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Holomorphic Maps of Levi-Degenerate Tube Hypersurfaces in $\mathbb{C}^3$

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Abstract—Locally biholomorphic maps between 2-nondegenerate smooth real tube hypersurfaces in $\mathbb{C}^3$ with Levi form of rank 1 are described. It is shown that, except for hypersurfaces that are locally equivalent to the boundary of the future tube, such maps must be affine. The proof uses the local holomorphic version of the fundamental theorem of projective geometry which was earlier proved by the author.

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1. INTRODUCTION

The general theory of Levi-nondegenerate smooth real hypersurfaces in a multidimensional complex space was long ago worked out by N. Tanaka [16] and S. S. Chern and J. Moser [1] (who developed further the results due to É. Cartan for real hypersurfaces in $\mathbb{C}^2$). On the other hand, if a hypersurface has zero Levi form, then the local structure of this hypersurface is trivial: it is a direct product of a smooth complex hypersurface and a (one-dimensional) real curve. In the case when the Levi form is everywhere degenerate but distinct from zero, it has long been known that the hypersurface is also foliated by complex submanifolds, which follow the null directions of the Levi form. However, the fact that such a *Levi foliation* exists could not be used so far to develop a geometric theory of hypersurfaces with degenerate Levi form, and there are not so many general results known for these surfaces.

The author is aware of the papers by A. Isaev and D. Zaitsev [4] and S. Pocchiola [12] (the latter paper was elaborated in [10]), where for 2-nondegenerate real hypersurfaces in $\mathbb{C}^3$ having a Levi form of rank 1 (in this case 2-nondegeneracy is equivalent to the impossibility to “straighten” the hypersurface, i.e., to represent it locally as a direct product of a Levi-nondegenerate real hypersurface in $\mathbb{C}^2$ and a complex curve) they developed a theory which is to a certain extent an (incomplete) analogue of the Cartan–Chern and Tanaka theories (note also that a slightly stronger result was announced in [9] and that a more general theory, covering also Levi-degenerate hypersurfaces in spaces of dimension greater than 5, was considered by C. Porter and I. Zelenko in their arXiv preprint [14]). In that theory the role of a “zero-curvature” hypersurface, which is similar to the role of hyperspheres (hyperquadrics) in the theory of Levi-nondegenerate hypersurfaces, is played by the tube hypersurface $T(F)$ in the space $\mathbb{C}^3$ with the coordinates $z = x + iy = (z_1, z_2, z_3)$, $z_j = x_j + iy_j$, that lies over the boundary F of the so-called future cone (the “light cone”):

$$T(F) = \big\{z \in \mathbb{C}^3\colon\ x_1^2 = x_2^2 + x_3^2,\ x_1 > 0\big\}. \tag{1}$$

(We will see below that, similarly to the real hypersphere, the hypersurface $T(F)$ has a fairly large group of holomorphic automorphisms.) We also mention the recent preprint [7] by M. Kolář and I. Kossovskiy, where the authors consruct normal forms of real analytic hypersurfaces in the above class, which are similar to Moser normal forms of Levi-nondegenerate hypersurfaces (see [1]).

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Definition 1. A hypersurface in $\mathbb{C}^3$ that has the form $\{z \in \mathbb{C}^3 : x \in S\}$, where S is a real hypersurface in $\mathbb{R}^3$, is called the *tube hypersurface with base* S; it is denoted by $T(S)$.

Here we also consider Levi-degenerate 2-nondegenerate smooth real hypersurfaces in $\mathbb{C}^3$, and we limit ourselves just to tube hypersurfaces, which form perhaps the simplest sufficiently rich class of such surfaces.

These hypersurfaces were considered earlier by other authors. We note here the rather lengthy paper [2] by G. Fels and W. Kaup, where, in particular, they found all locally homogeneous real hypersurfaces in this class. As had been clear before that, one of these is $T(F)$, but the other were apparently discovered in [2] for the first time. Note that, in fact, it was shown in [2] that any locally homogeneous Levi-degenerate 2-nondegenerate CR hypersurface of dimension 5 is locally CR equivalent to a tube hypersurface.

We also mention here Isaev's result in [3]: on the basis of the theory developed in [4], he showed there that if a tube hypersurface is locally holomorphically equivalent to $T(F)$, then it can be taken to (a part of) $T(F)$ by an affine map.

