ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2021, Vol. 313, pp. 17–26. © Pleiades Publishing, Ltd., 2021.
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On the Inversion Formula of Linear Quantization and the Evolution Equation for the Wigner Function

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Received January 20, 2021; revised February 5, 2021; accepted March 29, 2021

Abstract—We consider the inversion problem for linear quantization defined by an integral transformation relating the matrix of a quantum operator to its classical symbol. For an arbitrary linear quantization, we construct evolution equations for the density matrix and the Wigner function. It is shown that the Weyl quantization is the only one for which the evolution equation of the Wigner function is free of a quasi-probability source, which distinguishes this quantization as the only physically adequate one in the class under consideration. As an example, we give an exact stationary solution for the Wigner function of a harmonic oscillator with an arbitrary linear quantization, and construct a sequence of quantizations that approximate the Weyl quantization and tend to it in the weak sense so that the Wigner function remains positive definite.

DOI: 10.1134/S0081543821020036

1. INTRODUCTION

The Wigner function [21] is commonly understood as the Weyl symbol of the density matrix. In problems of quantum statistics, this function is used as quasi-probability in the sense that the average value of a quantum operator is obtained by averaging the corresponding classical symbol with the Wigner function in the phase space of classical mechanics. However, in the general case, the Wigner function is only real rather than nonnegative; moreover, it depends on the Planck constant, so the analogy with the classical probability density is formal. Nevertheless, the convenience of performing calculations in the classical phase space makes them more attractive compared to using the complex density matrix of a quantum system. Thus, in [11, 12] the tomographic representation of quantum mechanics was studied using the Wigner function as an analog of the density of the distribution function. The properties of the Wigner function have also been investigated for various quantum systems [14, 8] and for various commutation rules of canonical variables [9].

In many problems of quantum mechanics, relatively simple systems are considered: a set of nonrelativistic particles interacting with an external field or with each other through a pair potential. For such systems, the quantum Hamiltonian operator does not depend on the quantization rule, i.e., on the arrangement of noncommuting operators in the product of coordinates and momenta. However, the development of quantum theory has made it necessary to consider quantum equations in the context of choosing a quantization rule. There are two reasons for that. First, it is needed to formalize mathematically the transition from classical to quantum mechanics, which was begun by Weyl [20], Born and Jordan [7], von Neumann [15], and other founders of quantum theory and further developed by Berezin [2, 3]. Second, the questions of quantizing mechanical systems with a potential that depends not only on coordinates but also on momenta have become topical. These

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are the so-called weakly relativistic systems with delayed interaction [10, 19, 17]. This sparked interest in quantum equations written for a parametric class of quantizations containing various variants of symmetrization. A relevant approach was developed in [18]. It turned out that the Wigner function depends on the quantization rule, but the evolution equation was obtained only for the τ-quantization [17, 16]. Particular results were obtained for equilibrium distributions within the framework of Chernoff equivalent quantum semigroups [5]. In the present paper, we extend the evolution equation for the Wigner function to the case of an arbitrary linear quantization.

2. LINEAR QUANTIZATION AND INVERSION FORMULA

Let $A(q,p)$ be a dynamical quantity defined in the phase space $\mathbb{R}^2$ of classical Hamiltonian mechanics, so that q and p are generalized coordinate and momentum. Denote by $\widehat{A}$ the quantum operator obtained by quantizing the function $A(q,p)$. The operator $\widehat{A}$ acts in a Hilbert space L of functions $\psi(x)$, which is assumed to be the space $L_2(\mathbb{R})$. The function $A(q,p)$ is called the classical symbol of the operator $\widehat{A}$.

Among many different rules of correspondence between functions and operators, we will consider the so-called linear quantization, when the relation between the matrix of an operator and its symbol is given by a linear transformation. The idea of linear quantization was formulated in Berezin's papers [2, 3], where the correspondence between the matrix $\widetilde{A}(x,y)$ of an operator $\widehat{A}$ and its classical symbol $A(q,p)$ was established by an integral transformation of the symbol with kernel $K(q,p|x,y)$, called the quantization kernel:

