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Interpolation Theorems for Nonlinear Operators in General Morrey-Type Spaces and Their Applications

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Abstract—We prove new interpolation theorems for a sufficiently wide class of nonlinear operators in Morrey-type spaces. In particular, these theorems apply to Urysohn integral operators. We also obtain analogs of the Marcinkiewicz–Calderón and Stein–Weiss–Peetre interpolation theorems and establish a criterion of (p, q) quasiweak boundedness of the Urysohn operator.

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1. INTRODUCTION

Let (U, μ) be a space with measure μ and Ω be a set. By $G_\Omega = \{G_{t,y}\}_{t>0,\, y\in\Omega}$ we will denote a two-parameter family of μ-measurable sets $G_{t,y}$ satisfying the condition

$$G_{t,y} \subset G_{s,y} \qquad \text{for} \quad 0 < t < s \quad \text{and} \quad y \in \Omega.$$

We will call such a family a *net*. For $y \in \Omega$, we set $G_{\{y\}} = \{G_{t,y}\}_{t>0}$. We say that these nets are generated by the net G_Ω. If the sets $G_{t,y}$, $t > 0$, are independent of the parameter y, we denote such a net by $G = \{G_t\}_{t>0}$.

Let $0 < p, q \le \infty$ and $0 < \lambda < \infty$. Define the space $M^\lambda_{p,q}(G_\Omega, \mu)$ of all μ-measurable functions $f\colon U \to \mathbb{R}$ such that

$$\|f\|_{M^\lambda_{p,q}(G_\Omega,\mu)} = \left(\int\limits_0^\infty \left(t^{-\lambda} \sup_{y\in\Omega} \left(\int\limits_{G_{t,y}} |f(x)|^p \, d\mu \right)^{1/p} \right)^q \frac{dt}{t} \right)^{1/q} < \infty$$

for $q < \infty$ and

$$\|f\|_{M^\lambda_{p,\infty}(G_\Omega,\mu)} = \sup_{t>0,\, y\in\Omega} t^{-\lambda} \left(\int\limits_{G_{t,y}} |f(x)|^p \, d\mu \right)^{1/p} < \infty$$

for $q = \infty$.

If $U = \mathbb{R}^n$, μ is the Lebesgue measure, and $G_{t,y} = B(y, t)$ (ball of radius t centered at y), then we denote this space by $M^\lambda_{p,q,\Omega}$. In particular, for $q = \infty$ and $\Omega = \mathbb{R}^n$, this is the classical Morrey space M^λ_p.

If $U = \mathbb{R}^n$, μ is the Lebesgue measure, $\Omega = \{0\}$, and $G_t = G_{t,0} = B(0, t)$, then $M^\lambda_{p,q}(G_\Omega, \mu)$ is the local Morrey-type space $LM^\lambda_{p,q}$, which was introduced and used to study the properties of the

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maximal and fractional maximal operators by V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev [4–7]. If $\Omega = \{y\}$ and $G_t = G_{t,y} = B(y,t)$, then we denote the corresponding local Morrey-type space by $LM^{\lambda}_{p,q,y}$.

The problem of real interpolation of Morrey-type spaces was addressed in [26, 14, 23, 25, 3, 17]. It follows from the results of [23] that the interpolation space $(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta,\infty}$, where $1 \le p < \infty$ and $0 < \lambda_0, \lambda_1 < n/p$, satisfies the inclusion

$$\big(M_p^{\lambda_0}, M_p^{\lambda_1}\big)_{\theta,\infty} \subset M_p^{\lambda} \qquad \text{with} \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1, \quad 0 < \theta < 1.$$

In [25, 3], this inclusion was shown to be strict.

In [18], Lemarié-Rieusset proved that for $1 \le p_0, p_1 < \infty$, $0 < \lambda_0 < n/p_0$, and $0 < \lambda_1 < n/p_1$, the inclusion

$$\big(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1}\big)_{\theta,\infty} \subset M_p^{\lambda}, \qquad \text{where} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1, \quad 0 < \theta < 1,$$

holds if and only if $p_0 = p_1$.

In [10, 12, 8], it was shown that the scale of local Morrey-type spaces $LM^{\lambda}_{p,q}$ with fixed p (in contrast to the scale of Morrey spaces M_p^{λ}) is closed under interpolation; namely, it was proved that if $0 < p, q_0, q_1, q \le \infty$ and $\lambda_0 \ne \lambda_1$, $0 < \lambda_0, \lambda_1 < \infty$, then

$$\big(LM^{\lambda_0}_{p,q_0}, LM^{\lambda_1}_{p,q_1}\big)_{\theta,q} = LM^{\lambda}_{p,q}, \qquad \text{where} \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1, \quad 0 < \theta < 1.$$

In [12], the interpolation spaces were described for much more general spaces than $LM^{\lambda}_{p,q}$.

In [8, 9], the following interpolation theorem for quasi-additive operators was proved.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$. Let $0 < p, q, \sigma, \tau \le \infty$, $0 \le \alpha_0, \alpha_1 < \infty$ with $\alpha_0, \alpha_1 > 0$ if $\sigma < \infty$, $\alpha_0 \ne \alpha_1$, $0 \le \beta_0, \beta_1 < \infty$, $\beta_0 \ne \beta_1$, $0 < \theta < 1$, and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \qquad \beta = (1-\theta)\beta_0 + \theta\beta_1. \tag{1.1}$$

Let T be a quasi-additive operator on $\bigcup_{y\in\Omega}\big(LM^{\alpha_0}_{p,\sigma,y} + LM^{\alpha_1}_{p,\sigma,y}\big)$ with a quasi-additivity constant A.

If for some $M_0, M_1 > 0$ the inequalities

$$\|Tf\|_{LM^{\beta_i}_{q,\infty,y}} \le M_i \|f\|_{LM^{\alpha_i}_{p,\sigma,y}} \tag{1.2}$$

hold for all $y \in \Omega$ and all functions $f \in LM^{\alpha_i}_{p,\sigma,y}$, $i = 0, 1$, then the inequality

$$\|Tf\|_{M^{\beta}_{q,\tau,\Omega}} \le cAM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\alpha}_{p,\tau,\Omega}} \tag{1.3}$$

holds for all functions $f \in M^{\alpha}_{p,\tau,\Omega}$, where $c > 0$ depends only on α_0, α_1, β_0, β_1, q, σ, τ, and θ.

In this theorem, the strong estimate (1.3) is derived from the weak (in a sense) estimates (1.2).

In the present study, we significantly generalize this result: first, we consider a wider class of nonlinear operators, including the Urysohn integral operator

$$(Tf)(y) = \int_U K(f(x), x, y)\, d\mu;$$

second, we consider a much more general class of Morrey-type spaces; third, we consider a still weaker version of estimates of type (1.2), which leads to a strong estimate of type (1.3).

In Section 2, we present a number of properties of the spaces $M^{\lambda}_{p,q}(G_\Omega, \mu)$ and give various examples of these spaces. In particular, in Example 1, we demonstrate that the space $M^{\lambda}_{p,q}(G_\Omega, \mu)$ coincides with the Lorentz space $L_{r,q}(U, \mu)$ for a certain choice of the net G_Ω and under certain relations between the parameters p, q, and λ. In Section 3, we describe the classes of nonlinear operators under consideration (partially quasi-additive operators) and give examples of operators from these classes, including the Urysohn integral operator. In Section 4, we prove general interpolation theorems for partially quasi-additive operators in the spaces $M^{\lambda}_{p,q}(G_\Omega,\mu)$. In Section 5, using these theorems, we prove an analog of Calderón's interpolation theorem for partially quasi-additive operators. In Section 6, we prove a weak version of the Stein–Weiss–Peetre interpolation theorem for the class of operators under consideration, including the Urysohn integral operator satisfying the quasiweak boundedness condition. Finally, in Section 7, we prove a quasiweak boundedness criterion for a wide class of Urysohn integral operators. The results for Urysohn integral operators were given without proofs in the authors' note [11].

Let F and G be nonnegative functions defined on a set A. We say that G dominates F on A and write "$F \lesssim G$ on A" (or simply $F \lesssim G$ when it is clear what set A is meant) if there exists a $c>0$ such that $F(x)\le cG(x)$ for all $x\in A$. We also say that F and G are equivalent on A and write "$F \asymp G$ on A" if $F\lesssim G$ and $G \lesssim F$ on A.

2. PROPERTIES OF MORREY-TYPE SPACES AND EXAMPLES

Lemma 1. 1. *Let $0<q_0<q_1\le\infty$ and $0<p,\lambda<\infty$. Then the following continuous embedding holds*:

$$M^{\lambda}_{p,q_0}(G_\Omega,\mu)\hookrightarrow M^{\lambda}_{p,q_1}(G_\Omega,\mu). \tag{2.1}$$

2. *Let $0<p_1<p_2<\infty$, $0<q\le\infty$, and $0<\lambda_1,d<\infty$. Suppose the net $G_\Omega=\{G_{t,y}\}_{t>0,\ y\in\Omega}$ and measure μ are such that for some $C>0$*

$$\mu(G_{t,y})\le Ct^d$$

for all $y\in\Omega$ and $t>0$. Then the following continuous embedding holds:

$$M^{\lambda_2}_{p_2,q}(G_\Omega,\mu)\hookrightarrow M^{\lambda_1}_{p_1,q}(G_\Omega,\mu), \qquad \text{where} \quad \lambda_2=\lambda_1-d\biggl(\frac{1}{p_1}-\frac{1}{p_2}\biggr)>0.$$

The proof is similar to that of [9, Lemma 2.1], which was given for the case of $G_{t,y}=B(y,t)$.

Choosing the net G_Ω and the parameter λ, we can describe the quasinorms of various spaces using the quasinorm of the space $M^{\lambda}_{p,q}(G_\Omega,\mu)$. Here are a few examples. Based on these examples and the interpolation theorem for linear and some nonlinear operators in the spaces $M^{\lambda}_{p,q}(G_\Omega,\mu)$, we will then obtain, as a corollary, interpolation theorems for the corresponding spaces.

Let f be a μ-measurable function defined on U. The function

$$f^*(t)=\inf\bigl\{\sigma\ge 0\colon\ \mu(\{x\in U\colon\ |f(x)|>\sigma\})\le t\bigr\}, \qquad t\ge 0,$$

is called the nonincreasing rearrangement of f.

We say that a measure μ satisfies the regularity condition if for any μ-measurable set e and any $\alpha\in(0,\mu(e)/2]$ there exists a μ-measurable subset $w\subset e$ such that

$$\alpha\le\mu(w)\le 2\alpha. \tag{2.2}$$

Lemma 2. *Let μ be a regular measure and $f\in L_1^{\mathrm{loc}}(U,\mu)$. Suppose also that the function $f\colon U\to\mathbb{C}$ is integrable on all μ-measurable subsets of U of finite measure. Then there exists a net $G(f)=\{G_t(f)\}_{t>0}$ such that*

$$t\le\mu(G_t(f))\le 2t, \tag{2.3}$$

$$\int\limits_0^t f^*(s)\,ds\le\int\limits_{G_t(f)}|f(x)|\,d\mu\le\int\limits_0^{2t} f^*(s)\,ds, \tag{2.4}$$

and

$$\big(f\big(1-\chi_{G_t(f)}\big)\big)^*(s) \le f^*(t+s), \tag{2.5}$$

where χ_A *is the characteristic function of a set* $A \subset U$.

Proof. Let $t>0$. Recall that according to the properties of rearrangements we have

$$\mu\big(\{x\in U\colon |f(x)|\ge f^*(t)\}\big)\ge t \qquad \text{and} \qquad \mu\big(\{x\in U\colon |f(x)|> f^*(t)\}\big)\le t. \tag{2.6}$$

Put $\alpha = t-\mu(\{x\in U\colon |f(x)|>f^*(t)\})$. In view of (2.6), we have $\alpha\ge 0$. Let us construct the set $G_t(f)$.

If $\alpha>\mu(\{x\in U\colon |f(x)|=f^*(t)\})/2$, then we set $G_t(f)=\{x\in U\colon |f(x)|\ge f^*(t)\}$, while if $\alpha\le\mu(\{x\in U\colon |f(x)|=f^*(t)\})/2$, then we set $G_t(f)=\{x\in U\colon |f(x)|>f^*(t)\}\cup w$, where

$$w\subset\big\{x\in U\colon |f(x)|=f^*(t)\big\}, \qquad \alpha\le\mu(w)\le 2\alpha\le\mu\big(\{x\in U\colon |f(x)|=f^*(t)\}\big).$$

Such a set exists since the measure μ is regular.

