1}{3}\delta^i_\mu ea^2,\\ b_\mu{}^i\big|_{r\to0}&\approx-\frac{1}{6}\varepsilon_\mu{}^{ij}x_j\left(ea^2+\frac{e^2a^4r^2}{20}\right)\to-\frac{1}{6}\varepsilon_\mu{}^{ij}x_jea^2,\\ \varphi^i\big|_{r\to0}&\approx\frac{1}{3}x^i\left(ea^2-\frac{2e^3a^4r^2}{5}\right)\to\frac{1}{3}x^iea^2,\end{aligned} \tag{5.23}$$

$$f_\mu{}^i\big|_{r\to\infty}\approx-\frac{x_\mu x^i}{er^4}\to0,\qquad b_\mu{}^i\big|_{r\to\infty}\approx-\varepsilon_\mu{}^{ij}x_j\frac{1}{er^2}\to0,\qquad \varphi^i\big|_{r\to\infty}\approx\frac{x^i}{r}\left(a-\frac{1}{er}\right)\to\frac{x^i}{r}a.$$

It implies, in particular, that the energy integral (5.10) converges.

Thus the monopole solutions of the $\mathbb{SO}(3)$ gauge model describe continuous distributions of dislocations and disclinations in continuous media. There is no descriptive representation of such a distribution of defects by the displacement vector field and n-field, because they are not defined for continuous distributions of defects.

6. SPHERICALLY SYMMETRIC DISCLINATIONS

Let us consider the Chern–Simons action for the $\mathbb{SO}(3)$ connection as the free energy for disclinations [27, 28]. The point disclinations considered in the present section are described in [33].

Consider the three-dimensional Euclidean space with Cartesian coordinates $(x^\mu)\in\mathbb{R}^3$, $\mu=1,2,3$. Let components of the local $\mathbb{SO}(3)$ connection form $A_\mu{}^{ij}(x)=-A_\mu{}^{ji}(x)$, $i,j=1,2,3$ (the

Yang–Mills fields) be given. From the geometric point of view, we may assume that the topologically trivial manifold $\mathbb{R}^3$ is equipped with the Riemann–Cartan geometry defined by the flat vielbein $e_\mu{}^i$ satisfying the equality $\delta_{\mu\nu} = e_\mu{}^i e_\nu{}^j \delta_{ij}$ and the $\mathbb{SO}(3)$ connection $\omega_{\mu i}{}^j = A_{\mu i}{}^j$.

Since we have the third-rank totally antisymmetric tensor ε_{ijk} in three dimensions, the connection components can be parameterized by a field with two indices: $A_\mu{}^{ij} = A_\mu{}^k \varepsilon_k{}^{ij}$. The related components of the local curvature form for the $\mathbb{SO}(3)$ connection are

$$F_{\mu\nu k} := \frac{1}{2} F_{\mu\nu}{}^{ij} \varepsilon_{ijk} = \partial_\mu A_{\nu k} - \partial_\nu A_{\mu k} + A_\mu{}^i A_\nu{}^j \varepsilon_{ijk}. \tag{6.1}$$

We assume that the group of global $\mathbb{SO}(3)$ rotations acts simultaneously on the base $\mathbb{R}^3$ and on the Lie algebra $\mathfrak{so}(3)$, which is also the three-dimensional space $\mathbb{R}^3$ as a vector space. This means that if $S \in \mathbb{SO}(3)$ is an orthogonal matrix, then the transformation has the form

$$A_\mu{}^{ij} \mapsto S^{-1\nu}_{\ \ \mu} A_\nu{}^{kl} S_k{}^i S_l{}^j, \qquad S \in \mathbb{SO}(3).$$

The difference between Greek and Latin indices disappears under this assumption, but we will distinguish them if possible.

The most general spherically symmetric connection components are

$$A_\mu{}^i = \varepsilon_\mu{}^{ij} \frac{x_j}{r} \frac{K-1}{r} + \delta_\mu^i V(r) + \frac{x_\mu x^i}{r^2} U(r), \qquad r \geq 0, \tag{6.2}$$

where $K(r)$, $V(r)$, and $U(r)$ are arbitrary sufficiently smooth functions of radius. The case $V = U = 0$ corresponds to the 't Hooft–Polyakov monopole (5.14).