$$\widetilde{A}(x,y) = \int A(q,p)K(q,p|x,y)\,dq\,dp. \tag{2.1}$$

The operator $\widehat{A}$ itself acts on $\psi \in L$ as

$$\widehat{A}\psi(x) = \int \widetilde{A}(x,y)\psi(y)\,dy. \tag{2.2}$$

Here and in what follows, integrals are taken from $-\infty$ to ∞. In [16] the so-called τ-quantization kernel was introduced:

$$K_\tau(q,p|x,y) = \frac{1}{2\pi\hbar}\delta\big(q-(1-\tau)x-\tau y\big)\exp\left[\frac{ip(x-y)}{\hbar}\right], \qquad \tau\in[0,1], \tag{2.3}$$

where $\hbar$ is the Planck constant. When the quantization with kernel (2.3) is applied, the generalized coordinate and momentum are assigned the operators $\widehat{q}=x$ and $\widehat{p}=-i\hbar\,\partial/\partial x$ irrespective of the value of τ.

The matrix obtained by formula (2.1) with the quantization kernel (2.3) will be denoted by $\widetilde{A}_\tau(x,y)$. For example, for the symbol $\varphi(q)p^m$ the τ-quantization yields the operator

$$\widehat{A}_\tau = (i\hbar)^m\sum_{k=0}^{m} C_m^k\tau^k\varphi^{(k)}(x)\frac{\partial^{m-k}}{\partial x^{m-k}}. \tag{2.4}$$

The τ-quantization is Hermitian only for $\tau=1/2$. In this case, formula (2.3) defines the so-called Weyl quantization. Other versions of linear Hermitian quantizations can be obtained, for example, by taking linear combinations of kernels of the form (2.3), as is done in [16]. Such a combination can be represented as a linear integral transformation of the quantization kernel (2.3) with some generalized function $Q(\tau)$ satisfying the condition

$$\int Q(\tau)\,d\tau = 1. \tag{2.5}$$

The resulting quantization kernel can be represented as follows:

$$K(q,p|x,y)=\int_0^1 Q(\tau)K_\tau(q,p|x,y)\,d\tau,\qquad \widetilde{A}(x,y)=\int_0^1 Q(\tau)\widetilde{A}_\tau(x,y)\,d\tau. \tag{2.6}$$

We will call $Q(\tau)$ the symmetrization function. If it is symmetric with respect to the point $1/2$, then the quantum operator given by (2.6) is Hermitian. We also introduce the characteristic function $X_Q(z)$ of the quantization and the moments σ_n of the symmetrization function:

$$X_Q(z)=\int Q(\tau)\exp(iz\tau)\,d\tau=\sum_{n=0}^{\infty}\sigma_n\frac{(iz)^n}{n!}. \tag{2.7}$$

By specifying different symmetrization functions $Q(\tau)$ of the τ-quantization kernels, we obtain different quantum operators corresponding to the same symbol. For example, as applied to the symbol $A(q,p)=\varphi(q)p^m$, the quantization (2.6) yields

$$\widehat{A}=(i\hbar)^m\sum_{k=0}^{m}\binom{m}{k}\sigma_k\varphi^{(k)}(x)\frac{\partial^{m-k}}{\partial x^{m-k}}. \tag{2.8}$$

Thus, to define an operator corresponding to a classical polynomial (in momenta) symbol, it is sufficient to specify the moments σ_k.

Usually, one chooses nonnegative symmetrization functions $Q(\tau)$, so that an arbitrary linear quantization can be viewed as a probabilistic mixture of τ-quantizations. For example, for the Weyl quantization we have $Q(\tau)=\delta(\tau-1/2)$, for the Born quantization we have $Q(\tau)=1$, $\tau\in[0,1]$, $Q(\tau)=0$, $\tau\notin[0,1]$, and for the Jordan quantization we have $Q(\tau)=(\delta(\tau)+\delta(\tau-1))/2$. The characteristic functions of these quantizations are as follows: $X_Q(z)=e^{iz/2}$ for the Weyl quantization, $X_Q(z)=(e^{iz}-1)/(iz)$ for the Born quantization, and $X_Q(z)=(e^{iz}+1)/(iz)$ for the Jordan quantization.