Let us prove inequality (2.3). Indeed, in the first case, according to (2.6),

$$\begin{aligned} t\le\mu(G_t(f)) &= \mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)+\mu\big(\{x\in U\colon |f(x)|=f^*(t)\}\big)\\ &\le \mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)+2\alpha = 2t-\mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)\le 2t. \end{aligned}$$

In the second case,

$$\begin{aligned} t &= \alpha+\mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)\le\mu(G_t(f))=\mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)+\mu(w)\\ &\le \mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)+2\alpha\le 2t. \end{aligned}$$

Moreover,

$$\sup_{\mu(e)\le t}\int_e |f(x)|\,d\mu \le \int_{G_t(f)} |f(x)|\,d\mu. \tag{2.7}$$

Indeed,

$$\begin{aligned} t &= \alpha+\mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)\le\mu(w)+\mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)=\mu(G_t(f))\\ &\le \mu\big(\{x\in U\colon |f(x)|>f^*(t)\}\big)+2\alpha = t+\alpha\le 2t. \end{aligned}$$

Let e be an arbitrary μ-measurable set with $\mu(e)\le t$. It is clear from the definition of the set $G_t(f)$ that $|f(x)|\ge f^*(t)$ for $x\in G_t(f)\setminus e$ and $|f(x)|\le f^*(t)$ for $x\in e\setminus G_t(f)$. Therefore,

$$\begin{aligned} \int_{G_t(f)}|f(x)|\,d\mu-\int_e|f(x)|\,d\mu &= \int_{G_t(f)\setminus e}|f(x)|\,d\mu-\int_{e\setminus G_t(f)}|f(x)|\,d\mu\\ &\ge f^*(t)\big(\mu(G_t(f)\setminus e)-\mu(e\setminus G_t(f))\big)\ge f^*(t)\big(\mu(G_t(f))-\mu(e)\big). \end{aligned}$$

Since $\mu(G_t(f))\ge t$ and $\mu(e)\le t$, we obtain $f^*(t)(\mu(G_t(f))-\mu(e))\ge 0$, and so

$$\int_{G_t(f)}|f(x)|\,d\mu-\int_e|f(x)|\,d\mu\ge 0.$$

Thus, according to the properties of rearrangements and inequalities (2.7) and (2.3),

$$\int_0^t f^*(s)\,ds=\sup_{\mu(e)\le t}\int_e|f(x)|\,d\mu\le\int_{G_t(f)}|f(x)|\,d\mu\le\int_0^{\mu(G_t(f))}f^*(s)\,ds\le\int_0^{2t}f^*(s)\,ds;$$

i.e., (2.4) holds.

Next, for any $s,t>0$, we have $f^*(s+t)=\inf A_{s,t}$, where

$$\begin{aligned}A_{s,t}&=\big\{\sigma\ge 0\colon\ \mu(\{x\in U\colon |f(x)|>\sigma\})\le s+t\big\}\\&=\big\{\sigma\ge 0\colon\ \mu(\{x\in G_t(f)\colon |f(x)|>\sigma\})+\mu(\{x\in U\setminus G_t(f)\colon |f(x)|>\sigma\})\le s+t\big\}\\&=\big\{\sigma\ge 0\colon\ \mu(\{x\in G_t(f)\colon |f(x)|>\sigma\})+\mu\big(\big\{x\in U\colon \big|\big(f\big(1-\chi_{G_t(f)}\big)\big)(x)\big|>\sigma\big\}\big)\le s+t\big\}.\end{aligned}$$

It follows from the last representation of the set $A_{s,t}$ that either $\mu(\{x\in G_t(f)\colon |f(x)|>\sigma\})\le t$ or $\mu(\{x\in U\colon |(f(1-\chi_{G_t(f)}))(x)|>\sigma\})\le s$, i.e., $A_{s,t}\subset B_t\cup C_{s,t}$, where

$$\begin{aligned}B_t&=\big\{\sigma\ge 0\colon\ \mu(\{x\in G_t(f)\colon |f(x)|>\sigma\})\le t\big\},\\C_{s,t}&=\big\{\sigma\ge 0\colon\ \mu\big(\big\{x\in U\colon \big|\big(f\big(1-\chi_{G_t(f)}\big)\big)(x)\big|>\sigma\big\}\big)\le s\big\}.\end{aligned}$$

Since $|f(x)|\ge f^*(t)$ for $x\in G_t(f)$ and $\big|\big(f\big(1-\chi_{G_t(f)}\big)\big)(x)\big|\le f^*(t)$ for any $x\in U$, we have

$$\begin{aligned}f^*(t+s)=\inf A_{s,t}&\ge\inf(B_t\cup C_{s,t})=\min\{\inf B_t,\inf C_{s,t}\}\\&\ge\min\big\{f^*(t),\big(f\big(1-\chi_{G_t(f)}\big)\big)^*(t)\big\}=\big(f\big(1-\chi_{G_t(f)}\big)\big)^*(t).\qquad\square\end{aligned}$$

Example 1. Let $0<r<\infty$ and $0<q\le\infty$. The Lorentz space $L_{r,q}(U,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\|f\|_{L_{r,q}(U,\mu)}=\Bigg(\int_0^\infty\big(t^{1/r}f^*(t)\big)^q\,\frac{dt}{t}\Bigg)^{1/q}<\infty.$$

Let μ be a regular measure, $1<r<\infty$, and $\lambda=1/r'$, where $1/r+1/r'=1$. Then

$$\|f\|_{L_{r,q}(U,\mu)}\asymp\|f\|_{M_{1,q}^{1/r'}(G,\mu)},\tag{2.8}$$

where the implied equivalence constants depend only on the parameters q and r.

Indeed, it is well known that for $1<r<\infty$ the Lorentz space $L_{r,q}(U,\mu)$ can be equivalently defined in terms of the average of the nonincreasing rearrangement $f^{**}(t)=t^{-1}\int_0^t f^*(s)\,ds$. By Lemma 2 there exists a net $G(f)=\{G_t\}_{t>0}$ for which inequalities (2.4) hold. Therefore,

$$\begin{aligned}\|f\|_{L_{r,q}}&\asymp\Bigg(\int_0^\infty\big(t^{1/r}f^{**}(t)\big)^q\,\frac{dt}{t}\Bigg)^{1/q}=\Bigg(\int_0^\infty\Bigg(t^{1/r-1}\int_0^t f^*(s)\,ds\Bigg)^q\frac{dt}{t}\Bigg)^{1/q}\\&\asymp\Bigg(\int_0^\infty\Bigg(t^{1/r-1}\int_{G_t(f)}|f(x)|\,dx\Bigg)^q\frac{dt}{t}\Bigg)^{1/q}=\|f\|_{M_{1,q}^{1/r'}(G,\mu)}.\end{aligned}$$

Example 2. Let w be a positive μ-measurable function. Put $G_t=\{x\in U\colon 1/\omega(x)<t\}$ for $t>0$. Then, up to a constant factor, $\|f\|_{M_{p,p}^\lambda(G,\mu)}$ coincides with

$$\|f\|_{L_p(w^\lambda)}=\Bigg(\int_U\big(w^\lambda(x)|f(x)|\big)^p\,d\mu\Bigg)^{1/p}.$$

Example 3. Let $0<p\le\infty$, $U=\mathbb{R}^n$, and μ be the Lebesgue measure on $\mathbb{R}^n$. Let v be a positive strictly increasing locally absolutely continuous function on $(0,\infty)$ such that

$$\lim_{t\to+0}v(t)=0\qquad\text{and}\qquad\lim_{t\to+\infty}v(t)=\infty.$$

If $G=\{G_t\}_{t>0}$, where $G_t=B(0,v^{-1}(t))$, $t>0$, then the space $M^1_{p,q}(G)$ coincides with one of the variants of the general local Morrey-type space $LM^{v(\cdot)}_{p,q}$ of all functions $f\in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{v(\cdot)}_{p,q}}=\left(\int\limits_0^\infty\bigl((v(r))^{-1}\|f\|_{L_p(B(0,r))}\bigr)^q\,\frac{dv(r)}{v(r)}\right)^{1/q}$$

(with the usual modification for $q=\infty$). Setting $v(r)=t$, we have

$$\|f\|_{LM^{v(\cdot)}_{p,q}}=\|f\|_{M^1_{p,q}(G)}.$$

Moreover, for $0<\lambda<\infty$, we have $M^\lambda_{p,q}(G)=LM^{v^\lambda(\cdot)}_{p,q}$ and

$$\|f\|_{LM^{v^\lambda(\cdot)}_{p,q}}=\lambda^{1/q}\|f\|_{M^\lambda_{p,q}(G)}.$$

Example 4. Let $0<p,q\le\infty$ and w be a positive measurable function on $(0,\infty)$ such that $\|w\|_{L_q(t,\infty)}<\infty$ for any $t>0$,

$$\lim_{t\to+0}\|w\|_{L_q(t,\infty)}=\infty\qquad\text{and}\qquad\lim_{t\to+\infty}\|w\|_{L_\infty(t,\infty)}=0\quad\text{for}\ \ q=\infty$$

(for $q<\infty$, the equality $\lim_{t\to+\infty}\|w\|_{L_q(t,\infty)}=0$ is obvious). Let us set $v(t)=q^{-1/q}\|w\|^{-1}_{L_q(t,\infty)}$ in the previous example. Then the space $M^1_{p,q}(G)$ coincides with a variant of the general local Morrey-type space $LM_{p,q,w(\cdot)}$ of all functions $f\in L^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,q,w(\cdot)}}=\bigl\|w(r)\|f\|_{L_p(B(0,r))}\bigr\|_{L_q(0,\infty)}.$$

Moreover,

$$\|f\|_{LM_{p,q,w(\cdot)}}=\|f\|_{M^1_{p,q}(G)},$$

since $(v(r))^{-q-1}v'(r)=w(r)$ for a.e. $r>0$.

3. SOME CLASSES OF NONLINEAR OPERATORS, EXAMPLES

Let (U,μ) be a space with measure μ and $Z(U)$ be a linear space of μ-measurable functions $f\colon U\to\mathbb{R}$ such that $f\chi_w\in Z(U)$ for any function $f\in Z(U)$ and any μ-measurable set $w\subset U$. Let (V,ν) be a space with measure ν and $M(V)$ be the space of all ν-measurable functions $f\colon V\to\mathbb{R}$.

An operator T is said to be *quasi-additive* on $Z(U)$ if $T\colon Z(U)\to M(V)$ and there exists an $A\ge1$ such that

$$|(T(f+g))(x)|\le A\bigl(|(Tf)(x)|+|(Tg)(x)|\bigr)\qquad\text{for a.e. }\ x\in V \tag{3.1}$$

for any functions $f,g\in Z(U)$.

We say that an operator T is *partially additive* on $Z(U)$ if $T\colon Z(U)\to M(V)$ and for any function $f\in Z(U)$ and any μ-measurable set $w\subset U$ one has

$$(Tf)(x)=(T(f\chi_w))(x)+(T(f\chi_{U\setminus w}))(x)\qquad\text{for a.e. }\ x\in V. \tag{3.2}$$

We say that an operator T is *partially quasi-additive* on $Z(U)$ if $T\colon Z(U)\to M(V)$ and there exists an $A\ge1$ such that for any function $f\in Z(U)$ and any μ-measurable set $w\subset U$ one has

$$|(Tf)(x)|\le A\bigl(|(T(f\chi_w))(x)|+|(T(f\chi_{U\setminus w}))(x)|\bigr)\qquad\text{for a.e. }\ x\in V \tag{3.3}$$

(i.e., inequality (3.1) holds with f and g changed to $f\chi_w$ and $f\chi_{U\setminus w}$, respectively). Here A is a partial quasi-additivity constant.

If $A = 1$ in (3.1), then T is a subadditive operator; accordingly, if $A = 1$ in (3.3), then T is a partially subadditive operator.

Let $K\colon E \times U \times V \to \mathbb{R}$, where $E = \bigcup_{f\in Z} f(U)$, and $T\colon Z(U) \to M(V)$ be an operator defined as follows: for any $f \in Z(U)$,

$$(Tf)(y) = \int\limits_U K(f(x), x, y)\, d\mu \tag{3.4}$$

under the assumption that this integral exists and is finite for a.e. $y \in V$. This operator is called the Urysohn integral operator. For the properties of the Urysohn operator and, in particular, the conditions on K under which this integral exists and is finite for a.e. $y \in V$, we refer the reader to the book [16] and papers [21, 22].

In particular, if $f \in L_p(U, \mu)$,

$$|K(z, x, y)| \le c|z|^\alpha |K(1, x, y)| \qquad \text{for all} \quad x \in U, \quad y \in V, \quad z \in f(U)$$

with some $0 \le \alpha \le p$ and $c > 0$, and $\|K(1, \cdot, y)\|_{L_r(U,\mu)} < \infty$ for a.e. $y \in V$, where $1/r = 1 - \alpha/p$, then the integral in (3.4) exists and is finite for a.e. $y \in V$.