Note that in [2–4, 10, 12, 14] the authors did not use the (Levi) foliation by complex curves of the hypersurfaces under consideration. By contrast, we rely on the existence of this foliation. Using it we characterize the possible local holomorphic maps between tube hypersurfaces of the form under consideration. In particular, this characterization implies both Isaev's result in [3] and the result of Fels and Kaup in [2] that the identity component of the group of local holomorphic automorphisms of a locally homogeneous tube hypersurface of the form considered here, but distinct from $T(F)$, consists of affine maps.

Our presentation is organized as follows. In Section 2 we recall some facts on maps of the "spherical" tube hypersurface $T(F)$. In Section 3 we discuss the Levi foliation of the tube hypersurfaces under consideration and give a few auxiliary definitions. In Section 4 we establish an important property of local biholomorphic maps between such hypersurfaces, which will allow us to use the local version of the fundamental theorem of projective geometry that was established in [8]. In addition, in Lemma 3 in that section we refine Lemma 1 in [8], which results in a slight improvement of the central result in [8]. Using this, in Section 5 we prove our main result here, which gives a fairly complete description of the possible local holomorphic maps between smooth Levi-degenerate tube hypersurfaces in $\mathbb{C}^3$.

We assume that the hypersurfaces under consideration are sufficiently (for instance, C^3) smooth, so that the null subspace of the Levi form depends smoothly on the point on the surface. (In fact, in our case C^2-regularity is sufficient for the Levi foliation to exist if the Levi form is degenerate; see Remark 1 in Section 3.)

We state all of our results for locally defined holomorphic maps. In fact, these could be locally defined smooth CR maps which extend holomorphically to a one-sided neighbourhood of the hypersurface. Note that if the Levi form at the point is not identically zero, such a one-sided extension does exist.

2. THE FUTURE TUBE AND ITS BOUNDARY $T(F)$

It is known that the future tube in $\mathbb{C}^3$ is the tube domain

$$\left\{z \in \mathbb{C}^3 :\ x_1^2 > x_2^2 + x_3^2,\ x_1 > 0\right\}.$$

Its boundary is the tube hypersurface $T(F)$ introduced in Section 1.

The holomorphic automorphism group of this domain was described by H. Klingen [6] in 1955. To present his result, we make the change of variables $w_1 = iz_1 + iz_2$, $w_2 = iz_1 - iz_2$, $w_3 = iz_3$, and with a point z in the future tube we associate the 2×2 matrix $W = \left(\begin{smallmatrix} w_1 & w_3 \\ w_3 & w_2 \end{smallmatrix}\right)$. Using this notation, we can represent holomorphic automorphisms of the future tube by matrix linear fractional maps of the form

$$W' = (AW + B)(CW + D)^{-1}, \tag{2}$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is the 4×4 real matrix consisting of the 2×2 blocks A, B, C and D that belongs to the 10-dimensional symplectic group $\mathrm{Sp}(4, \mathbb{R})$ (so that ${}^tAC = {}^tCA$, ${}^tBD = {}^tDB$ and ${}^tAD - {}^tCB = E$). In the original z-variables these automorphisms become (bi)rational maps of a rather involved form.

The future tube is homogeneous under this action. As concerns its boundary $T(F)$, it is homogeneous even under the action of the subgroup of affine transformations of the holomorphic automorphism group of the future tube. However, it was shown in [5] that formula (2) with A, B, C and D as above describes all local CR maps of $T(F)$ into itself. Thus the general local holomorphic map of $T(F)$ into itself is birational rather than affine.

3. LEVI FOLIATIONS ON TUBE HYPERSURFACES WITH DEGENERATE LEVI FORM

We show that on a real tube hypersurface in $\mathbb{C}^3$ with degenerate Levi form the Levi foliation is formed by complex lines. This fact was known before, but for lack of a suitable reference we present a proof here.