Using the characteristic function of the quantization, $X_Q(z)$, one can represent the quantization kernel (2.6) (taking account of (2.3)) as

$$K(q,p|x,y)=\frac{1}{(2\pi)^2\hbar}\int\exp\biggl[\frac{iz(q-x)+ip(x-y)}{\hbar}\biggr]X_Q(z(x-y))\,dz. \tag{2.9}$$

Let us obtain an inversion formula for the quantization (2.1) in the case when the quantization kernel has the form (2.9). To this end we use the fact that an inverse transformation exists for any τ-quantization. Namely, if the relation between the symbol and the matrix of the operator is given by the quantization rule (2.1) with kernel (2.3), then for any τ there is an inverse transformation of the form

$$\begin{gathered}A(q,p)=\int L_\tau(x,y|q,p)\widetilde{A}_\tau(x,y)\,dx\,dy,\\ L_\tau(x,y|q,p)=\delta\bigl(q-(1-\tau)x-\tau y\bigr)\exp\biggl[-\frac{ip(x-y)}{\hbar}\biggr].\end{gathered} \tag{2.10}$$

Thus, the kernel $L_\tau(x,y|q,p)$ is related to the kernel $K_\tau(q,p|x,y)$ as $L_\tau(x,y|q,p)=2\pi\hbar K_\tau^*(q,p|x,y)$, where the asterisk means complex conjugation. We emphasize that for an arbitrary linear combination of τ-quantizations with a kernel of the form (2.9), the inverse transformation has not been previously considered.

For a linear quantization, the connection between the matrix of the operator and the symbol is given by a kernel of the inverse transformation, $L(x,y|q,p)$, so that

$$A(q,p)=\int L(x,y|q,p)\widetilde{A}(x,y)\,dx\,dy. \tag{2.11}$$

We represent the sought kernel $L(x, y|q, p)$ as a linear integral transformation of the inverse kernels (2.10) with an unknown symmetrization function $S(\tau)$, which is to be determined:

$$L(x, y|q, p) = \int S(\tau) L_\tau(x, y|q, p)\, d\tau = \frac{1}{2\pi} \int \exp\left[\frac{iz(q-x) - ip(x-y)}{\hbar}\right] X_S(z(x-y))\, dz. \quad (2.12)$$

To find conditions on the characteristic function $X_S(z)$, we substitute expressions (2.12) and (2.1) into (2.11). Introducing the new variables $u = x - y$ and $v = x + y$, we transform the integral (2.11) to

$$A(q, p) = \frac{1}{4\pi^2\hbar} \int \exp\left[\frac{ik(q-\xi) - iu(p-\eta)}{\hbar}\right] A(\xi, \eta) X_S(uk) X_Q^*(uk)\, d\xi\, d\eta\, dk\, du. \quad (2.13)$$

This implies that if the characteristic function $X_S(z)$ corresponding to the symmetrization function of the inverse kernels of τ-quantizations (2.12) is related to the characteristic function $X_Q(z)$ by the condition

$$X_S(z) X_Q^*(z) = 1, \quad (2.14)$$

then (2.13) becomes an identity. Indeed, in this case the integration with respect to dk and du can be performed independently, so that we obtain $2\pi\delta(q-\xi)$ and $2\pi\hbar(p-\eta)$, respectively. As a result, the remaining integral in (2.13) is transformed to $\int A(\xi, \eta)\delta(q-\xi)\delta(p-\eta)\, d\xi\, d\eta = A(q, p)$. One can easily check that for the τ-quantization, when $Q(\lambda) = \delta(\lambda - \tau)$, the characteristic function is $X_Q(z) = \exp(iz\tau)$, so for the inverse transformation we have $X_S(z) = X_Q(z)$, just as it should be according to (2.14).

Denote the moments of the inverse symmetrization function $S(\tau)$ by μ_n, so that

$$X_S(z) = \sum_{n=0}^{\infty} \mu_n \frac{(iz)^n}{n!}. \quad (2.15)$$

It then follows from (2.14) and (2.8) that

$$\sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} \sigma_k \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \mu_n = 1. \quad (2.16)$$

Equating the coefficients of the same powers of z in (2.16), we arrive at a triangular system of equations, which allows us to successively express μ_k in terms of σ_k:

$$\mu_0\sigma_0 = 1 \ \Rightarrow\ \mu_0 = 1, \qquad \sum_{k=0}^{n} (-1)^k \frac{\mu_{n-k}\sigma_k}{k!(n-k)!} = 0, \quad n \geq 1. \quad (2.17)$$