Lemma 3. *For any function $f \in Z(U)$, any μ-measurable set $w \subset U$, and any $y \in V$,*

$$(Tf)(y) = (T(f\chi_w))(y) + (T(f\chi_{U\setminus w}))(y) - (T(0))(y). \tag{3.5}$$

Proof. Since the integral is additive with respect to the measure,

$$\begin{aligned}
(Tf)(y) - (T(f\chi_w))(y) &= \int\limits_U K(f(x), x, y)\, d\mu - \int\limits_U K\big(f(x)\chi_w(x), x, y\big)\, d\mu \\
&= \int\limits_{U\setminus w} K\big(f(x)\chi_{U\setminus w}(x), x, y\big)\, d\mu + \int\limits_w K\big(f(x)\chi_w(x), x, y\big)\, d\mu \\
&\quad - \int\limits_{U\setminus w} K(0, x, y)\, d\mu - \int\limits_w K\big(f(x)\chi_w(x), x, y\big)\, d\mu \\
&= \int\limits_U K\big(f(x)\chi_{U\setminus w}(x), x, y\big)\, d\mu - \int\limits_w K(0, x, y)\, d\mu - \int\limits_{U\setminus w} K(0, x, y)\, d\mu \\
&= \int\limits_U K\big(f(x)\chi_{U\setminus w}(x), x, y\big)\, d\mu - \int\limits_U K(0, x, y)\, d\mu = (T(f\chi_{U\setminus w}))(y) - (T(0))(y). \qquad \square
\end{aligned}$$

Corollary 1. *If*

$$(T(0))(y) = \int\limits_U K(0, x, y)\, d\mu = 0 \qquad \textit{for a.e.} \;\; y \in V, \tag{3.6}$$

then T is a partially additive operator.

Corollary 2. *If, for any $f \in Z(U)$,*

$$(Tf)(y) \ge 0 \qquad \textit{for a.e.} \;\; y \in V \tag{3.7}$$

(*in particular, if the kernel K is nonnegative*), *then T is a partially subadditive operator.*

Proof. According to (3.5), we have

$$\begin{aligned}
|(Tf)(y)| &= (Tf)(y) = (T(f\chi_w))(y) + (T(f\chi_{U\setminus w}))(y) - (T(0))(y) \\
&\le (T(f\chi_w))(y) + (T(f\chi_{U\setminus w}))(y) = |(T(f\chi_w))(y)| + |(T(f\chi_{U\setminus w}))(y)|. \qquad \square
\end{aligned}$$

Corollary 3. *If there is a $B > 0$ such that for any $f \in Z(U)$*

$$|(T(0))(y)| \le B|(Tf)(y)| \qquad \text{for a.e. } y \in V, \tag{3.8}$$

then T is a partially quasi-additive operator with partial quasi-additivity constant $A = B + 1$.

Proof. According to (3.5), we have

$$\begin{aligned}|(Tf)(y)| &\le |(T(f\chi_w))(y)| + |(T(f\chi_{U\setminus w}))(y)| + |(T(0))(y)|\\ &\le |(T(f\chi_w))(y)| + |(T(f\chi_{U\setminus w}))(y)| + B|(T(f\chi_w))(y)|\\ &\le (B+1)\big(|(T(f\chi_w))(y)| + |(T(f\chi_{U\setminus w}))(y)|\big). \qquad \square\end{aligned}$$

Remark 1. If we are interested in the classical interpolation theorems for the Urysohn integral operator and, hence, in estimates of the form

$$\|Tf\|_Y \le C\|f\|_X, \qquad f \in X, \tag{3.9}$$

where X and Y are some quasinormed function spaces and $C > 0$ is independent of f, then condition (3.6) is necessary. (If we take $f = 0$ in (3.9), then $(Tf)(y) = 0$ for a.e. $y \in V$.)

4. INTERPOLATION THEOREMS FOR MORREY-TYPE SPACES

Lemma 4 (Hardy inequalities). *Let $\xi > 0$ and $0 < \sigma \le \tau \le \infty$. Then, for any nonnegative Lebesgue measurable function g on $(0, \infty)$,*

$$\left(\int\limits_0^\infty \left(t^{-\xi}\left(\int\limits_0^t (g(r))^\sigma \frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau} \le (\xi\sigma)^{-1/\sigma}\left(\int\limits_0^\infty \big(t^{-\xi}g(t)\big)^\tau \frac{dt}{t}\right)^{1/\tau}$$

and

$$\left(\int\limits_0^\infty \left(t^{\xi}\left(\int\limits_t^\infty (g(r))^\sigma \frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau} \le (\xi\sigma)^{-1/\sigma}\left(\int\limits_0^\infty \big(t^{\xi}g(t)\big)^\tau \frac{dt}{t}\right)^{1/\tau}.$$

Let (U, μ) be a space with measure μ, $s > 0$, and $G_\Omega = \{G_{t,y}\}_{t>0,\, y\in\Omega}$ be a net. Define the nets $\check{G}^s_\Omega = \{\check{G}^s_{t,y}\}_{t>0,\, y\in\Omega}$, $\widehat{G}^s_\Omega = \{\widehat{G}^s_{t,y}\}_{t>0,\, y\in\Omega}$ and $\check{G}^s_{\{y\}} = \{\check{G}^s_{t,y}\}_{t>0}$, $\widehat{G}^s_{\{y\}} = \{\widehat{G}^s_{t,y}\}_{t>0}$ for $y \in \Omega$, where

$$\check{G}^s_{t,y} = \begin{cases} G_{t,y} & \text{if } s \ge t,\\ G_{s,y} & \text{if } s < t\end{cases} \qquad \text{and} \qquad \widehat{G}^s_{t,y} = \begin{cases} \varnothing & \text{if } s \ge t,\\ G_{t,y} & \text{if } s < t.\end{cases} \tag{4.1}$$

Remark 2. 1. If $G = \{G_t\}_{t>0}$ is a filtration, i.e., G is a system of expanding σ-algebras of measurable sets, then, in the theory of stochastic processes, the procedure defined by the first relation in (4.1) is called a stop corresponding to the moment s, while the procedure defined by the second relation in (4.1) defines a start corresponding to the moment s. These transformations play an important role in constructing interpolation methods for stochastic processes [1, 2, 19]. In this section, we present the main results of the study, in which these transformations also play a significant role.

2. Note that

$$\|f\|_{M^\alpha_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)} = \left(\int\limits_s^\infty \big(t^{-\alpha}\|f\|_{L_p(G_{t,y},\mu)}\big)^\sigma \frac{dt}{t}\right)^{1/\sigma}$$

and

$$\|f\|_{M^{\alpha}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)} = \left(\int_0^s \bigl(r^{-\alpha}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r} + \int_s^\infty \bigl(r^{-\alpha}\|f\|_{L_p(G_{s,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma}$$

$$= \left(\int_0^s \bigl(r^{-\alpha}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r} + \|f\|^{\sigma}_{L_p(G_{s,y},\mu)}\int_s^\infty r^{-\alpha\sigma}\,\frac{dr}{r}\right)^{1/\sigma}$$

$$= \left(\int_0^s \bigl(r^{-\alpha}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r} + (\alpha\sigma)^{-1}s^{-\alpha\sigma}\|f\|^{\sigma}_{L_p(G_{s,y},\mu)}\right)^{1/\sigma}$$

$$\asymp s^{-\alpha}\|f\|_{L_p(G_{s,y},\mu)} + \left(\int_0^s \bigl(r^{-\alpha}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma}.$$

Lemma 5. *Let $0 < p,\sigma,\tau \le \infty$ and $0 \le \alpha_0 < \alpha < \alpha_1$ with $\alpha_0 > 0$ if $\sigma < \infty$, and let $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$. Then, for arbitrary $y\in\Omega$ and $s>0$,*

$$\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)} < \infty \qquad \text{and} \qquad \bigl\|f\bigl(1-\chi_{G_{s,y}}\bigr)\bigr\|_{M^{\alpha_1}_{p,\sigma}(G_{\{y\}},\mu)} \le \|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)} < \infty.$$

Proof. Let $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$, $y\in\Omega$, and $s>0$. Then

$$\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}_{\{y\}},\mu)} \asymp s^{-\alpha_0}\|f\|_{L_p(G_{s,y},\mu)} + \left(\int_0^s \bigl(r^{-\alpha_0}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma}$$

$$\le s^{-\alpha_0}\|f\|_{L_p(G_{s,y},\mu)} + \|f\|_{M^{\alpha}_{p,\infty}(G_\Omega,\mu)}\left(\int_0^s r^{(\alpha-\alpha_0)\sigma}\,\frac{dr}{r}\right)^{1/\sigma} < \infty.$$

To obtain the second relation, notice that

$$\bigl\|f\bigl(1-\chi_{G_{s,y}}\bigr)\bigr\|_{M^{\alpha_1}_{p,\sigma}(G_{\{y\}},\mu)} = \left(\int_s^\infty \bigl(r^{-\alpha_1}\bigl\|f\bigl(1-\chi_{G_{s,y}}\bigr)\bigr\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma}$$

$$\le \left(\int_s^\infty \bigl(r^{-\alpha_1}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma} = \|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)}$$

and

$$\|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)} = \left(\int_s^\infty \bigl(r^{-\alpha_1}\|f\|_{L_p(G_{r,y},\mu)}\bigr)^{\sigma}\,\frac{dr}{r}\right)^{1/\sigma} \le M^{\alpha}_{p,\infty}(G_\Omega,\mu)\left(\int_s^\infty r^{(\alpha-\alpha_1)\sigma}\,\frac{dr}{r}\right)^{1/\sigma} < \infty. \qquad \square$$

Theorem 2. *Let (U,μ) and (V,ν) be spaces with measures μ and ν, and let $\Omega\subset\mathbb{R}^n$.*

Let $G_\Omega = \{G_{t,y}\}_{t>0,\,y\in\Omega}$ be a net in U, $\check{G}^s_\Omega$ and $\widehat{G}^s_\Omega$ for $s>0$ be the nets in U defined in (4.1), and $\check{G}^s_{\{y\}}$ and $\widehat{G}^s_{\{y\}}$ for $s>0$ and $y\in\Omega$ be the nets generated by $\check{G}^s_\Omega$ and $\widehat{G}^s_\Omega$ in U.

Let $F_\Omega = \{F_{t,y}\}_{t>0,\,y\in\Omega}$ be a net in V and $F_{\{y\}}$ for $y\in\Omega$ be the nets generated by F_Ω in V.

Let $0<p,q,\sigma,\tau\le\infty$, $0\le\alpha_0,\alpha_1<\infty$ with $\alpha_0,\alpha_1>0$ if $\sigma<\infty$, $\alpha_0\ne\alpha_1$, $0\le\beta_0,\beta_1<\infty$, $\beta_0\ne\beta_1$, and $0<\theta<1$, and let α and β be as defined in (1.1).

Let $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$, $U_f = \{f\chi_w : w \subset U$ is a μ-measurable subset$\}$, and T be a quasi-additive operator on U_f with quasi-additivity constant A.

If for some $M_0, M_1 > 0$ the inequalities

$$\big\|T\big(f\chi_{G_{s,y}}\big)\big\|_{M^{\beta_0}_{q,\infty}(F_{\{y\}},\nu)} \le M_0\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)}, \tag{4.2}$$

$$\big\|T\big(f\big(1-\chi_{G_{s,y}}\big)\big)\big\|_{M^{\beta_1}_{q,\infty}(F_{\{y\}},\nu)} \le M_1\|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)} \tag{4.3}$$

hold for all $y \in \Omega$ and $s > 0$, then the inequality

$$\|Tf\|_{M^{\beta}_{q,\tau}(F_\Omega,\nu)} \le cAM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\alpha}_{p,\tau}(G_\Omega,\mu)} \tag{4.4}$$

holds with a constant $c > 0$ depending only on α_0, α_1, β_0, β_1, q, σ, τ, and θ.

Remark 3. 1. According to Lemma 5, for a fixed function $f \in M^{\alpha}_{p,q}(G_\Omega,\mu)$, the norms in the hypotheses of the theorem are well defined.

2. This theorem, albeit similar to the classical interpolation theorems, has a significant difference. The point is that the hypotheses and the statement of the theorem are formulated for a fixed function $f \in M^{\alpha}_{p,q}(G_\Omega,\mu)$. We can say that here we deal not with interpolation of operators but rather with interpolation of inequalities for a fixed function. This fact makes the statement more universal for applications (see Section 5). In particular, the sets $G_{s,y}$ can be chosen depending on f.

3. Consider the nonlinear integral operator

$$(Tf)(y) = \int\limits_U K(f(x),x,y)\,d\mu \qquad \text{such that} \quad (T(0))(y) = \int\limits_U K(0,x,y)\,d\mu \ne 0.$$

The classical interpolation theorems do not apply to this operator (see Section 3). At the same time, if T satisfies one of the conditions (3.7) or (3.8), Theorem 2 makes sense.

Proof of Theorem 2. Without loss of generality, we assume that $\alpha_0 < \alpha_1$. Since the embedding $M^{\alpha}_{p,\sigma_1}(G_{\{y\}},\mu) \hookrightarrow M^{\alpha}_{p,\sigma_2}(G_{\{y\}},\mu)$ holds for $\sigma_1 < \sigma_2$, we can also assume without loss of generality that $0 < \sigma \le \tau \le \infty$.