Straightforward calculations of the components of the spherically symmetric curvature tensor yield

$$\begin{aligned} F_{\mu\nu}{}^i = {} & \frac{\varepsilon_{\mu\nu}{}^i}{r} \big[K' + rV(V+U)\big] + \frac{\varepsilon_{\mu\nu}{}^j x_j x^i}{r^3} \left(-K' + \frac{K^2-1}{r} - rVU\right) \\ & + \frac{x_\mu \delta_\nu^i - x_\nu \delta_\mu^i}{r^2} \big[rV' - U - (K-1)(V+U)\big]. \end{aligned} \tag{6.3}$$

We assume that the expression for the free energy of the $\mathbb{SO}(3)$ connection is given by the Chern–Simons action [7], which can be conveniently written using the notation of differential forms,

$$S_{\text{CS}} := \int_{\mathbb{R}^3} \operatorname{tr}\left(dA \wedge A - \frac{2}{3} A \wedge A \wedge A\right), \tag{6.4}$$

where $dx^\mu A_{\mu i}{}^j(x)$ is the matrix-valued local connection 1-form, the symbol $\wedge$ denotes external multiplication, and matrix indices are dropped. The Euler–Lagrange equations for the Chern–Simons action (6.4) are nonlinear: $F_{\mu\nu}{}^i = 0$ (flat connection). In the spherically symmetric case, these equations reduce to the following system:

$$K' + rV(V+U) = 0, \tag{6.5}$$

$$-K' + \frac{K^2-1}{r} - rVU = 0, \tag{6.6}$$

$$rV' - U - (K-1)(V+U) = 0, \tag{6.7}$$

because the tensor structures in (6.3) are functionally independent.

Theorem 6.1. *A general solution to the system of equations* (6.5)–(6.7) *is*

$$K=\cos f,\qquad V=\frac{\sin f}{r},\qquad U=\frac{rf'-\sin f}{r},\tag{6.8}$$

where $f(r)$ *is an arbitrary sufficiently smooth function of the radius* $r\geq 0$.

The proof is given in [33].

Thus a general spherically symmetric solution of the Euler–Lagrange equations is

$$A_\mu{}^i=\frac{\varepsilon_\mu{}^{ij}x_j}{r^2}(\cos f-1)+\delta^i_\mu\frac{\sin f}{r}+\frac{x_\mu x^i}{r^3}(rf'-\sin f),\tag{6.9}$$

where $f(r)$ is an arbitrary function. If the function f is smooth and tends to zero sufficiently fast as $r\to 0$, then the curvature for the $\mathbb{SO}(3)$ connection is identically zero on the whole $\mathbb{R}^3$ and there is no disclination. If $f(0)\neq 0$, then disclinations may appear at the origin of the coordinate system. To understand their structure, we have to find the unit vector field $n(x)$.

6.1. Point disclinations. In the simply connected domains of the Euclidean space with vanishing curvature, the connection components are pure gauge, $A_\mu=\partial_\mu S^{-1}S$, where $S\in\mathbb{SO}(3)$ and matrix indices are dropped. Our aim is to find the orthogonal matrix S for a given connection (6.9). The equation for S has the form $\partial_\mu S^{-1}=A_\mu S^{-1}$ and coincides with the condition of parallel displacement of vectors. In the case of zero curvature, the parallel displacement does not depend on the curve along which it is made. Therefore, we consider an arbitrary curve $\gamma=x(t)$, $t\in[0,b]$, starting at $x_0:=x(0)$ and ending at $x_b:=x(b)$. Then for the matrix S, we obtain an ordinary differential equation along γ:

$$\dot S^{-1}=\dot x^\mu A_\mu S^{-1}.\tag{6.10}$$

When the curve passes through the point $x(t)$, the solution of this equation is given by the path-ordered exponential:

$$S^{-1}(x(t))=\mathsf{P}\exp\left(\int_0^t ds\,\dot x^\mu(s)A_\mu(s)\right)S_0^{-1},\tag{6.11}$$

where S_0 is the orthogonal matrix at the initial point x_0.