Note that the inversion of a linear quantization is closely related to the Bernoulli numbers. As is well known (see, e.g., [1]), the Bernoulli numbers B_n are the coefficients of the Taylor series expansion of the generating function

$$\frac{x}{\exp(x) - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \quad (2.18)$$

In this connection, consider the inversion of the Born quantization, with the characteristic function $X_Q(z) = (\exp(iz) - 1)/(iz)$. According to (2.14), the characteristic function of the symmetrization of the inverse kernel in this case has the form $X_S(z) = -iz/(\exp(-iz) - 1)$. Therefore, it is the generating function of the Bernoulli numbers, but with alternating signs, so for this quantization we have

$$\mu_n = (-1)^n B_n, \qquad n > 0. \quad (2.19)$$

In fact, since only the even Bernoulli numbers are nonzero (apart from the first one), formula (2.19) is only meaningful for the first moment; for the other moments we have $\mu_n = B_n$, $n > 1$.

The Jordan quantization is also of practical interest. The moments μ_n of $S(\tau)$ in (2.17) for this quantization turn out to be in a sense dual to the Bernoulli numbers. Since the characteristic function for the Jordan quantization is $X_Q(z) = (1 + \exp(iz))/2$, by inverting this quantization we find that the moments μ_n of the corresponding function $S(\tau)$ have a generating function $X_S(z) = 2/(1 + \exp(-iz))$. It follows from (2.17) that μ_n satisfy the relation

$$\mu_n + \sum_{k=0}^{n} (-1)^k C_n^k \mu_{n-k} = 0, \qquad n \geq 1. \tag{2.20}$$

In (2.20) the zeroth-order moment is equal to one: $\mu_0 = 1$ (just as the Bernoulli number $B_0 = 1$); all moments of even order $2k$, $k \geq 1$, vanish (all Bernoulli numbers with odd indices $2k + 1$, $k \geq 1$, vanish); the moments of odd order $2n + 1$ have the sign $(-1)^n$ (the Bernoulli numbers with even indices $2n$ have the sign $(-1)^{n+1}$). Here are the first few nonzero numbers μ_k obtained recursively from system (2.20) (these are their exact values):

$$\begin{gathered} \mu_0 = 1, \quad \mu_1 = 0.5, \quad \mu_3 = -0.25, \quad \mu_5 = 0.5, \quad \mu_7 = -2.125, \\ \mu_9 = 15.5, \quad \mu_{11} = -172.75, \quad \mu_{13} = 2730.5, \quad \mu_{15} = 58\,098.0625. \end{gathered} \tag{2.21}$$

Note that linear quantizations admit a statistical interpretation. These quantizations are a statistical mixture of τ-quantizations with a positive definite generalized probability density $Q(\tau)$; i.e., any Hermitian quantization can be interpreted as a result of averaging τ-quantizations. Then the concept of random τ-quantization defined by the kernel (2.3) arises naturally, and its average (2.9) provides the quantization of the dynamical system.

3. EVOLUTION EQUATION FOR THE WIGNER FUNCTION

We use formula (2.14) to impart a closed form to the evolution equation of the Wigner function for the linear quantization defined by (2.1), (2.3), and (2.9). In the particular case of Weyl quantization, the evolution equation for the Wigner function was obtained by Moyal [13].

Denote by $\widetilde{\rho}(x,y)$ the density matrix. The average value of an operator $\widehat{A}$ in the state $\widehat{\rho}$ is given by

$$\langle \widehat{A} \rangle = \operatorname{Tr} \widehat{\rho}\widehat{A} = \int \widetilde{\rho}(x,y)\widetilde{A}(y,x)\,dx\,dy. \tag{3.1}$$

Substituting formula (2.1) for the matrix $\widetilde{A}(x,y)$ into (3.1), we obtain

$$\langle \widehat{A} \rangle = \int W(q,p)A(q,p)\,dq\,dp, \tag{3.2}$$

where the function $W(q,p)$ is a generalization of the Wigner function [21] to linear quantization:

$$W(q,p) = \int \widetilde{\rho}(x,y)K(q,p|y,x)\,dx\,dy = \int W_\tau(q,p)Q(\tau)\,d\tau, \tag{3.3}$$

$$W_\tau(q,p) = \int K_\tau(q,p|y,x)\widetilde{\rho}(x,y)\,dx\,dy. \tag{3.4}$$

Using the inversion formula (2.14), we obtain a representation of the density matrix in terms of the Wigner function:

$$\widetilde{\rho}(x,y) = \int W(q,p)L(y,x|q,p)\,dq\,dp = \int S(\tau)\,d\tau \int W(q,p)L_\tau(y,x|q,p)\,dq\,dp. \tag{3.5}$$