Let $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$, $y \in \Omega$, and $s > 0$. Since $f\chi_w \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$ for any μ-measurable set $w \subset U$ and the operator T is quasi-additive on the set U_f, it follows that

$$\begin{aligned}\|Tf\|_{L_q(F_{t,y},\nu)} &\le A\big\|\,|T(f\chi_{G_{s,y}})| + |T(f(1-\chi_{G_{s,y}}))|\,\big\|_{L_q(F_{t,y},\nu)}\\ &\le 2^{(1/q-1)_+}A\big(\|T(f\chi_{G_{s,y}})\|_{L_q(F_{t,y},\nu)} + \|T(f(1-\chi_{G_{s,y}}))\|_{L_q(F_{t,y},\nu)}\big).\end{aligned}$$

From condition (4.2), we have

$$\begin{aligned}\|T(f\chi_{G_{s,y}})\|_{L_q(F_{t,y},\nu)} &= t^{\beta_0}t^{-\beta_0}\|T(f\chi_{G_{s,y}})\|_{L_q(F_{t,y},\nu)} \le t^{\beta_0}\sup_{r>0} r^{-\beta_0}\|T(f\chi_{G_{s,y}})\|_{L_q(F_{r,y}\nu)}\\ &= t^{\beta_0}\|T(f\chi_{G_{s,y}})\|_{M^{\beta_0}_{q,\infty,y}(F,\nu)} \le M_0t^{\beta_0}\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}_{\{y\}},\mu)}\\ &\asymp M_0t^{\beta_0}\left(s^{-\alpha}\|f\|_{L_p(G_{s,y},\mu)} + \left(\int\limits_0^s \big(r^{-\alpha}\|f\|_{L_p(G_{r,y},\mu)}\big)^{\sigma}\frac{dr}{r}\right)^{1/\sigma}\right).\end{aligned}$$

Hence, the following inequality holds for arbitrary $s > 0$ and $y \in \Omega$:

$$\|T(f\chi_{G_{s,y}})\|_{L_q(F_{t,y},\nu)} \le c_1M_0t^{\beta_0}\left[\left(\int\limits_0^s \big(r^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{r,y},\mu)}\big)^{\sigma}\frac{dr}{r}\right)^{1/\sigma} + s^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{s,y},\mu)}\right],$$

where $c_1 > 0$ depends only on α_0, α_1, and σ.

Let us use condition (4.3) to estimate the second term:

$$\|T(f(1-\chi_{G_{s,y}}))\|_{L_q(F_{t,y},\nu)} = t^{\beta_1}t^{-\beta_1}\|T(f(1-\chi_{G_{s,y}}))\|_{L_q(F_{t,y},\nu)} \le M_1 t^{\beta_1}\|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)}$$
$$= M_1 t^{\beta_1}\left(\int\limits_s^\infty \big(t^{-\alpha_1}\|f\|_{L_p(G_{t,y},\mu)}\big)^\sigma \frac{dt}{t}\right)^{1/\sigma} \le M_1 t^{\beta_1}\left(\int\limits_s^\infty \Big(t^{-\alpha_1}\sup_{y\in\Omega}\|f\|_{L_p(G_{t,y},\mu)}\Big)^\sigma \frac{dt}{t}\right)^{1/\sigma}.$$

Thus, for all $t>0$, $s>0$, and $y\in\Omega$, we obtain

$$\|Tf\|_{L_q(F_{t,y})} \le c_2 A\left(M_0 t^{\beta_0}\left[\left(\int\limits_0^s \Big(r^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{r,y})}\Big)^\sigma \frac{dr}{r}\right)^{1/\sigma} + s^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{s,y},\mu)}\right]\right.$$
$$\left. + M_1 t^{\beta_1}\left(\int\limits_s^\infty \Big(t^{-\alpha_1}\sup_{y\in\Omega}\|f\|_{L_p(G_{t,y},\mu)}\Big)^\sigma \frac{dt}{t}\right)^{1/\sigma}\right),$$

where $c_2>0$ depends only on q, α_0, α_1, and σ.

Set $s=ct^\gamma$, where $\gamma=(\beta_1-\beta_0)/(\alpha_1-\alpha_0)$ and $c>0$ is a constant to be chosen later. Then

$$\|Tf\|_{M^\beta_{q,\tau}(G_\Omega,\mu)} = \left(\int\limits_0^\infty \Big(t^{-\beta}\sup_{x\in\Omega}\|Tf\|_{L_q(F_{t,y})}\Big)^\tau \frac{dt}{t}\right)^{1/\tau} \le 3^{(1/\tau-1)_+}c_2 A(M_0I_1+M_0I_2+M_1I_3),$$

where

$$I_1 = \left(\int\limits_0^\infty \left(t^{\beta_0-\beta}\left(\int\limits_0^{ct^\gamma}\Big(r^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{r,y})}\Big)^\sigma\frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau},$$

$$I_2 = \left(\int\limits_0^\infty \Big(t^{\beta_0-\beta}(ct^\gamma)^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{ct^\gamma,y},\mu)}\Big)^\tau \frac{dt}{t}\right)^{1/\tau},$$

$$I_3 = \left(\int\limits_0^\infty \left(t^{\beta_1-\beta}\left(\int\limits_{ct^\gamma}^{\infty}\Big(r^{-\alpha_1}\sup_{y\in\Omega}\|f\|_{L_p(G_{r+ct^\gamma,y})}\Big)^\sigma\frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau}.$$

Making the change $ct^\gamma\to t$, we obtain

$$I_1=\gamma^{-1/\tau}c^{\theta(\alpha_1-\alpha_0)}J_1,\qquad I_2=\gamma^{-1/\tau}c^{\theta(\alpha_1-\alpha_0)}J_2,\qquad I_3=\gamma^{-1/\sigma}c^{-(1-\theta)(\alpha_1-\alpha_0)}J_3,$$

where

$$J_1 = \left(\int\limits_0^\infty \left(t^{-\theta(\alpha_1-\alpha_0)}\left(\int\limits_0^{t}\Big(r^{-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{r,y},\mu)}\Big)^\sigma\frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau},$$

$$J_2 = \left(\int\limits_0^\infty \Big(t^{-\theta(\alpha_1-\alpha_0)-\alpha_0}\sup_{y\in\Omega}\|f\|_{L_p(G_{t,y},\mu)}\Big)^\tau \frac{dt}{t}\right)^{1/\tau} = \|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)},$$

$$J_3 = \left(\int\limits_0^\infty \left(t^{(1-\theta)(\alpha_1-\alpha_0)}\left(\int\limits_0^{\infty}\Big(r^d(r+t)^{-\alpha_1-d}\sup_{y\in\Omega}\|f\|_{L_p(G_{r+t,y})}\Big)^\sigma\frac{dr}{r}\right)^{1/\sigma}\right)^\tau \frac{dt}{t}\right)^{1/\tau}.$$

To estimate J_1 and J_3, we apply the Hardy inequalities (Lemma 4), according to which we have

$$J_1 \le (\theta(\alpha_1-\alpha_0)\sigma)^{-1/\sigma}\left(\int\limits_0^\infty\left(t^{-\alpha}\sup_{y\in\Omega}\|f\|_{L_p(G_{t,y})}\right)^\tau\frac{dt}{t}\right)^{1/\tau} = (\theta(\alpha_1-\alpha_0)\sigma)^{-1/\sigma}\|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)},$$

$$J_2 \le (\theta(\alpha_1-\alpha_0)\sigma)^{-1/\sigma}\|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)},$$

$$J_3 \le ((1-\theta)(\alpha_1-\alpha_0)\sigma)^{-1/\sigma}\|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)}.$$

For all $c>0$, we obtain

$$\|Tf\|_{M^\beta_{q,\tau}(F_\Omega,\nu)} \le c_3\big(M_0c^{\theta(\alpha_1-\alpha_0)}+M_1c^{-(1-\theta)(\alpha_1-\alpha_0)}\big)\|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)},$$

where $c_3>0$ depends only on α_0, α_1, β_0, β_1, q, σ, τ, and θ.

Now we set $c=(M_1/M_0)^{1/(\alpha_1-\alpha_0)}$. Then

$$\|Tf\|_{M^\beta_{q,\tau}(F_\Omega,\nu)} \le 2c_3M_0^{1-\theta}M_1^\theta\|f\|_{M^\alpha_{p,\tau}(G_\Omega,\mu)}. \qquad \square$$

Remark 4. Theorem 2 implies, in particular, Theorem 1 formulated in the Introduction. Indeed, if $U=V=\mathbb{R}^n$, μ and ν are the Lebesgue measures, $G_{t,y}=B(y,t)$, $t>0$, $y\in\mathbb{R}^n$, and $\Omega=\mathbb{R}^n$, then inequalities (1.2) and Theorem 1 with f replaced by $f\chi_{G_{s,y}}$ for $i=0$ and by $f(1-\chi_{G_{s,y}})$ for $i=1$ imply inequalities (4.2) and (4.3). Hence, inequality (4.4) holds, which leads to inequality (1.3).

Let $U=V=\mathbb{R}^n$, μ and ν be the Lebesgue measures, $\Omega\subset\mathbb{R}^n$, $0<p,q\le\infty$, and $0\le\lambda<\infty$ with $\lambda>0$ if $q<\infty$. Let v be a positive locally absolutely continuous strictly increasing function defined on $(0,\infty)$. Define the spaces $M^\lambda_{p,q,\Omega}(v)$ in the spirit of Example 3:

$$M^\lambda_{p,q,\Omega}(v)=\left\{f\in L^{\mathrm{loc}}_p(\mathbb{R}^n)\colon\ \|f\|_{M^\lambda_{p,q,\Omega}(v)}=\left(\int\limits_0^\infty\left((v(r))^{-\lambda}\sup_{y\in\Omega}\|f\|_{L_p(B(y,r))}\right)^q\frac{dv(r)}{v(r)}\right)^{1/q}<\infty\right\}$$

for $0<q<\infty$ and

$$M^\lambda_{p,\infty,\Omega}(v)=\left\{f\colon\ \|f\|_{M^\lambda_{p,\infty,\Omega}(v)}=\sup_{r>0,\ y\in\Omega}(v(r))^{-\lambda}\|f\|_{L_p(B(y,r))}<\infty\right\}$$

for $q=\infty$.

Corollary 4. *Let $\Omega\subset\mathbb{R}^n$, $0<p,q,\sigma,\tau\le\infty$, $0\le\alpha_0,\alpha_1<\infty$ with $\alpha_0,\alpha_1>0$ if $\sigma<\infty$, $\alpha_0\ne\alpha_1$, $0\le\beta_0,\beta_1<\infty$, $\beta_0\ne\beta_1$, and $0<\theta<1$, and let α and β be as defined in* (1.1).

Suppose that the functions v and w satisfy the conditions indicated above, and let T be a partially quasi-additive operator on $M^\alpha_{p,\tau,\Omega}(w)$ with partial quasi-additivity constant A.

If for some $M_0,M_1>0$ the inequalities

$$\|Tf\|_{M^{\beta_i}_{q,\infty,y}(v)} \le M_i\|f\|_{M^{\alpha_i}_{p,\sigma,y}(w)}$$

hold for all $y\in\Omega$ and all functions $f\in LM^{\alpha_i}_{p,\sigma,y}(w)$, $i=0,1$, then the inequality

$$\|Tf\|_{M^\beta_{q,\tau,\Omega}(v)} \le cAM_0^{1-\theta}M_1^\theta\|f\|_{M^\alpha_{p,\tau,\Omega}(w)}$$

holds for all functions $f\in LM^\alpha_{p,\tau,\Omega}(w)$, where $c>0$ depends only on α_0, α_1, β_0, β_1, q, σ, τ, and θ.

Proof. Let μ be the Lebesgue measure in $\mathbb{R}^n$. Let v^{-1} and w^{-1} be the inverse functions of v and w, respectively. Consider the nets $F=\{F_{t,y}\}_{t>0,\ y\in\Omega}$ and $G=\{G_{t,y}\}_{t>0,\ y\in\Omega}$ with

$$F_{t,y}=B(y,v^{-1}(t)) \qquad \text{and} \qquad G_{t,y}=B(y,w^{-1}(t)).$$

Then $M^{\lambda}_{p,q,\Omega}(w)$ and $M^{\lambda}_{p,q,y}(w)$ coincide with the spaces $M^{\lambda}_{p,q}(G_\Omega,\mu)$ and $M^{\lambda}_{p,q}(G_{\{y\}},\mu)$, respectively. Next, we apply Theorem 2. $\square$

The following interpolation theorem is purely a theorem on interpolation of inequalities (it does not contain any operator).