Lemma 1. *Let S be a real C^3 surface in $\mathbb{R}^3(x)$. Suppose that the rank of the Levi form of the tube hypersurface $T(S)$ in $\mathbb{C}^3(z)$ is everywhere equal to* 1*. Then S is a developable surface and $T(F)$ is foliated by complex lines that are the complexifications of the real lines in $\mathbb{R}^3$ forming S.*

Proof. Without loss of generality we assume that in a neighbourhood of some point the C^3 surface S is the graph of a real function $g(x_1, x_2)$, so that the difference $g(x_1, x_2) - x_3$ is a local defining function of S. However, then this is also a local defining function of the tube hypersurface $T(S)$. Calculating the Levi form of $T(S)$ in terms of this defining function, it is easy to see that the Levi form is degenerate if and only if

$$\det \begin{pmatrix} g_{x_1x_1} & g_{x_1x_2} \\ g_{x_1x_2} & g_{x_2x_2} \end{pmatrix} = 0,$$

that is, if g solves the homogeneous Monge–Ampère equation.

Hence the level curves of any partial derivative of g (for instance, g_{x_1}) are level curves of the entire gradient of g. They are the projections onto the plane x_1Ox_2 of curves on S such that the tangent planes to S at points on these curves are parallel.

It is a classical result that these curves are in fact straight lines. This has several proofs. Here we present a simple proof, which seems to be new and uses ideas from complex geometry.

Consider the direction field V on S such that the above curves are integral curves of it. Let $x_0 \in S$, and let v_0 be the direction at this point; then $z_0 = x_0 + i0$ lies on $T(S)$ and the direction iv_0 is tangent to the real 3-plane $T(x_0)$. Note that the Levi form of $T(S)$ at z_0 vanishes at the complex line spanned by v_0 and iv_0. Performing this construction at each point in the plane $T(x_0)$, we obtain the field of directions parallel to iv_0 on this plane, and these are null directions of the Levi form. Hence the integral trajectories of this field lie on leaves of the Levi foliation, and therefore these leaves are complex lines. Their projections onto the base S are real lines such that the tangent planes to S at points in these lines are parallel. This completes the proof. □

Remark 1. If, following A. V. Pogorelov (see, for instance, [13]), one considers the Gauss map of S, then one can establish a similar result also in the C^2-smooth case.

As S is a connected surface and the direction field of level curves of the gradient of g depends continuously on the point, the complexified directions of the generators of S form a continuous real curve (or a connected union of several continuous curves) γ in $\mathbb{CP}^2$. Note that a point $v \in \gamma$ can be identified with the point on the real part of the plane at infinity of $\mathbb{C}^3$ at which the complex lines parallel to the direction v intersect. The complex lines in the Levi foliation of $T(S)$ that intersect at such a point form at least a two-dimensional family.

Let Γ denote the Zariski closure of γ.

Here are the possible forms of the projective variety Γ (we have compiled this list bearing in mind the statements in [8] that we use in what follows):

(a) Γ is a point. In this case S is a cylinder, a direct product of a real line and a real curve; accordingly, $T(S)$ is the direct product of a Levi-nondegenerate tube hypersurface and a complex line, so that $T(S)$ can be straightened (and hence is not 2-nondegenerate). Since 2-nondegeneracy is a holomorphically invariant property, this case is outside the scope of our considerations here;

(b) Γ is a (complex) line. Then all generators of S are parallel to the same real plane. Without loss of generality we will assume that this is the coordinate plane x_1Ox_2;

(c) Γ is a pair of (complex) lines;

(d) Γ is a nondegenerate quadric;

(e) Γ is a curve of degree greater than 2 or coincides with $\mathbb{C}\mathrm{P}^2$.

4. THE ACTION OF HOLOMORPHIC MAPS BETWEEN TUBE HYPERSURFACES ON COMPLEX LINES

In this section we prove that local holomorphic diffeomorphisms between tube hypersurfaces of the form under consideration take complex lines with directions in γ (or more precisely, the parts of these lines lying in the domain of definition of the diffeomorphism) to complex lines.