Let us now derive the evolution equation for the Wigner function. The time-dependent density matrix and Wigner function will be denoted by the same symbols but with a time argument added. The starting point is the quantum Liouville equation [4]; for a given Hamiltonian $\widehat{H}$, this equation reads $i\hbar\,\partial\widehat{\rho}/\partial t=[\widehat{H},\widehat{\rho}]$. In terms of the matrices of operators, the equation has the form

$$\begin{aligned} i\hbar\frac{\partial\widetilde{\rho}(x,y,t)}{\partial t}&=\int\big(\widetilde{H}(x,z)\widetilde{\rho}(z,y,t)-\widetilde{H}(z,y)\widetilde{\rho}(x,z,t)\big)\,dz\\ &=\frac{1}{2\pi\hbar}\int Q(\tau)\int H(\tau z+(1-\tau)x,p)\widetilde{\rho}(z,y,t)e^{ip(x-z)/\hbar}\,dz\,dp\,d\tau\\ &\quad-\frac{1}{2\pi\hbar}\int Q(\tau)\int H(\tau y+(1-\tau)z,p)\widetilde{\rho}(x,z,t)e^{ip(z-y)/\hbar}\,dz\,dp\,d\tau. \end{aligned}\tag{3.6}$$

Differentiating (3.3) with respect to t and substituting the right-hand side of the Liouville equation (3.6) for $i\hbar\partial\widetilde{\rho}(x,y,t)/\partial t$, we find

$$\begin{aligned} i\hbar\frac{\partial W(q,p,t)}{\partial t}&=\int K(q,p|y,x)\big(\widetilde{H}(x,z)\widetilde{\rho}(z,y,t)-\widetilde{H}(z,y)\widetilde{\rho}(x,z,t)\big)\,dx\,dy\,dz\\ &=\int\big(U(q,p;q',p';q'',p'')-V(q,p;q',p';q'',p'')\big)H(q',p')W(q'',p'',t)\,dq'\,dp'\,dq''\,dp'', \end{aligned}\tag{3.7}$$

where

$$\begin{aligned} U(q,p;q',p';q'',p'')&=\int K(q,p|y,x)K(q',p'|x,z)L(y,z|q'',p'')\,dx\,dy\,dz,\\ V(q,p;q',p';q'',p'')&=\int K(q,p|y,x)K(q',p'|z,y)L(z,x|q'',p'')\,dx\,dy\,dz. \end{aligned}\tag{3.8}$$

Taking account of (2.9) and (2.12), we transform the integrals in (3.8) by making the change of variables $u=x-y$, $v=x+y$, $z'=z-x$. This results in the following expressions:

$$\begin{aligned} U(q,p;q',p';q'',p'')&=\int\frac{X_Q^*(su)X_Q(-s'z')}{X_Q^*(-s''(u+z'))}\,\frac{d(v/2)\,du\,dz'}{(2\pi\hbar)^2}\,\frac{ds\,ds'\,ds''}{(2\pi)^2}\\ &\times\exp\Big\{is\Big(q-\frac{v}{2}+\frac{u}{2}\Big)+is'\Big(q'-\frac{v}{2}-\frac{u}{2}\Big)+is''\Big(q''-\frac{v}{2}+\frac{u}{2}\Big)-\frac{ipu}{\hbar}-\frac{ip'z'}{\hbar}+\frac{ip''(z'+u)}{\hbar}\Big\},\\ V(q,p;q',p';q'',p'')&=\int\frac{X_Q^*(su)X_Q(s''(u+z'))}{X_Q^*(s'z')}\,\frac{d(v/2)\,du\,dz'}{(2\pi\hbar)^2}\,\frac{ds\,ds'\,ds''}{(2\pi)^2}\\ &\times\exp\Big\{is\Big(q-\frac{v}{2}+\frac{u}{2}\Big)+is'\Big(q'-\frac{v}{2}-\frac{u}{2}-z'\Big)+is''\Big(q''-\frac{v}{2}-\frac{u}{2}-z'\Big)-\frac{ipu}{\hbar}-\frac{ip'z'}{\hbar}+\frac{ip''(z'+u)}{\hbar}\Big\}. \end{aligned}\tag{3.9}$$