Let γ be the ray starting at the infinite point and ending at a point x, i.e., $\gamma=(x^\mu t)$, $t\in[1,\infty]$, and $S_0:=S(\infty):=\mathbb{1}$. Then the equality $\dot x^\mu A_{\mu i}{}^j=f'x^k\varepsilon_{ki}{}^j$ holds for the connection (6.9). Now we can easily check that the matrices in the integrand in the path-ordered exponential commute: $[\dot x^\mu A_\mu,\dot x^\nu A_\nu]=0$. Consequently, the path-ordered exponential coincides with the usual one, and the integral (6.11) can be easily calculated:

$$\int_\infty^1 ds\,\dot x^\mu A_{\mu i}{}^j=\int_\infty^1 ds\,f'x^k\varepsilon_{ki}{}^j=\int_\infty^1 ds\,\frac{df}{d(rs)}\,x^k\varepsilon_{ki}{}^j=\frac{x^k\varepsilon_{ki}{}^j}{r}\big[f(r)-f(\infty)\big].$$

That is, the solution of (6.10) is

$$S^{-1}{}_i^j=\exp(-f^k\varepsilon_{ki}{}^j),\qquad f^k:=\frac{x^k}{r}\big[f(\infty)-f(r)\big].\tag{6.12}$$

The vector (f^k) is an element of the Lie algebra $\mathfrak{so}(3)$. Its direction coincides with the rotational axis in the isotopic space and its length is equal to the rotation angle. The exponential map for the $\mathbb{SO}(3)$ group is well known:

$$S_i{}^j=\exp(f^k\varepsilon_{ki}{}^j)=\delta_i^j\cos F+\frac{f^k\varepsilon_{ki}{}^j}{F}\sin F+\frac{f_if^j}{F^2}(1-\cos F),\tag{6.13}$$

where $F^2:=f^if_i=[f(\infty)-f(r)]^2$.

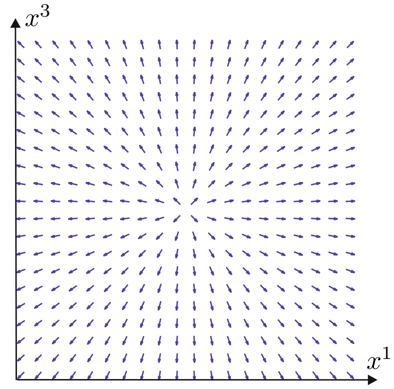

Fig. 5. Spherically symmetric "hedgehog" disclination. The section $x_2 = 0$ is shown in the figure.

6.2. Examples of point disclinations. The rotational matrix (6.13) is defined by the difference $f(\infty) - f(r)$, where $f(r)$ is an arbitrary sufficiently smooth function. Without loss of generality we put $f(\infty) = 0$ and change the sign of $f(r)$. Then we can choose $F(r) = f(r)$, and the spherically symmetric rotational matrix takes the form

$$S_i{}^j = \exp(f^k \varepsilon_{ki}{}^j) = \delta_i^j \cos f + \frac{f^k \varepsilon_{ki}{}^j}{f} \sin f + \frac{f_i f^j}{f^2}(1 - \cos f), \tag{6.14}$$

where $f^i = x^i f(r)/r$ with an arbitrary function $f(r)$ equal to zero at infinity.

Example 6.1 ("hedgehog" disclination)**.** Let us choose the spherically symmetric boundary condition at infinity: $n^i(r = \infty) = x^i/r$. Then the n-field has the same form in the whole space $\mathbb{R}^3$: $n^i(r) := n^j(\infty) S_j{}^i(r) = x^i/r$ for an arbitrary function f. We see that the n-field is directed along the radius everywhere and has the unit length. The distribution of the n-field is shown in Fig. 5.

Now we consider spherically asymmetric disclinations. Fix the vector $n_0 := (0, 0, 1)$ at infinity, and thus break the spherical symmetry. Then the components of the n-field are

$$\begin{aligned} n_1 &= -\frac{x_2}{r} \sin f + \frac{x_1 x_3}{r^2}(1 - \cos f), \\ n_2 &= \frac{x_1}{r} \sin f + \frac{x_2 x_3}{r^2}(1 - \cos f), \\ n_3 &= \cos f + \frac{x_3^2}{r^2}(1 - \cos f), \end{aligned}$$

where we lowered the coordinate indices for simplicity to distinguish them from exponents. Let us pass to the spherical coordinates, $(x_1, x_2, x_3) \mapsto (r, \theta, \varphi)$. Then the components of the n-field are

$$\begin{aligned} n_1 &= -\sin\theta \sin\varphi \sin f + \sin\theta \cos\theta \cos\varphi (1 - \cos f), \\ n_2 &= \sin\theta \cos\varphi \sin f + \sin\theta \cos\theta \sin\varphi (1 - \cos f), \\ n_3 &= \cos f + \cos^2\theta (1 - \cos f). \end{aligned}$$

This implies that the limit of the n-field at the origin does not depend on the path along which the limit $r \to 0$ is taken if and only if $f(0) = 0, \pi$. This is the degenerate case, when the n-field is continuous at zero and disclinations are absent. If $f(0) \neq 0, \pi$, then the limit n-field does depend on the path to the origin along which the limit is taken. Consequently, in the general case, the origin is an essential singularity, and the model describes point disclinations located at the origin.