Lemma 2. *Let $T(S)$ and $T(S')$ be 2-nondegenerate smooth real tube hypersurfaces in $\mathbb{C}^3$ such that their Levi forms have rank 1. Let f be an invertible holomorphic map defined in a connected neighbourhood of a point $z_0 \in T(S)$ such that it takes $T(S)$ to $T(S')$. Let $v_0 \in \gamma \subset \mathbb{C}\mathrm{P}^2$ be the direction of the complex line in the Levi foliation on $T(S)$ that passes through z_0. Then f maps each complex line in the direction v_0 that intersects its domain of definition into a complex line.*

Proof. The map f takes the Levi foliation of $T(S)$ to the Levi foliation of $T(S')$. In particular, it takes the complex lines in the direction v_0 that pass through points of the form $z_0 + iy$, $y \in \mathbb{R}^3$, on $T(S)$ to some complex lines. Let ν be a nonzero vector in $\mathbb{C}^3$ which has direction v_0. Then the condition that f takes the complex line in the direction v_0 through a point z into a complex line is equivalent to the following one: for each $a \in \mathbb{C}$ the vectors $f(z+\nu) - f(z)$ and $f(z+a\nu) - f(z)$ are linearly dependent. This latter condition can be expressed as the vanishing of the determinants of the three matrices formed by the pairs of corresponding coordinates of these two vectors.

By assumption, for each a these determinants vanish at the points of the form $z_0 + iy$, which is a generic submanifold of the domain of definition of f. Since f is holomorphic, they vanish identically for each a, as required. $\square$

It was shown in [8] that the set of those $v \in \mathbb{C}\mathrm{P}^2$ for which a fixed holomorphic map takes each line in the direction v (intersecting its domain of definition) to a complex line is a projective variety in $\mathbb{C}\mathrm{P}^2$. Thus, we have the following result.

Corollary 1. *Under the assumptions of Lemma 2 the diffeomorphism f maps each complex line with direction in Γ that intersects the domain of definition of f into a complex line.*

This shows that in investigations of holomorphic maps between tube hypersurfaces we can use the results in [8]. However, in Subsection 5.2 we need to know slightly more about the maps considered in [8] than it is immediately clear from the statements there. The following result is a refinement of Lemma 1 in [8].

Lemma 3. *Let f be an invertible holomorphic map in a domain U in $\mathbb{C}^3$. Let $\Gamma \subset \mathbb{CP}^2$ be a set with the following property*: *for $v \in \Gamma$, f maps all complex lines in the direction v that intersect the domain of definition of f into some complex lines. If v_0 is a nonisolated point in Γ, then for each line l_0 in the direction v_0 that does not lie in the set of indeterminacy of f the map $f|_{l_0}\colon l_0 \to f(l_0)$ is rational of degree* 1.

Proof. In fact, one can extract this result from the proof of Lemma 1 in [8]. However, it was not stated there explicitly, and thus we will in fact retrieve some arguments from that proof. It was shown in that lemma that $f|_{l_0}$ is a rational map, so we need to show only that it has multiplicity one. Moreover, it is sufficient to look at the lines in some open family of lines in the direction v_0; for example, we can consider the lines intersecting a sufficiently small ball $U_0 \subset U$ such that the ball $2U_0$ with the same centre and twice the radius also lies in U and $f(2U_0)$ is a convex domain.

So assume that l_0 intersects the ball U_0. Consider the subdomain $W = f(l_0 \cap U_0)$ of $f(l_0)$; we show that no finite point on l_0 which lies outside U_0 goes to W.

Since f is injective in U, consider a point $z \in l_0 \setminus U$. Also let $v \in \Gamma$ be a point sufficiently close to v_0 such that the complex line l through z in the direction v intersects U_0. By assumption $f(l)$ is also a complex line. This line $f(l)$ does not intersect $f(l_0)$ within $f(2U_0)$ because $f(2U_0)$ is convex; however, arguing as in the proof of Lemma 1 in [8], we see that these two lines intersect in $\mathbb{CP}^3$ and their intersection point is the image of z with respect to the restriction $f|_{l_0}$. Hence the image of z lies outside W, as required. $\square$

5. MAPS OF TUBE HYPERSURFACES

5.1. Case (e). It is easy to see that if Γ is a curve of degree 3 or higher, then in fact f takes any complex lines intersecting its domain of definition into lines. We used this fact in [8] without stating it explicitly or proving. We prove it here.