Theorem 3. *Let (U,μ) and (V,ν) be spaces with measures μ and ν, and let $\Omega\subset\mathbb{R}^n$.*

Let $G_\Omega=\{G_{t,y}\}_{t>0,\, y\in\Omega}$ be a net in U, $\check{G}^s_\Omega$ and $\widehat{G}^s_\Omega$ for $s>0$ be the nets in U defined in (4.1), *and $\check{G}^s_{\{y\}}$ and $\widehat{G}^s_{\{y\}}$ for $s>0$ and $y\in\Omega$ be the nets generated by $\check{G}^s_\Omega$ and $\widehat{G}^s_\Omega$ in U.*

Let $F_\Omega=\{F_{t,y}\}_{t>0,\, y\in\Omega}$, $F^0_\Omega(s)=\{(F^0(s))_{t,y}\}_{t>0,\, y\in\Omega}$, and $F^1_\Omega(s)=\{(F^1(s))_{t,y}\}_{t>0,\, y\in\Omega}$ for $s>0$ be nets in V, and let $F^0_{\{y\}}(s)$ and $F^1_{\{y\}}(s)$ for $s>0$ and $y\in\Omega$ be the nets in V generated by $F^0_\Omega(s)$ and $F^1_\Omega(s)$; in addition, suppose that

$$F_{t,y}\subset (F^0(s))_{t,y}\cup(F^1(s))_{t,y} \tag{4.5}$$

for all $s,t>0$ and $y\in\Omega$.

Let $0<p,q,\sigma,\tau\le\infty$, $0\le\alpha_0,\alpha_1<\infty$ with $\alpha_0,\alpha_1>0$ if $\sigma<\infty$, $\alpha_0\ne\alpha_1$, $0\le\beta_0,\beta_1<\infty$, $\beta_0\ne\beta_1$, and $0<\theta<1$, and let α and β be as defined in (1.1).

Let $f\in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$ and g be a ν-measurable function on V.

If for some $M_0,M_1>0$ the inequalities

$$\|g\|_{M^{\beta_0}_{q,\infty}(F^0_{\{y\}}(s),\nu)}\le M_0\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)}, \tag{4.6}$$

$$\|g\|_{M^{\beta_1}_{q,\infty}(F^1_{\{y\}}(s),\nu)}\le M_1\|f\|_{M^{\alpha_1}_{p,\sigma}(\widehat{G}^s_{\{y\}},\mu)} \tag{4.7}$$

hold for all $y\in\Omega$ and $s>0$, then the inequality

$$\|g\|_{M^{\beta}_{q,\tau}(F_\Omega,\nu)}\le cM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\alpha}_{p,\tau}(G_\Omega,\mu)} \tag{4.8}$$

holds with a constant $c>0$ depending only on q, τ, σ, α_0, α_1, β_0, β_1, and θ.

Proof. Without loss of generality, we may assume that $\alpha_0<\alpha_1$. Since the embedding $M^{\alpha}_{p,\sigma_1}(G_{\{y\}},\mu)\hookrightarrow M^{\alpha}_{p,\sigma_2}(G_{\{y\}},\mu)$ holds for $\sigma_1<\sigma_2$, we can also assume without loss of generality that $0<\sigma\le\tau\le\infty$.

Let $f\in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$, $y\in\Omega$, and $s>0$. According to (4.5),

$$\|g\|_{L_q(F_{t,y},\nu)}\le 2^{(1/q-1)_+}\big(\|g\|_{L_q((F^0(s))_{t,y},\nu)}+\|g\|_{L_q((F^1(s))_{t,y},\nu)}\big).$$

From conditions (4.6) and (4.7), we have

$$\begin{aligned}\|g\|_{L_q((F^0(s))_{t,y},\nu)}&=t^{\beta_0}t^{-\beta_0}\|g\|_{L_q((F^0(s))_{t,y},\nu)}\le t^{\beta_0}\sup_{r>0}r^{-\beta_0}\|g\|_{L_q((F^0(s))_{t,y},\nu)}\\&=A^{\beta_0}t^{\beta_0}\|g\|_{M^{\beta_0}_{q,\infty}(F^0_{\{y\}}(s),\nu)}\le M_0t^{\beta_0}\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)}\\&\asymp M_0t^{\beta_0}\left(\left(\int_0^s\big(r^{-\alpha_0}\|f\|_{L_p(G^s_{r,y},\mu)}\big)^\sigma\frac{dr}{r}\right)^{1/\sigma}+s^{-\alpha_0}\|f\|_{L_p(G_{s,y},\mu)}\right)\end{aligned}$$

and

$$\|f\|_{L_q((F^1(s))_{t,y},\nu)}\le M_1t^{\beta_1}\|f\|_{M_{p,\sigma}(\widehat{G}^s_{\{y\}})}=M_1t^{\beta_1}\left(\int_s^\infty\big(r^{-\alpha_0}\|f\|_{L_p(G^s_{r,y},\mu)}\big)^\sigma\frac{dr}{r}\right)^{1/\sigma}.$$

Thus, for all $t > 0$, $s > 0$, and $y \in \Omega$, we obtain

$$\|g\|_{L_q(F_{t,y})} \le c_2 \left(M_0 t^{\beta_0} \left[\left(\int_0^s \left(r^{-\alpha_0} \sup_{y\in\Omega} \|f\|_{L_p(G_{r,y})} \right)^\sigma \frac{dr}{r} \right)^{1/\sigma} + s^{-\alpha_0} \sup_{y\in\Omega} \|f\|_{L_p(G_{s,y},\mu)} \right] \right.$$
$$\left. + M_1 t^{\beta_1} \left(\int_s^\infty \left(t^{-\alpha_1} \sup_{y\in\Omega} \|f\|_{L_p(G_{t,y},\mu)} \right)^\sigma \frac{dt}{t} \right)^{1/\sigma} \right),$$

where $c_2 > 0$ depends only on q, α_0, α_1, and σ.

The further arguments are similar to those in the proof of Theorem 2. $\square$

Remark 5. The hypotheses of Theorem 3 contain the nets F_Ω, $F^0_\Omega(s)$, and $F^1_\Omega(s)$ that satisfy condition (4.5). We now give an example of such nets, independent of Ω, which will be used in Sections 5 and 6.

Let (U,μ), $Z(U)$, (V,ν), and $M(V)$ be as in Section 3. Let $T\colon Z(U) \to M(V)$ be a partially quasi-additive operator with partial quasi-additivity constant A, and let the nets F, $F^0(s)$, and $F^1(s)$ be defined by the following sets:

$$F_t = \{x \in V\colon\ |(Tf)(x)| > t^{-1}h(x)\}, \qquad t > 0,$$
$$F^0_t(s) = \{x \in V\colon\ |(T(f\chi_{G_s}))(x)| > (2At)^{-1}h(x)\}, \qquad t > 0, \quad s > 0,$$
$$F^1_t(s) = \{x \in V\colon\ |(T(f(1-\chi_{G_s})))(x)| > (2At)^{-1}h(x)\}, \qquad t > 0, \quad s > 0,$$

where h is a positive function on V.

Now, let us show that the partial quasi-additivity of T implies the inclusion

$$F_t \subset F^0_t(s) \cup F^1_t(s), \qquad s > 0, \quad t > 0.$$

Indeed, if $x \in F_t$, then $x \in V$ and

$$(T(f\chi_{G_s}))(x) > \frac{h(x)}{2At} \qquad \Leftrightarrow \qquad x \in F^0_t(s)$$

or

$$(T(f(1-\chi_{G_s})))(x) > \frac{h(x)}{2At} \qquad \Leftrightarrow \qquad x \in F^1_t(s),$$

since otherwise

$$(T(f\chi_{G_s}))(x) \le \frac{h(x)}{2At}, \qquad (T(f(1-\chi_{G_s})))(x) \le \frac{h(x)}{2At}$$

and

$$|(Tf)(x)| \le A\big(|(T(f\chi_{G_s}))(x)| + |(T(f(1-\chi_{G_s})))(x)|\big) \le \frac{h(x)}{t},$$

which contradicts the fact that $x \in F_t$.

5. MARCINKIEWICZ–CALDERÓN TYPE INTERPOLATION THEOREM

Inspecting the standard proof of the Marcinkiewicz theorem, one can easily verify that the condition of quasi-additivity of the operator T is applied in the proof only to sums of the form $f\chi_w + f(1-\chi_w)$ rather than to arbitrary sums $f_1 + f_2$; i.e., it suffices to assume that the operator T is partially quasi-additive. Taking this fact into account, we present the Marcinkiewicz interpolation theorem in the following form.

Theorem 4. *Let (U,μ) and (V,ν) be spaces with measures μ and ν.*

Let $1 \le p_0 \le p_1 < \infty$, $1 \le q_0, q_1 < \infty$, $q_0 \ne q_1$, $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \ge \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T be a partially quasi-additive operator on $L_{p_0}(U,\mu) + L_{p_1}(U,\mu)$ with partial quasi-additivity constant A.

If for some $M_0, M_1 > 0$ the inequalities

$$\|Tf\|_{L_{q_i,\infty}(V,\nu)} \le M_i \|f\|_{L_{p_i}(U,\mu)} \tag{5.1}$$

hold for all functions $f \in L_{p_i}(U,\mu)$, $i = 0, 1$, then the inequality

$$\|Tf\|_{L_q(V,\nu)} \le cAM_0^{1-\theta} M_1^{\theta} \|f\|_{L_p(U,\mu)} \tag{5.2}$$

holds for all $f \in L_p(U,\mu)$, where $c > 0$ depends only on p_0, p_1, q_0, q_1, and θ.

The well-known Calderón theorem [13] generalizes and, in a sense, strengthens the Marcinkiewicz interpolation theorem in the case when the operator is quasi-additive. There are various proofs of this theorem, but all of them employ the quasi-additivity condition in an essential way and are not generally valid in the case of just partial quasi-additivity.

We will prove an analog of the Calderón theorem for partially quasi-additive operators T, while requiring the regularity of the corresponding measures.

Theorem 5. *Let (U,μ) and (V,ν) be spaces with measures μ and ν satisfying the regularity condition* (2.2).

Let $1 \le p_0 < p_1 < \infty$, $1 \le q_0, q_1 < \infty$, $q_0 \ne q_1$, $0 < \sigma, \tau \le \infty$, $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{5.3}$$

Let T be a partially quasi-additive operator on $L_{p_0,\sigma}(U,\mu) + L_{p_1,\sigma}(U,\mu)$ with partial quasi-additivity constant A.

If for some $M_0, M_1 > 0$ the inequalities

$$\|Tf\|_{L_{q_i,\infty}(V,\nu)} \le M_i \|f\|_{L_{p_i,\sigma}(U,\mu)} \tag{5.4}$$

hold for all $f \in L_{p_i,\sigma}(U,\mu)$, $i = 0, 1$, then the inequality

$$\|Tf\|_{L_{q,\tau}(V,\nu)} \le cAM_0^{1-\theta} M_1^{\theta} \|f\|_{L_{p,\tau}(U,\mu)} \tag{5.5}$$

holds for all $f \in L_{p,\tau}(U,\mu)$, where $c > 0$ depends only on p_0, p_1, q_0, q_1, σ, τ, and θ.

Proof. Let $f \in L_{p,\tau}(U,\mu)$. It follows from Lemma 2 and Example 1 that there exist nets $G(f) = \{G_t(f)\}_{t>0}$ and $H(Tf) = \{H_t(Tf)\}_{t>0}$ that satisfy the conditions

$$\mu(G_t(f)) \asymp t \qquad \text{and} \qquad \nu(H_t(Tf)) \asymp t$$

and are such that

$$\int_0^t (Tf)^*(s)\,ds \le \int_{H_t(Tf)} |(Tf)(x)|\,d\nu \le \int_0^{2t} (Tf)^*(s)\,ds,$$

$$\int_0^t f^*(s)\,ds \le \int_{G_t(f)} |f(x)|\,d\mu \le \int_0^{2t} f^*(s)\,ds,$$

$$\|Tf\|_{L_{q,\tau}(V,\nu)} \asymp \|Tf\|_{M_{1,\tau}^{1/q'}(H(Tf),\nu)}, \qquad \text{and} \qquad \|f\|_{L_{p,\tau}(U,\mu)} \asymp \|f\|_{M_{1,\tau}^{1/p'}(G(f),\mu)},$$

where the corresponding constants depend only on the parameter p, q, and τ.