Integrating each of the expressions in (3.9) with respect to $dv/2$ gives rise to the delta function $2\pi\delta(s+s'+s'')$, which allows us to integrate with respect to ds''. Thus, the evolution equation of the Wigner function for linear quantization has a rather cumbersome form and depends on the quantization rule. This dependence appears in the form of three characteristic functions. The evolution itself is determined by the difference $U-V$ in (3.7). The further transformations in (3.7) and (3.8) are made by passing to the Fourier transforms $H_{k\omega}$ and $\Lambda(\xi,\eta,t)$ for the Hamilton and Wigner functions:

$$H(q',p')=\int H_{k\omega}e^{-ikq'-i\omega p'}\,dk\,d\omega\qquad\text{and}\qquad\Lambda(\xi,\eta,t)=\int W(q,p,t)e^{-i\xi q+i\eta p}\,dq\,dp.\tag{3.10}$$

Substituting (3.10) into (3.7) and using (3.9), after integrating with respect to $dq'\,dp'$ we obtain the evolution equation for the Wigner function:

$$\frac{i\hbar}{X_Q^*(\hbar\xi\eta)}\frac{\partial\Lambda(\xi,\eta,t)}{\partial t}=\int\frac{X_Q(\hbar k\omega)e^{-i\hbar k\eta}}{X_Q^*(\hbar(\xi+k)(\eta-\omega))}H_{k\omega}\Lambda(\xi+k,\eta-\omega,t)\,dk\,d\omega$$
$$-\int\frac{X_Q(\hbar k\omega)e^{i\hbar\xi\omega}}{X_Q^*(\hbar(\xi+k)(\eta-\omega))}H_{k\omega}\Lambda(\xi+k,\eta-\omega,t)\,dk\,d\omega.$$

One can see that instead of the Fourier transform of the Wigner function, it is convenient to introduce the function

$$F(\xi,\eta,t)=\frac{\Lambda(\xi,\eta,t)}{X_Q^*(\hbar\xi\eta)},\tag{3.11}$$

and instead of the Fourier transform of the Hamiltonian, the function

$$G(k,\omega)=X_Q(\hbar k\omega)H_{k\omega}.\tag{3.12}$$

Then we obtain a more compact final expression for the generalization of the Moyal equation to an arbitrary linear quantization of the form (2.9):

$$i\hbar\frac{\partial F(\xi,\eta,t)}{\partial t}=\int\big(e^{-i\hbar k\eta}-e^{i\hbar\xi\omega}\big)G(k,\omega)F(\xi+k,\eta-\omega,t)\,dk\,d\omega.\tag{3.13}$$

Traditionally, equation (3.13) is written for the Weyl quantization (in which case it is called the Moyal equation). This equation has the form

$$\frac{\partial W(q,p,t)}{\partial t}=\frac{i}{\hbar}\int H_{k\omega}e^{ikq+i\omega p}\bigg(W\Big(q+\frac{\hbar\omega}{2},p-\frac{\hbar k}{2},t\Big)-W\Big(q-\frac{\hbar\omega}{2},p+\frac{\hbar k}{2},t\Big)\bigg)\,dk\,d\omega.\tag{3.14}$$

It is easy to check that for the Weyl quantization equations (3.14) and (3.13) coincide. Indeed, passing to the Fourier transform of the Wigner function in (3.14), we obtain the equation

$$\hbar\frac{\partial\Lambda(\xi,\eta,t)}{\partial t}=-\int H_{k\omega}\Lambda(\xi+k,\eta-\omega,t)\sin\frac{\hbar(\xi\omega+k\eta)}{2}\,dk\,d\omega,\tag{3.15}$$

which coincides with (3.13) in view of (3.11) and (3.12). For other linear quantizations of the form (2.6), the evolution equation of the Wigner function differs significantly from (3.15). However, we emphasize that the difference is not related to the physics of quantization of the Hamiltonian or other observables but follows from the mathematical definition of the Wigner function (3.3), which itself depends on quantization, even if the Hamiltonian operator does not depend on it.