After fixing the vector n_0, we still have the invariance under rotations in the x_1, x_2 plane. Therefore, to visualize disclinations, it is sufficient to put $x_2 = 0$, that is, to analyze the distribution

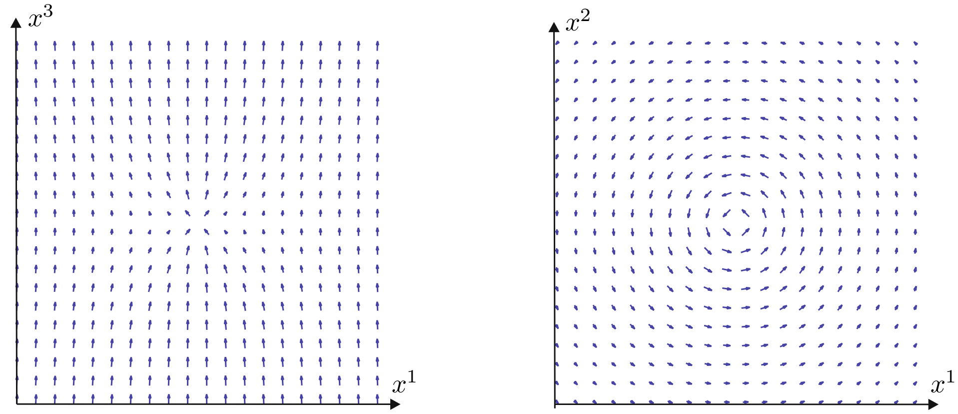

Fig. 6. Two sections ($x_2 = 0$ and $x_3 = 0$) of the disclination for $f(r) := \pi \mathrm{e}^{-r}/2$. The arrows are the projections of the vector n on the corresponding plane. If the length of an arrow is less then unity, then the vector has a component in the perpendicular direction. The spherical symmetry is broken by the boundary condition $n(\infty) := (0, 0, 1)$.

of the n-field in the x_1, x_3 plane:

$$n_1 = \frac{x_1 x_3}{r^2}(1 - \cos f), \qquad n_2 = \frac{x_1}{r}\sin f, \qquad n_3 = \cos f + \frac{x_3^2}{r^2}(1 - \cos f).$$

We see that in general the vector n has a nonzero component in the direction perpendicular to the x_1, x_3 plane, which slightly obscures the pictures.

Various distributions of the n-field depend on the choice of the function $f(r)$. We put $f(\infty) = 0$. Then the n-field coincides with n_0 at infinity. If $f(0) = 0, \pi$, the unit vector field is continuous at zero and disclinations are absent. In the opposite case, there are disclinations with an essential singularity at the origin.

Example 6.2. Set $f(r) := \pi \mathrm{e}^{-r}/2$, which implies $f(0) = \pi/2$ and $f(\infty) = 0$. In this case, the vector field n in the plane $x_2 = 0$ has all three nontrivial components. Therefore, we draw the projections of the n-field on the two planes $x_2 = 0$ and $x_3 = 0$ in Fig. 6 for visualization. The projection of the vector field has unit length on the plane $x_2 = 0$ at infinity, because the perpendicular component is absent. The projection becomes less at internal points because the perpendicular component arises. Conversely, the projections of the vectors n on the plane $x_3 = 0$ are zero at infinity and nontrivial at internal points, which is clear from the picture.

The equilibrium equations for the disclinations described above hold everywhere in $\mathbb{R}^3$ except the origin, where the $\mathbb{SO}(3)$ connection is singular. The analysis of this singularity is difficult in general, because the equations are nonlinear, and we postpone it for further investigations. We consider linear disclinations in the next section, for which the equations become linear. The singularity for these disclinations is proportional to the δ-function with support located along the disclination line.

7. LINEAR DISCLINATIONS

We assume that the expression for the free energy is given by the Chern–Simons action (6.4), as in the previous section. To describe linear disclinations, we add a source term to the action:

$$S_{\mathrm{CS}}[A] + S_{\mathrm{int}} = \int\limits_{\mathbb{R}^3} \left(\frac{1}{2} dA^i \wedge A_i + \frac{1}{6} A^i \wedge A^j \wedge A^k \epsilon_{ijk} - A^i \wedge J_i \right), \tag{7.1}$$

where J is the 2-form of the disclinations source, which is not specified here. The interaction term is similar to the minimal coupling of the electric charge to the electromagnetic field in electrodynamics.