From condition (5.4), we have

$$\|T(f\chi_{G_s(f)})\|_{M^{1/q_0'}_{1,\infty}(H(Tf),\nu)} = \sup_{t>0} t^{-1/q'} \int\limits_{G_t(Tf)} \big|\big(T\big(f\chi_{G_s(f)}\big)\big)(x)\big|\,d\mu$$

$$\le \sup_{t>0} t^{-1/q'} \sup_{\mu(e)=2t} \int\limits_e \big|\big(T\big(f\chi_{G_s(f)}\big)\big)(x)\big|\,d\mu \le \sup_{t>0} t^{-1/q'} \int\limits_0^{2t} \big(T\big(f\chi_{G_s(f)}\big)\big)^*(t)\,dt$$

$$\lesssim \big\|T\big(f\chi_{G_s(f)}\big)\big\|_{L_{q_0,\infty}(D,\nu)} \le M_0\big\|f\chi_{G_s(f)}\big\|_{L_{p_0,\sigma}(U,\mu)}$$

$$\lesssim M_0\big\|f\chi_{G_s(f)}\big\|_{M^{1/p_0}_{1,\sigma}(G(f),\mu)} = M_0\|f\|_{M^{1/p_0}_{1,\sigma}(\check G^s(f),\mu)}.$$

According to inequalities (2.4) and (2.5), by Lemma 2 we obtain

$$\big\|T\big(f\big(1-\chi_{G_s(f)}\big)\big)\big\|_{M^{1/q_1}_{1,\infty}(H(Tf),\nu)} \le \big\|T\big(f\big(1-\chi_{G_s(f)}\big)\big)\big\|_{L_{q_1,\infty}(V,\nu)}$$

$$\le M_1\big\|f\big(1-\chi_{G_s(f)}\big)\big\|_{L_{p_1,\sigma}(U,\mu)} \le M_1\left(\int\limits_0^\infty \big(t^{1/p_1}f^*(t+s)\big)^\sigma\,\frac{dt}{t}\right)^{1/\sigma}$$

$$\le M_1\left(\int\limits_0^\infty\left(t^{1/p_1}\frac{1}{s+t}\int\limits_0^{s+t} f^*(\xi)\,d\xi\right)^\sigma\frac{dt}{t}\right)^{1/\sigma} \lesssim M_1\left(\int\limits_0^\infty\left(t^{1/p_1}\frac{1}{t+s}\int\limits_{G_{t+s}(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}$$

$$\le M_1 2^{(1/\sigma-1)_+}$$

$$\times\left(\left(\int\limits_0^s\left(t^{1/p_1}\frac{1}{t+s}\int\limits_{G_{t+s}(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma} + \left(\int\limits_s^\infty\left(t^{1/p_1}\frac{1}{t+s}\int\limits_{G_{t+s}(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}\right)$$

$$\le M_1 2^{(1/\sigma-1)_+}\left(\int\limits_{G_{2s}(f)}|f(x)|\,d\mu\left(\int\limits_0^s\left(t^{1/p_1}\frac{1}{t+s}\right)^\sigma\frac{dt}{t}\right)^{1/\sigma} + \left(\int\limits_s^\infty\left(t^{1/p_1}\frac{1}{t}\int\limits_{G_{2t}(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}\right).$$

Taking into account that

$$\left(\int\limits_s^\infty\left(t^{-1/p_1'}\int\limits_{G_{2t}(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma} = 2^{1/p_1'}\left(\int\limits_{2s}^\infty\left(t^{-1/p_1'}\int\limits_{G_t(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}$$

$$\le 2^{1/p_1'}\|f\|_{M^{1/p_1'}_{1,\sigma}(\widehat G^s,\mu)},$$

$$\left(\int\limits_0^s\left(t^{1/p_1}\frac{1}{t+s}\right)^\sigma\frac{dt}{t}\right)^{1/\sigma} = s^{-1/p_1'}\left(\int\limits_0^1\left(t^{1/p_1}\frac{1}{t+1}\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}$$

and

$$s^{-1/p_1'}\|f\|_{L_1(G_{2s}(f),\mu)} \le 2^{1/p_1'}\left(\frac{\sigma}{p_1'}\right)^{1/\sigma}\left(\int\limits_{2s}^\infty\left(t^{-1/p_1'}\int\limits_{G_t(f)}|f(x)|\,d\mu\right)^\sigma\frac{dt}{t}\right)^{1/\sigma}$$

$$\le 2^{1/p_1'}\left(\frac{\sigma}{p_1'}\right)^{1/\sigma}\|f\|_{M^{1/p_1'}_{1,\sigma}(\widehat G^s,\mu)},$$

we find

$$\big\|T\big(f\big(1-\chi_{G_s(f)}\big)\big)\big\|_{M^{1/q_1}_{1,\infty}(G(Tf),\nu)} \lesssim M_1\|f\|_{M^{1/p_1'}_{1,\sigma}(\widehat{G}^s,\mu)}.$$

Let us apply Theorem 2:

$$\|Tf\|_{L_{q,\tau}(D,\nu)} \lesssim \|Tf\|_{M^{1/q}_{1,\tau}(G(Tf),\nu)} \lesssim M_0^{1-\theta}M_1^{\theta}\|f\|_{M^{1/p}_{1,\tau}(G(f),\mu)} \lesssim M_0^{1-\theta}M_1^{\theta}\|f\|_{L_{p,\tau}(U,\mu)}.$$

In the above considerations, the constants implied by the signs $\lesssim$ are independent not only of f but also of M_0 and M_1. $\square$

Corollary 5. *Let* $1 \le p_0 < p_1 < \infty$, $1 \le q_0, q_1 < \infty$, $q_0 \ne q_1$, $0 < \sigma, \tau \le \infty$, *and* $0 < \theta < 1$, *and let* p *and* q *be defined by* (5.3).

Let

$$(Tf)(y) = \int\limits_U K(f(x),x,y)\,d\mu, \qquad y \in V.$$

If for some $M_0, M_1 > 0$ *the inequalities*

$$\|Tf\|_{L_{q_i,\infty}(V,\nu)} \le M_i\|f\|_{L_{p_i,\sigma}(U,\mu)} \tag{5.6}$$

hold for all $f \in L_{p_i,\sigma}(U,\mu)$, $i = 0, 1$, *then the inequality*

$$\|Tf\|_{L_{q,\tau}(V,\nu)} \le cM_0^{1-\theta}M_1^{\theta}\|f\|_{L_{p,\tau}(U,\mu)}$$

holds for all $f \in L_{p,\tau}(U,\mu)$, *where* $c > 0$ *depends only on* p_0, p_1, q_0, q_1, σ, θ, *and* τ.

Proof. From condition (5.6), we have

$$\sup_{t>0} t^{1/p_0}(Tf)^*(t) \le M_0\|f\|_{L_{p_0,1}}.$$

Setting $f = 0$, we obtain

$$\int\limits_U K(0,x,y)\,d\mu = 0$$

almost everywhere in V; in this case, as shown in Section 3, the operator T is partially additive. Hence, we can apply Theorem 5. $\square$

6. STEIN–WEISS–PEETRE TYPE INTERPOLATION THEOREMS

Let μ be a measure on U satisfying the regularity condition (2.2) and w be a positive μ-measurable (weight) function on U.

For $0 < p \le \infty$, denote by $L_p(U,w,\mu)$ the space of all μ-measurable functions on U such that

$$\|f\|_{L_p(U,w,\mu)} = \left(\int\limits_U \big(w(x)|f(x)|\big)^p\,d\mu\right)^{1/p} < \infty.$$

If $w \equiv 1$, then $L_p(U,1,\mu) \equiv L_p(U,\mu)$; if μ is the Lebesgue measure, then $L_p(U,w,\mu) \equiv L_p(U,w)$ and $L_p(U,\mu) \equiv L_p(U)$.

Recall the Stein–Weiss–Peetre theorem.

Theorem 6. *Let* $1 \le p_0 \le p_1 < \infty$, $0 < \theta < 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{6.1}$$

Let w_0 and w_1 be positive μ-measurable functions on U, and let T be a subadditive operator on $L_{p_0}(U,w_0,\mu)+L_{p_1}(U,w_1,\mu)$.

If for some $M_0, M_1>0$ the inequalities

$$\|Tf\|_{L_{p_i}(U,w_i,\mu)} \le M_i\|f\|_{L_{p_i}(U,w_i,\mu)} \tag{6.2}$$

hold for all $f\in L_{p_i}(U,w_i,\mu)$, $i=0,1$, then the inequality

$$\|Tf\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)} \le cM_0^{1-\theta}M_1^{\theta}\|f\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)} \tag{6.3}$$

holds for all $f\in L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)$, where $c>0$ depends only on p_0, p_1, and θ.

Remark 6. In the case when T is a linear operator and $p_0=p_1=p$, Theorem 6 was proved by Stein and Weiss [27]. In this case, the constant c in inequality (6.3) is equal to 1. In the present version, the theorem was proved by Peetre [24].

Theorem 7. *Let $0<p\le q<\infty$ and $0<\theta<1$. Let w_0 and w_1 be positive μ-measurable functions on U, and let T be a partially quasi-additive operator on $L_p(U,w_0,\mu)+L_p(U,w_1,\mu)$ with partial quasi-additivity constant A.*

If for some $M_0, M_1>0$ the inequalities

$$\|Tf\|_{L_q(U,w_i,\mu)} \le M_i\|f\|_{L_p(U,w_i,\mu)}$$

hold for all $f\in L_p(U,w_i,\mu)$, $i=0,1$, then the inequality

$$\|Tf\|_{L_q(U,w_0^{1-\theta}w_1^{\theta},\mu)} \le cAM_0^{1-\theta}M_1^{\theta}\|f\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)}$$

holds for all $f\in L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)$, where $c>0$ depends only on q, α_0, α_1, λ_0, λ_1, and θ.

Proof. Let $0<\lambda_0,\lambda_1<\infty$ and $0<\alpha_0<\alpha_1<\infty$. We define nets $G_{\lambda_0,\lambda_1}=\{G_{t,\lambda_0,\lambda_1}\}_{t>0}$ and $F_{\alpha_0,\alpha_1}=\{F_{t,\alpha_0,\alpha_1}\}_{t>0}$ by setting

$$G_{t,\lambda_0,\lambda_1}=\bigl\{x\in U\colon\ w_0^{1/(\lambda_1-\lambda_0)}(x)w_1^{1/(\lambda_0-\lambda_1)}(x)<t\bigr\},\qquad t>0,$$

$$F_{t,\alpha_0,\alpha_1}=\bigl\{x\in V\colon\ v_0^{1/(\alpha_1-\alpha_0)}(x)v_1^{1/(\alpha_0-\alpha_1)}(x)<t\bigr\},\qquad t>0,$$

and let

$$d\mu_{\lambda_0,\lambda_1}=\bigl(w_0^{\lambda_1/(\lambda_1-\lambda_0)}(x)w_1^{\lambda_0/(\lambda_0-\lambda_1)}(x)\bigr)^p\,d\mu,\qquad d\nu_{\alpha_0,\alpha_1}=\bigl(v_0^{\alpha_1/(\alpha_1-\alpha_0)}(x)v_1^{\alpha_0/(\alpha_0-\alpha_1)}(x)\bigr)^q\,d\nu.$$

Then, according to Example 8 and Theorem 7 from [12],

$$M_{p,1}^{\lambda_0}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})\hookrightarrow M_{p,p}^{\lambda_0}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})=L_p(U,w_0,\mu),$$

$$M_{p,1}^{\lambda_1}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})\hookrightarrow M_{p,p}^{\lambda_1}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})=L_p(U,w_1,\mu),$$

$$M_{q,\infty}^{\alpha_0}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})\hookleftarrow M_{q,q}^{\alpha_0}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})=L_q(U,w_0,\nu),$$

$$M_{q,\infty}^{\alpha_1}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})\hookleftarrow M_{q,q}^{\alpha_1}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})=L_q(U,w_1,\nu),$$

$$M_{p,p}^{\lambda}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})=L_p(U,w_0^{1-\theta}w_1^{\theta},\mu),\qquad \lambda=(1-\theta)\lambda_0+\theta\lambda_0,$$

$$M_{q,q}^{\alpha}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})=L_q(U,w_0^{1-\theta}w_1^{\theta},\nu),\qquad \alpha=(1-\theta)\alpha_0+\theta\alpha_0.$$

From the hypotheses of the theorem, we obtain

$$\|Tf\|_{M_{q,\infty}^{\alpha_i}(F_{\alpha_0,\alpha_1},\nu_{\alpha_0,\alpha_1})}\le c_1M_i\|f\|_{M_{p,1}^{\lambda_i}(G_{\lambda_0,\lambda_1},\mu_{\lambda_0,\lambda_1})},$$

where $c_1>0$ depends only on the parameters α_0, α_1, λ_0, and λ_1.