The further analysis depends on the specific form of the Hamiltonian defining the integral kernel in (3.13). For example, if we consider a particle of unit mass with momentum p in a field with potential $U(q)$, then $H(q,p)=p^2/2+U(q)$ and so $H_{k\omega}=-\delta(k)\delta''(\omega)/2+\delta(\omega)U_k$. For this model, equation (3.13) takes the form

$$\frac{\partial\Lambda(\xi,\eta,t)}{\partial t}+\xi\frac{\partial\Lambda(\xi,\eta,t)}{\partial\eta}-\frac{i\hbar}{2}\xi^2\Lambda(\xi,\eta,t)\bigg(1-2i\frac{X_Q'^*(\hbar\xi\eta)}{X_Q^*(\hbar\xi\eta)}\bigg)$$
$$=-\frac{i}{\hbar}X_Q^*(\hbar\xi\eta)\int\big(X_Q(-\hbar k\eta)e^{-i\hbar\eta k}-1\big)U_k\frac{\Lambda(\xi+k,\eta,t)}{X_Q^*(\hbar(\xi+k)\eta)}\,dk.\tag{3.16}$$

It follows from (3.16) that in the general case of an arbitrary linear quantization, the left-hand side of the equation contains a source term, which is absent only if the parenthesized expression with the logarithmic derivative of the characteristic function of the quantization is equal to zero, that is, if $2iX'^*(z)=X^*(z)$. This condition holds only for the Weyl quantization, for which $X^*(z)=e^{-iz/2}$. Only in this case the source term vanishes.

4. HARMONIC OSCILLATOR

To illustrate the differences between quantizations in equation (3.16), we consider the example of a harmonic oscillator. In this case $U_k = -1/2\delta''(k)$, so that (3.16) becomes

$$\frac{\partial\Lambda(\xi,\eta,t)}{\partial t} + \xi\frac{\partial\Lambda(\xi,\eta,t)}{\partial\eta} - \eta\frac{\partial\Lambda(\xi,\eta,t)}{\partial\xi} = \frac{i\hbar}{2}(\xi^2-\eta^2)\Lambda(\xi,\eta,t)\left(1 - 2i\frac{X_Q'^*(\hbar\xi\eta)}{X_Q^*(\hbar\xi\eta)}\right). \tag{4.1}$$

Let us find the general stationary solution of equation (4.1) as the sum of a particular solution and the general solution of the homogeneous equation. The general solution to the homogeneous equation (4.1) is an arbitrary differentiable function of the sum of squared arguments, $g(\xi^2/2+\eta^2/2)$. A particular solution of this equation for the function $w = \ln\Lambda(\xi,\eta)$ is $i\hbar\xi\eta/2 + \ln X_Q^*(\hbar\xi\eta)$. As a result, we obtain the stationary Wigner function (more precisely, its Fourier transform) for the harmonic oscillator in the form

$$\Lambda(\xi,\eta) = e^{i\hbar\xi\eta/2}X_Q^*(\hbar\xi\eta)\exp\left(g\left(\frac{\xi^2}{2}+\frac{\eta^2}{2}\right)\right). \tag{4.2}$$

The factor $e^{i\hbar\xi\eta/2}X_Q^*(\hbar\xi\eta)$ multiplying the exponential of the function $g(\xi^2/2+\eta^2/2)$ in (4.2) corresponds to the choice of symmetrization in the quantization rule. This factor is equal to one only if $X_Q^*(z) = e^{-iz/2}$, i.e., for the Weyl quantization. In all other cases, we obtain a solution with a source of quasi-probability.

For example, for the Jordan quantization (2.14), the solution (4.2) has the form

$$\Lambda(\xi,\eta) = \cos\left(\frac{\hbar}{2}\xi\eta\right)\exp(g). \tag{4.3}$$

Because of the cosine factor in (4.3), the Wigner function for the equilibrium harmonic oscillator is of alternating sign, which is not very natural for its interpretation as an analog of the probability density. Thus, the Jordan quantization is not very convenient for describing the harmonic oscillator.