The equilibrium equations for the action (7.1) are

$$F_{\mu\nu}{}^k = J_{\mu\nu}{}^k, \tag{7.2}$$

where $J_{\mu\nu}{}^k$ are the components of the source for the $\mathbb{SO}(3)$ connection.

The first two terms in the action (7.1) change by an external differential under local $\mathbb{SO}(3)$ rotations. Therefore, we have to impose the condition $DJ^k = 0$, where $DJ^k := dJ^k + J^j \wedge \omega_j{}^k$ is the external covariant derivative, for the self-consistency of the Euler–Lagrange equations.

Consider one linear disclination $q^\mu(t) \in \mathbb{R}^3$, where $t \in \mathbb{R}$ is a parameter along the disclination core. The interaction term is written in the form

$$S_{\text{int}} := \int dq^\mu A_{\mu i} J^i = \int dt\, \dot{q}^\mu A_{\mu i} J^i. \tag{7.3}$$

This action is invariant with respect to coordinate changes in $\mathbb{R}^3$ (up to boundary terms) and an arbitrary reparameterization of the curve $q^\mu(t)$. We assume that the disclination is located in such a way that the inequality $\dot{q}^3 \neq 0$ holds everywhere. To obtain the variation of this action with respect to the $\mathbb{SO}(3)$ connection, we insert the three-dimensional δ-function into the integrand:

$$S_{\text{int}} = \int dt\, d^3x\, \dot{q}^\mu A_{\mu i} J^i \delta^3(x-q) = \int d^3x\, \frac{\dot{q}^\mu}{\dot{q}^3} A_{\mu i} J^i \delta^2(x-q),$$

where we have integrated with respect to t using one δ-function $\delta(x^3 - q^3(t))$ and $\delta^2(x-q) := \delta(x^1 - q^1)\delta(x^2 - q^2)$ denotes the two-dimensional δ-function on the plane x^1, x^2. Then the variation of the interaction term is

$$\frac{\delta S_{\text{int}}}{\delta A_{\mu i}} = \frac{\dot{q}^\mu}{\dot{q}^3} J^i \delta^2(x-q). \tag{7.4}$$

We consider equations (7.2) on the topologically trivial manifold $\mathbb{M} \approx \mathbb{R}^3$ with Cartesian coordinate system $x^1 = x$, $x^2 = y$ and x^3. The disclination is supposed to be straight and coinciding with the x^3 axis, i.e., $q^1 = q^2 = 0$ and $q^3 = t$. We are looking for solutions of (7.2) which are invariant with respect to translations along the x^3 axis and describe rotations only in the x, y plane. In this case, the $\mathbb{SO}(3)$ connection has only two nontrivial components $A_x{}^3$ and $A_y{}^3$, which depend on a point on the plane $(x, y) \in \mathbb{R}^2 = \mathbb{C}$. To find the solution, we introduce the complex coordinate $z := x + iy$. Then the two real components of the $\mathbb{SO}(3)$ connection combine into a complex one:

$$A_z{}^3 = \frac{1}{2}A_x{}^3 - \frac{i}{2}A_y{}^3, \qquad A_{\overline{z}}{}^3 = \frac{1}{2}A_x{}^3 + \frac{i}{2}A_y{}^3. \tag{7.5}$$

The corresponding curvature tensor (field strength) has only one linearly independent complex component

$$F_{z\overline{z}}{}^3 = 2\big(\partial_z A_{\overline{z}}{}^3 - \partial_{\overline{z}} A_z{}^3\big), \tag{7.6}$$

which is linear in the connection. This is a consequence of the fact that the rotational $\mathbb{SO}(2)$ group acting on the x, y plane is abelian and nonlinear terms in the curvature tensor disappear.

In our case, the quadratic terms in the curvature vanish identically, and we are able to consider sources of the δ-function form because the equilibrium equations (7.2) become linear. Now we fix the sources

$$F_{z\overline{z}}{}^3 = 4\pi i D\delta(z), \qquad D \in \mathbb{R}, \tag{7.7}$$

where $\delta(z)$ is the two-dimensional δ-function on the complex plane. It is clear that this source has rotational symmetry.

The solution of equation (7.7) describes a new type of geometric singularity. If this equation was considered as a second-order equation for the metric, then its solution would describe a conical singularity on the x, y plane. In this case, the solution corresponds to a wedge dislocation in the geometric theory of defects [34]. Now the situation is different. We consider this equation as a first-order one for the $\mathbb{SO}(3)$ connection and show that it describes the defect of the unit vector field (disclination), the metric being Euclidean.