To complete the proof, we apply Theorem 2. $\square$

Lemma 6. *Let $0 < p_0, p_1 < \infty$, $p_0 \neq p_1$, and $0 < \theta < 1$, and let p be defined by* (6.1). *Let w_0 and w_1 be positive μ-measurable functions on U and*

$$h_1(x) = \left(\frac{w_0(x)}{w_1(x)}\right)^{p_0 p_1/(p_1-p_0)}, \qquad h_2(x) = \left(\frac{w_0^{p_0}(x)}{w_1^{p_1}(x)}\right)^{1/(p_1-p_0)}, \qquad x \in U, \qquad d\widetilde{\mu} = h_1\, d\mu.$$

Let $f\colon U \to \mathbb{R}$ be a μ-measurable function and $G(f) = \{G_t(f)\}_{t>0}$ with

$$G_t(f) = \left\{x \in U\colon\ |f(x)| > \frac{h_2(x)}{t}\right\}.$$

Then the following equalities hold for an arbitrary μ-measurable set $e \subset U$:

$$\|\chi_e\|_{M_{1,1}^{p_0}(G(f),\widetilde{\mu})} = p_0^{-1}\|f\chi_e\|_{L_{p_0}(U,w_0,\mu)}^{p_0}, \tag{6.4}$$

$$\|1-\chi_e\|_{M_{1,1}^{p_1}(G(f),\widetilde{\mu})} = p_1^{-1}\|f(1-\chi_e)\|_{L_{p_1}(U,w_1,\mu)}^{p_1}, \tag{6.5}$$

$$\|1\|_{M_{1,1}^{p}(G(f),\widetilde{\mu})} = p^{-1}\|f\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)}^{p}. \tag{6.6}$$

Remark 7. The functions h_1 and h_2 and the sets G_t were used in [12] (see Example 9, Lemmas 3 and 4, and Theorem 8 there).

Proof of Lemma 6. Let $U_0(f) = \{x \in U\colon f(x) \neq 0\}$. Then $\bigcup_{t>0} G_t(f) = U_0(f)$ and

$$\|f\chi_e\|_{L_{p_0}(U,w_0,\mu)}^{p_0} = \int\limits_{U_0(f)} \big(|f(x)\chi_e(x)|w_0(x)\big)^{p_0}\, d\mu$$

$$= \int\limits_{U_0(f)} \left(\frac{w_0(x)}{w_1(x)}\right)^{p_0p_1/(p_1-p_0)} \left(\chi_e(x)|f(x)|\left(\frac{w_0^{p_0}(x)}{w_1^{p_1}(x)}\right)^{1/(p_0-p_1)}\right)^{p_0} d\mu$$

$$= \int\limits_{U_0(f)} \chi_e(x)h_1(x)\big(|f(x)|h_2^{-1}(x)\big)^{p_0}\, d\mu = p_0 \int\limits_{U_0(f)} \chi_e(x)h_1(x)\left(\int\limits_{h_2(x)/|f(x)|}^{\infty} t^{-p_0-1}\, dt\right) d\mu$$

$$= p_0 \int\limits_0^{\infty} t^{-p_0}\left(\int\limits_{h_2(x)/|f(x)|<t} \chi_e(x)h_1(x)\, d\mu\right)\frac{dt}{t} = p_0\int\limits_0^{\infty} t^{-p_0}\left(\int\limits_{G_t(f)} \chi_e(x)h_1(x)\, d\mu\right)\frac{dt}{t}$$

$$= \|\chi_e\|_{M_{1,1}^{p_0}(G(f),\widetilde{\mu})},$$

which implies equality (6.4). Next,

$$\|f(1-\chi_e)\|_{L_{p_1}(U,w_1,\mu)}^{p_1} = \int\limits_{U_0(f)} \big(|f(x)(1-\chi_e(x))|w_1(x)\big)^{p_0}\, d\mu$$

$$= \int\limits_{U_0(f)} \left(\frac{w_0(x)}{w_1(x)}\right)^{p_0p_1/(p_1-p_0)} \left((1-\chi_e(x))|f(x)|\left(\frac{w_0^{p_0}(x)}{w_1^{p_1}(x)}\right)^{1/(p_0-p_1)}\right)^{p_1} d\mu$$

$$= \int\limits_{U_0(f)} (1-\chi_e(x))h_1(x)\big(|f(x)|h_2^{-1}(x)\big)^{p_1}\, d\mu = p_1 \int\limits_{U_0(f)} (1-\chi_e(x))h_1(x)\left(\int\limits_{h_2(x)/|f(x)|}^{\infty} t^{-p_1-1}\, dt\right) d\mu$$

$$= p_1 \int_0^\infty t^{-p_1} \left(\int_{h_2(x)/|f(x)|<t} (1-\chi_e(x))h_1(x)\,d\mu \right) \frac{dt}{t} = p_1 \int_0^\infty t^{-p_1} \left(\int_{G_t(f)} (1-\chi_e(x))h_1(x)\,d\mu \right) \frac{dt}{t}$$
$$= \|\chi_e\|_{M^{p_1}_{1,1}(G(f),\widetilde{\mu})}$$

and

$$\|1\|_{M^p_{1,1}(G(f),\widetilde{\mu})} = \int_0^\infty t^{-p} \int_{G_t(f)} h_1(x)\,d\mu\,\frac{dt}{t} = \int_{U_0(f)} h_1(x) \int_0^{|f(x)|/h_2(x)} t^{-p}\,\frac{dt}{t}\,d\mu$$
$$= p^{-1} \int_U \bigl(|f(x)|w_0^{1-\theta}w_1^\theta\bigr)^p\,d\mu. \qquad \square$$

Theorem 8. *Let $0<p_0<p_1<\infty$ and $0<\theta<1$, and let p be defined by* (6.1).

Let the functions w_0, w_1, h_1, and h_2 and the measure $\widetilde{\mu}$ be the same as in Lemma 6, *and let T be a partially quasi-additive operator on $L_{p_0}(U,w_0,\mu)+L_{p_1}(U,w_1,\mu)$ with partial quasi-additivity constant A.*

If for some $M_0,M_1>0$ the inequalities

$$\|h_2^{-1}Tf\|_{L_{p_i,\infty}(U,\widetilde{\mu})} \le M_i\|f\|_{L_{p_i}(U,w_i,\mu)} \tag{6.7}$$

hold for all $f\in L_{p_i}(U,w_i,\mu)$, $i=0,1$, then the inequality

$$\|Tf\|_{L_p(U,w_0^{1-\theta}w_1^\theta,\mu)} \le cAM_0^{1-\theta}M_1^\theta\|f\|_{L_p(U,w_0^{1-\theta}w_1^\theta,\mu)} \tag{6.8}$$

holds for all $f\in L_p(U,w_0^{1-\theta}w_1^\theta,\mu)$, where $c>0$ depends only on p_0, p_1, and θ.

Remark 8. Note that conditions (6.7) are weaker than conditions (6.2).

Indeed, for example, for $i=0$,

$$\|h_2^{-1}Tf\|_{L_{p_i,\infty}(U,\widetilde{\mu})} = \sup_{t>0} t\bigl(\widetilde{\mu}\bigl(\{x\in U\colon |(Tf)(x)|>th_2(x)\}\bigr)\bigr)^{1/p_0}$$
$$= \sup_{t>0} t\left(\int_{\{x\in U\colon |(Tf)(x)|>th_2(x)\}} h_1(x)\,d\mu\right)^{1/p_0}$$
$$\le \sup_{t>0}\left(\int_{\{x\in U\colon |(Tf)(x)|>th_2(x)\}} \left|\frac{(Tf)(x)}{h_2(x)}\right|^{p_0} h_1(x)\,d\mu\right)^{1/p_0} \le \|Tf\|_{L_p(U,w_0,\mu)}.$$

Proof of Theorem 8. Let $s>0$ and $f\in L_p(U,w_0^{1-\theta}w_1^\theta,\mu)$. We will use the terminology of Lemma 6. Let G, F, $F^0(s)$, and $F^1(s)$ be the nets defined by the following sets:

$$G_t=\bigl\{x\in U\colon |f(x)|>th_2(x)\bigr\}, \qquad F_t=\bigl\{x\in U\colon |(Tf)(x)|>th_2(x)\bigr\}, \qquad t>0,$$
$$F_t^0(s)=\bigl\{x\in U\colon |(T(f\chi_{G_s}))(x)|>th_2(x)\bigr\}, \qquad t>0, \quad s>0,$$
$$F_t^1(s)=\bigl\{x\in U\colon |(T(f(1-\chi_{G_s})))(x)|>th_2(x)\bigr\}, \qquad t>0, \quad s>0.$$

These nets coincide with the nets considered in Remark 5, where $h(x)=h_2(x)$. Therefore, the condition of partial quasi-additivity of T for these nets implies inclusion (4.5).

Using conditions (6.7), we obtain

$$\begin{aligned}\|1\|_{M^{p_0}_{1,\infty}(F^0(s),\widetilde{\mu})} &= \|1\|_{M^{p_0}_{1,\infty}(G(T(f\chi_{G_s})),\widetilde{\mu})} = \sup_{t>0} t^{-p_0}\|1\|_{L_1(G_t(T(f\chi_{G_s})),\widetilde{\mu})}\\
&= \sup_{t>0} t^{-p_0}\widetilde{\mu}\big(\big\{x\in U\colon \big|\big(T\big(f\chi_{G_s(f)}\big)\big)(x)\big| > t^{-1}h_2(x)\big\}\big)\\
&= \bigg(\sup_{t>0} t\big(\widetilde{\mu}\big(\big\{x\in U\colon \big|\big(T\big(f\chi_{G_s(f)}\big)\big)(x)\big| > th_2(x)\big\}\big)\big)^{1/p_0}\bigg)^{p_0} = \|h_2^{-1}Tf\|_{L_{p_0,\infty}(U,\widetilde{\mu})}\\
&\le M_0^{p_0}\big\|f\chi_{G_s(f)}\big\|^{p_0}_{L_{p_0}(U,w_0,\mu)} = p_0M_0^{p_0}\big\|\chi_{G_s(f)}\big\|_{M^{p_0}_{1,1}(G,\widetilde{\mu})} = p_0M_0^{p_0}\|1\|_{M^{p_0}_{1,1}(\check{G}^s,\widetilde{\mu})}\end{aligned}$$

and

$$\begin{aligned}\|1\|_{M^{p_1}_{1,\infty}(F^1(s),\widetilde{\mu})} &= \sup_{t>0} t^{-p_1}\widetilde{\mu}\big(\big\{x\in U\colon \big|\big(T\big(1-f\chi_{G_s(f)}\big)\big)(x)\big| > t^{-1}h_2(x)\big\}\big)\\
&= \bigg(\sup_{t>0} t\big(\widetilde{\mu}\big(\big\{x\in U\colon \big|\big(T\big(1-f\chi_{G_s(f)}\big)\big)(x)\big| > th_2(x)\big\}\big)\big)^{1/p_1}\bigg)^{p_1} = \|h_2^{-1}Tf\|_{L_{p_1,\infty}(U,\widetilde{\mu})}\\
&\le M_1^{p_1}\big\|f\big(1-\chi_{G_s(f)}\big)\big\|^{p_1}_{L_{p_1}(U,w_1,\mu)} = M_1^{p_1}\big\|1-\chi_{G_s(f)}\big\|_{M^{p_1}_{1,1}(G(f),\widetilde{\mu})}\\
&= p_1M_1^{p_1}\int_0^\infty t^{-p_1}\big\|1-\chi_{G_s(f)}\big\|_{L_1(G_t(f),\widetilde{\mu})}\frac{dt}{t} \le p_1M_1^{p_1}\int_s^\infty t^{-p_1}\|1\|_{L_1(G_t(f),\widetilde{\mu})}\frac{dt}{t}\\
&= p_1M_1^{p_1}\|1\|_{M^{p_1}_{1,1}(\widehat{G}^s,\widetilde{\mu})}.\end{aligned}$$

Thus, all the hypotheses of Theorem 3 are satisfied. Hence, applying Theorem 3 combined with Lemma 6, we obtain

$$\|Tf\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)} \le cAM_0^{(1-\eta)p_0/p}M_1^{\eta p_1/p}\|f\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)},$$

where η is such that $p=(1-\eta)p_0+\eta p_1$. In view of (6.1), we have the equalities $\eta p_1/p=\theta$ and $(1-\eta)p_0/p=1-\theta$, so we arrive at inequality (6.8). $\square$

Corollary 6. *Let the parameters p_0, p_1, and θ, the functions w_0, w_1, h_1, and h_2, and the measure $\widetilde{\mu}$ be the same as in Theorem 8.*

Let T be the Urysohn integral operator defined as

$$(Tf)(y) = \int_U K(f(x),x,y)\,d\mu.$$

If for some $M_0, M_1>0$ the inequalities

$$\|h_2^{-1}Tf\|_{L_{p_i,\infty}(U,\widetilde{\mu})} \le M_i\|f\|_{L_{p_i}(U,w_i,\mu)} \tag{6.9}$$

hold for all $f\in L_{p_i}(U,w_i,\mu)$, $i=0,1$, then the inequality

$$\|Tf\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)} \le cM_0^{1-\theta}M_1^{\theta}\|f\|_{L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)} \tag{6.10}$$

holds for all $f\in L_p(U,w_0^{1-\theta}w_1^{\theta},\mu)$, where $c>0$ depends only on p_0, p_1, and θ.