Let us find out how the function $\Lambda(\xi,\eta)$ changes in (4.3) if we start to bring the delta functions closer to the middle of the interval $[0,1]$, that is, instead of the Jordan quantization, use the quantization shifted by $\pm a$ from the point $\tau = 1/2$:

$$Q(\tau) = \frac{1}{2}\left(\delta\left(\tau - a - \frac{1}{2}\right) + \delta\left(\tau + a - \frac{1}{2}\right)\right), \qquad X_Q(z) = e^{iz/2}\cos(az). \tag{4.4}$$

For this quantization, instead of (4.3) we obtain a solution of the form

$$\Lambda(\xi,\eta) = \cos(a\hbar\xi\eta)\exp(g). \tag{4.5}$$

As expected, for $a \to 0$, (4.5) leads to the result for the Weyl quantization; however, it should be noted that the convergence is not uniform in $\xi\eta$ for large values of $\xi\eta$, that is, for small values of qp, which is important precisely in the quantum domain. Therefore, for arbitrarily small values of a, the function (4.5) will always be nonpositive. Thus, the model of delta functions symmetrically approaching the point 1/2 exhibits properties that are fundamentally different from those of the Weyl quantization, and therefore cannot serve for the approximation of the latter. The need for such an approximation arises because the delta functions cannot be realized exactly in physics but are approximated by “caps,” i.e., are somewhat “smeared.”

Then, instead of a set of delta functions, consider an approximating model in the form of a triangular distribution. Let

$$Q(\tau) = \begin{cases} \dfrac{\tau - 1/2 + a}{a^2}, & \dfrac{1}{2} - a \le \tau \le \dfrac{1}{2}, \\ \dfrac{1/2 + a - \tau}{a^2}, & \dfrac{1}{2} < \tau \le \dfrac{1}{2} + a, \\ 0, & \left|\tau - \dfrac{1}{2}\right| > a. \end{cases} \tag{4.6}$$

The moments of this distribution are as follows:

$$\sigma_n(a) = \int\limits_{1/2-a}^{1/2+a} \tau^n Q(\tau)\, d\tau = \frac{1}{2^n(n+1)(n+2)} \sum_{k=2}^{n+2} C_{n+2}^k (1 + (-1)^k)(2a)^{k-2}. \tag{4.7}$$

Since $\lim_{a\to 0} \sigma_n(a) = 2^{-n}$, this distribution also approximates the Weyl quantization. Its characteristic function is

$$X_Q(z) = e^{iz/2} \frac{\sin^2(az/2)}{(az/2)^2}. \tag{4.8}$$

Therefore, the solution (4.2) in this case takes the form

$$\Lambda(\xi,\eta) = \frac{\sin^2(a\hbar\xi\eta/2)}{(a\hbar\xi\eta/2)^2} \exp(g(\xi^2 + \eta^2)) \tag{4.9}$$

and is nonnegative, in contrast to the solution (4.5). Thus, it preserves the possibility of probabilistic interpretation and shows that if the Weyl quantization is physically realized in the form of a "weakly smeared" delta function, then no paradox with negative quasi-probability arises. Of course, the convergence to the solution for the Weyl quantization as $a \to 0$ is not uniform.

5. CONCLUSIONS

For every linear quantization given by a superposition of τ-quantizations, an inversion formula for the corresponding kernel can be obtained, which relates the matrix of the operator to its symbol. This superposition can be interpreted as a probabilistic mixture of τ-quantizations, i.e., as their averaging with respect to some measure or pseudo-measure. The characteristic functions of the densities of these measures are tied by the inversion condition. On this basis, a generalization of the Moyal equation is obtained, which describes the evolution of the Wigner function for an arbitrary linear quantization. It is shown that among linear quantizations only the Weyl quantization is free of a probability source in the evolution equation of the Wigner function. Various approximations for the kernel of the Weyl quantization are constructed. It is shown that there is an approximation weakly converging to the Weyl kernel for which the stationary solution of the Moyal equation for the harmonic oscillator is positive, which agrees with the physical meaning of the solution.

In our recent paper [6], we derived the generalized Moyal equation and gave an example of its application to the problem of anharmonic oscillator as a demonstration of a real physical system in which the order of noncommuting operators affects the form of the evolution equation.

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