Equation (7.7) has the solution

$$A_z{}^3 = -\frac{iD}{z}, \qquad A_{\bar{z}}{}^3 = \frac{iD}{\bar{z}}. \tag{7.8}$$

To check that this is indeed a solution, one can use the well-known formula (see, e.g., [57])

$$\partial_z \frac{1}{\bar{z}} = \pi\delta(z) \qquad \Leftrightarrow \qquad \partial_{\bar{z}} \frac{1}{z} = \pi\delta(z). \tag{7.9}$$

The corresponding real components are

$$A_x{}^3 = -\frac{2Dy}{x^2+y^2}, \qquad A_y{}^3 = \frac{2Dx}{x^2+y^2}. \tag{7.10}$$

Outside the x^3 axis the curvature is flat, and therefore the connection is given by partial derivatives of some function. This function is the rotation angle field $\theta(x, y)$ of the unit vector field on the plane in the geometric theory of defects. This field must satisfy the following system of equations:

$$\partial_x\theta = -\frac{2Dy}{x^2+y^2}, \qquad \partial_y\theta = \frac{2Dx}{x^2+y^2}. \tag{7.11}$$

The integrability conditions for this system outside the disclination axis, $\partial_{xy}\theta = \partial_{yx}\theta$, are fulfilled, and one can easily write down a general solution

$$\theta = -2D \arctan\frac{x}{y} + C, \qquad C = \text{const}. \tag{7.12}$$

We fix the constant of integration to be $C := \pi D$. Then the solution takes the form

$$\tan\frac{\theta}{2D} = \frac{y}{x} = \tan\varphi, \tag{7.13}$$

where φ is the ordinary polar angle on the plane $(x, y) \in \mathbb{R}^2$. The polar angle changes by 2π along a contour C around the x^3 axis. We must impose the quantization condition $D = n/2$, $n \in \mathbb{Z}$, in order that the rotation angle field $\theta(x, y)$ be well defined.

Thus the rotation angle field takes the form $\theta = n\varphi$, where φ is the ordinary polar angle on the x, y plane. It is defined everywhere except the cut along a half-plane, say, $y = 0$, $x \geq 0$. The corresponding $\mathbb{SO}(3)$ connection has only two nontrivial components

$$A_x{}^{12} = -\frac{ny}{x^2+y^2} = -\frac{n}{r}\sin\varphi, \qquad A_y{}^{12} = \frac{nx}{x^2+y^2} = \frac{n}{r}\cos\varphi,$$

where $r := \sqrt{x^2+y^2}$ is the polar radius. The $\mathbb{SO}(3)$ connection is defined everywhere on the x, y plane except the origin, where its curl has a δ-function singularity (7.7). We see that the $\mathbb{SO}(3)$ connection behaves much better than the respective rotation angle field, as it should be in the geometric theory of defects.

Thus, along a closed contour C encircling the x^3 axis, the rotation angle field changes from 0 to $2\pi n$, where $|2\pi n| = |\Omega|$ is the magnitude of the Frank vector. It is exactly the linear disclination of the unit vector field with the core coinciding with the x^3 axis. For $n = 0$ the disclination is absent. This case requires separate treatment: for $D = 0$, the equality $\theta = 0$ must hold as a consequence of (7.12). Two simple examples of linear disclinations for $n = 1$ and $n = 2$ are presented in Fig. 4, where the distribution of the rotation angle field is shown on the x, y plane.

8. CONCLUSIONS

We give a review of the geometric theory of defects in this paper. Currently known examples of disclinations are described. Since the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, the static solutions of $\mathbb{SU}(2)$ gauge models have a straightforward physical interpretation in the framework of the geometric theory of defects. In particular, the 't Hooft–Polyakov monopole has a physical interpretation in crystals: it describes continuous distributions of dislocations and disclinations.

We have shown that the Chern–Simons action is well suited for describing single disclinations in the geometric theory of defects. To describe point disclinations, we have found the most general spherically symmetric $\mathbb{SO}(3)$ connection containing one arbitrary function of radius. Two examples are given: a spherically symmetric "hedgehog" disclination and a point disclination with a constant value of the n-field at infinity and an essential singularity at the origin. The Chern–Simons action also describes linear disclinations. As an example, we have considered straight linear disclinations with Frank vector being a multiple of 2π.

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