Proof. From condition (6.9) we have

$$\int_U K(0,x,y)\,d\mu = 0$$

almost everywhere in V. Hence, the operator T is partially additive, which allows us to apply Theorem 8. $\square$

7. CRITERION OF QUASIWEAK BOUNDEDNESS OF THE URYSOHN INTEGRAL OPERATOR

Let (U,μ) and (V,ν) be spaces with measures μ and ν. We say that an operator T is of (p,q) *quasiweak type* if it is bounded from $L_{p,1}(U,\mu)$ to $L_{q,\infty}(V,\nu)$. If the operator T is bounded from $L_{p,\tau}(U,\mu)$ to $L_{q,\tau}(V,\nu)$, then we say that it is of (p,q) *strong type*.

Let

$$(Tf)(y)=\int\limits_U K(f(x),x,y)\,d\mu,\qquad y\in V, \tag{7.1}$$

be an Urysohn integral operator.

It follows from Corollary 5 that if the operator (7.1) is of (p_0,q_0) and (p_1,q_1) quasiweak type, then it is of (p,q) strong type, where p and q are defined by (5.3) with $\theta\in(0,1)$. Thus, deriving weak-type estimates for these operators is of particular interest. In this section, we give necessary and sufficient conditions for the operator (7.1) to be of (p,q) quasiweak type provided that certain a priori assumptions on the kernel K hold. We need some lemmas.

Lemma 7. *Let $f\colon U\to\mathbb{R}$ be a μ-measurable function taking finitely many values,*

$$f(x)=\sum_{k=1}^n \lambda_k\chi_{w_k}(x),$$

where $w_k\subset U$ are μ-measurable sets of finite measure and $\lambda_k\in\mathbb{R}$ with $|\lambda_1|>|\lambda_2|>\ldots>|\lambda_n|>0$. Then

$$\|f\|_{L_{p,1}(U,\mu)}=p\sum_{k=1}^n(|\lambda_k|-|\lambda_{k+1}|)\Biggl(\mu\biggl(\bigcup_{i=1}^k w_i\biggr)\Biggr)^{1/p},$$

where $\lambda_{n+1}=0$.

Proof. Notice that the nonincreasing rearrangement f^* of f has the form

$$f^*(t)=\sum_{k=1}^n|\lambda_k|\chi_{[t_{k-1},t_k)}(t),$$

where $t_0=0$ and $t_k-t_{k-1}=\mu(w_k)$. Then, applying the Abel transformation, we obtain

$$\|f\|_{L_{p,1}(U,\mu)}=\sum_{k=1}^n|\lambda_k|\int\limits_{t_{k-1}}^{t_k}t^{1/p-1}\,dt=\sum_{k=1}^n(|\lambda_k|-|\lambda_{k+1}|)\int\limits_0^{t_k}t^{1/p-1}\,dt$$
$$=p\sum_{k=1}^n(|\lambda_k|-|\lambda_{k+1}|)\Biggl(\mu\biggl(\bigcup_{i=1}^k w_i\biggr)\Biggr)^{1/p}.\qquad\square$$

Lemma 8. *Suppose that the measure μ satisfies the regularity condition* (2.2) *and the function $K\colon U\to\mathbb{R}$ is integrable on all μ-measurable subsets of U of finite measure. Then, for any set e of positive finite measure, there exists a set $w\subset e$ such that $\mu(w)\ge\mu(e)/3$ and*

$$\int\limits_e|K(x)|\,d\mu\le 4\Biggl|\int\limits_w K(x)\,d\mu\Biggr|. \tag{7.2}$$

Proof. For an arbitrary set e such that $0<\mu(e)<\infty$, define the sets

$$e_+:=\bigl\{x\in e\colon\ K(x)\ge 0\bigr\}\qquad\text{and}\qquad e_-:=\bigl\{x\in e\colon\ K(x)<0\bigr\}.$$

Then

$$\int\limits_e |K(x)|\,d\mu = \int\limits_{e_+} K(x)\,d\mu - \int\limits_{e_-} K(x)\,d\mu \le 2\max\left\{\int\limits_{e_+} K(x)\,d\mu, \left|\int\limits_{e_-} K(x)\,d\mu\right|\right\}.$$

For definiteness, assume that

$$\int\limits_{e_+} K(x)\,d\mu \ge \left|\int\limits_{e_-} K(x)\,d\mu\right|.$$

Two cases are possible: either $\mu(e_+)\ge\mu(e)/2$ or $\mu(e_+)<\mu(e)/2$. In the first case, $w=e_+$.

In the second case, there exists an $\eta\subset e_-$ such that $\mu(\eta)\ge\mu(e_-)/3$ and

$$\left|\int\limits_{\eta} K(x)\,d\mu\right| \le \frac{1}{2}\left|\int\limits_{e_-} K(x)\,d\mu\right|.$$

Indeed, according to the regularity condition (2.2) with $\alpha=\mu(e_-)/3$, there exists a set $\eta'\subset e_-$ such that $\mu(e_-)/3\le\mu(\eta')\le 2\mu(e_-)/3$ and, hence, $\mu(e_-)/3\le\mu(e_-\setminus\eta')\le 2\mu(e_-)/3$. Since

$$\left|\int\limits_{e_-} K(x)\,d\mu\right| = \left|\int\limits_{\eta'} K(x)\,d\mu\right| + \left|\int\limits_{\eta''} K(x)\,d\mu\right|$$

for $\eta''=e_-\setminus\eta'$, at least one of the terms on the right-hand side does not exceed half the left-hand side. If this is the first term, then $\eta=\eta'$, and if the second, then $\eta=\eta''$.

Set $w=\eta\cup e_+$; then $\mu(w)=\mu(\eta)+\mu(e)\le(\mu(e_-)+\mu(e_+))/3=\mu(e)/3$ and

$$\left|\int\limits_{w} K(x)\,d\mu\right| = \left|\int\limits_{e_+} K(x)\,d\mu + \int\limits_{\eta} K(x)\,d\mu\right| \ge \left|\int\limits_{e_+} K(x)\,d\mu\right| - \left|\int\limits_{\eta} K(x)\,d\mu\right| \ge \frac{1}{2}\left|\int\limits_{e_+} K(x)\,d\mu\right|.$$

Thus, we have

$$\int\limits_e |K(x)|\,d\mu \le 2\left|\int\limits_{e_+} K(x)\,d\mu\right| \le 4\left|\int\limits_{w} K(x)\,d\mu\right|. \qquad \square$$

Theorem 9. *Let $1<p,q<\infty$, (U,μ) and (V,ν) be spaces with measures μ and ν satisfying the regularity condition* (2.2), *and $K\colon\mathbb{R}\times U\times V\to\mathbb{R}$.*

Suppose that the Urysohn operator (7.1) *is continuous from $L_{p,\tau}(U,\mu)$ to $L_{q,\infty}(V,\nu)$ and, for some $B>0$ and for an arbitrary ν-measurable set $e\subset V$,*

$$\left|\int\limits_e K(z,x,y)\,d\nu\right| \le B|z|\left|\int\limits_e K(1,x,y)\,d\nu\right| \tag{7.3}$$

for a.e. $x\in U$ and any $z\in\mathbb{R}$.

Then, for any $0<\tau<1$,

$$\|T\|_{L_{p,\tau}(U,\mu)\to L_{q,\infty}(V,\nu)} \asymp \|T\|_{L_{p,1}(U,\mu)\to L_{q,\infty}(V,\nu)}$$

$$\asymp \sup_{\nu(e)>0,\ \mu(w)>0} \frac{1}{(\nu(e))^{1/q'}(\mu(w))^{1/p}}\left|\int\limits_e\int\limits_w K(1,x,y)\,d\mu\,d\nu\right| \equiv F_{p,q}(K).$$

Proof. Since $0 < \tau < 1$ and so $L_{p,\tau}(U,\mu) \hookrightarrow L_{p,1}(U,\mu)$, we have

$$\|T\|_{L_{p,\tau}(U,\mu)\to L_{q,\infty}(V,\nu)} \lesssim \|T\|_{L_{p,1}(U,\mu)\to L_{q,\infty}(V,\nu)}.$$

Suppose that the operator $T\colon L_{p,\tau}(U,\mu) \to L_{q,\infty}(V,\nu)$ is bounded. Let $e \subset V$ and $w \subset U$ be, respectively, ν- and μ-measurable sets such that $0 < \nu(e), \mu(w) < \infty$. Then

$$\frac{1}{(\nu(e))^{1/q'}}\left|\int_e (Tf)(y)\,d\nu\right| \le \frac{1}{(\nu(e))^{1/q'}}\int_0^{\nu(e)} (Tf)^*(t)\,dt \le q'\|Tf\|_{L_{q,\infty}}$$
$$\le q'\|T\|_{L_{p,\tau}(U,\mu)\to L_{q,\infty}(V,\nu)}\|f\|_{L_{p,\tau}}.$$

Let $f(x) = \chi_w(x)$. Taking into account that $\|\chi_w(x)\|_{L_{p,\tau}} \asymp (\mu(w))^{1/p}$, we have

$$\|T\|_{L_{p,\tau}(U,\mu)\to L_{q,\infty}(V,\nu)} \ge \frac{1}{(\nu(e))^{1/q'}(\mu(w))^{1/p}}\left|\int_e\int_U K(\chi_w(x),x,y)\,d\mu\,d\nu\right|.$$

It follows from (7.3) that

$$\int_U\int_e K(\chi_w(x),x,y)\,d\nu\,d\mu = \int_w\int_e K(1,x,y)\,d\nu\,d\mu + \int_{U\setminus w}\int_e K(0,x,y)\,d\nu\,d\mu = \int_w\int_e K(1,x,y)\,d\nu\,d\mu.$$

Thus, since the choice of the sets e and w is arbitrary, we obtain

$$F_{p,q}(K) \lesssim \|T\|_{L_{p,\tau}(U,\mu)\to L_{q,\infty}(V,\nu)} \lesssim \|T\|_{L_{p,1}(U,\mu)\to L_{q,\infty}(V,\nu)}.$$

It remains to show that

$$\|T\|_{L_{p,1}(U,\mu)\to L_{q,\infty}(V,\nu)} \lesssim F_{p,q}(K).$$

Let f be an arbitrary function satisfying the hypotheses of Lemma 7:

$$f(x) = \sum_{k=1}^n \lambda_k \chi_{w_k}(x).$$

In view of condition (7.3), for an arbitrary ν-measurable set e we have

$$\left|\int_e (Tf)(y)\,d\nu\right| \le B\sum_{k=1}^n |\lambda_k| \int_{w_k}\left|\int_e K(\chi_{w_k},x,y)\,d\nu\right|d\mu.$$

Let us apply Lemma 7:

$$\left|\int_e (Tf)(y)\,d\nu\right| \lesssim \sum_{k=1}^n (|\lambda_k| - |\lambda_{k+1}|)\int_{\bigcup_{i=1}^k w_i}\left|\int_e K(1,x,y)\,d\nu\right|d\mu$$
$$\le \widetilde{F}_{p,q}(K)(\nu(e))^{1/q'}\sum_{k=1}^n (|\lambda_k| - |\lambda_{k+1}|)\left(\mu\left(\bigcup_{i=1}^k w_i\right)\right)^{1/p} = p^{-1}\widetilde{F}_{p,q}(K)(\nu(e))^{1/q'}\|f\|_{L_{p,1}}, \quad (7.4)$$

where $\lambda_{n+1} = 0$ and

$$\widetilde{F}_{p,q}(K) = \sup_{\nu(e)>0,\ \mu(w)>0} \frac{1}{(\nu(e))^{1/q'}(\mu(w))^{1/p}}\int_w\left|\int_e K(1,x,y)\,d\nu\right|d\mu.$$

It follows from Lemma 8 that $\widetilde{F}_{p,q}(K) \asymp F_{p,q}(K)$. According to [15, Lemma 1] (see also [20]), we have

$$\|Tf\|_{L_{q,\infty}} \asymp \sup_{\nu(e)>0} \frac{1}{(\nu(e))^{1/q'}} \left| \int_e (Tf)(y)\, d\nu \right|.$$

Thus, from (7.4) we obtain

$$\|Tf\|_{L_{q,\infty}} \le F_{p,q}(K)\|f\|_{L_{p,1}}.$$

The functions satisfying the hypotheses of Lemma 7 form a dense set in the space $L_{p,1}(U,\mu)$. Due to the continuity of the operator $T\colon L_{p,1}(U,\mu) \to L_{q,\infty}(V,\nu)$, inequality (7.4) remains valid for any function $f \in L_{p,1}(U,\mu)$; i.e.,

$$\|T\|_{L_{p,1}\to L_{q,\infty}} \lesssim F_{p,q}(K). \qquad \square$$

Remark 9. In the case when T is a linear integral operator

$$(Tf)(y) = \int_U K(x,y)f(x)\, d\mu,$$

the statement of Theorem 9 follows from the results of [15, 20], which, in particular, include necessary and sufficient conditions for the quasiweak boundedness of linear integral operators in net spaces.

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