Lemma 4. *Let f be an invertible holomorphic map defined in a domain in $\mathbb{C}^3$. Let $\Gamma \subset \mathbb{CP}^2$ be a projective curve of degree* 3 *or higher that has the following property*: *for $v \in \Gamma$, f takes all complex lines in the direction v that intersect its domain of definition into some complex lines. Then in fact complex lines in any direction that intersect the domain of definition are taken to complex lines by f.*

Proof. Let $p \subset \mathbb{CP}^2$ be a complex line intersecting Γ at least at three distinct points v_1, v_2 and v_3. Let Π be a complex plane in $\mathbb{C}^3$ that is parallel to the directions in p and intersects the domain of definition of f. Consider the three holomorphic families of complex lines in Π which have directions v_1, v_2 and v_3. They contain open subfamilies such that triples of lines in these subfamilies intersect pairwise within the domain of definition of f. Since $v_1, v_2, v_3 \in \Gamma$, these triples of complex lines are taken to triples of pairwise intersecting lines, which determine a plane Π' in $\mathbb{C}^3$. Thus f takes almost all complex planes to planes. As it is a local diffeomorphism, f must take all complex planes intersecting its domain of definition into planes.

However, each complex line is the locus of intersection of two complex planes. Since f maps it into the set of intersection of the images of these planes, f has the required property. $\square$

We see that in case (e) all complex lines intersecting the domain of definition of f are mapped into lines. However, then f is a projective map: for locally defined maps this was apparently first shown in [11, 15]. Alternatively, one can use a result in [8], where we showed that in cases (c)–(e) the map f is birational; hence the classical fundamental theorem of projective geometry is applicable.

Now let $T(S)$ and $T(S')$ be tube hypersurfaces of the form under consideration, and assume that $T(S)$ is covered by case (e). Let f be a local biholomorphic map between these two hypersurfaces. We have just shown that f is a projective transformation. For a point $x \in S$ the fibre $T(x)$ of the hypersurface over this point is a real 3-plane. All complex lines in the Levi foliation of $T(S)$

which pass through points in this fibre are parallel. If f takes them to parallel lines, then it must take all lines in this direction to parallel lines. If the same holds for all points in S, that is, for all complex lines having directions in γ, then this must also hold for all complex lines having directions in Γ. Note that the direction of the image of a complex line through a point is determined by the differential of f at this point, that is, by a linear operator. However, if we have four vectors any three of which are linearly independent, and if two linear operators $\mathbb{C}^3 \to \mathbb{C}^3$ take these vectors to pairwise parallel ones, then these operators are proportional. Thus the differentials of f at all the points are proportional, and hence the projective map f takes any pair of parallel lines to parallel ones. Hence f is an affine map.

Now suppose that the images of lines in the Levi foliation that pass through points in $T(x)$ are not parallel in general. Then the projections of these images are different generators of the surface S'. However, if the projective transformation f does not take some family of parallel lines to parallel ones, then it takes them to lines intersecting at some finite point. Hence all generators of S' intersect at one point, so that S' is part of a conic surface with some apex x_0. In a similar way, looking at the projective map f^{-1}, we see that S is also part of a conic surface.

We see that if at least one of S and S' is not part of a cone, then f must be an affine map. Assume now that both S and S' are parts of cones with apices at the origin, and suppose that f is projective but not affine. Then f takes almost the whole plane at infinity to $\mathbb{C}^3$, so that points in γ are taken to points at which different complex lines in the Levi foliation of $T(S')$ intersect, that is, to points lying on the real 3-plane $T(O)$ over the origin of $\mathbb{R}^3(x)$. However, each point in this plane is the point of intersection of a real one-parameter family of lines in the Levi foliation of $T(S')$, whereas there must be at least a real two-parameter family of such lines through the image of any point in γ, because, as we mentioned in Section 3, there is at least a real two-parameter family of lines in the Levi foliation of $T(S)$ through each point in γ.

Thus f must be an affine map.

5.2. Case (d). In this subsection we will occasionally use the same notation for an affine line in $\mathbb{C}^3$ and the projective line equal to its closure in $\mathbb{C}\mathrm{P}^3$.

Let $T(S)$ and $T(S')$ be tube hypersurfaces of the above form such that $T(S)$ is covered by case (d). Let f be a local biholomorphic map between these hypersurfaces. Then, as we showed in [8], f is rational.

As in the previous case, assume first that f takes parallel complex lines having directions in γ to parallel lines. Then the same holds for complex lines having directions in Γ.

Now let p be a complex line in $\mathbb{C}\mathrm{P}^2$ intersecting the quadric Γ at two distinct points v_1 and v_2. Then each complex plane Π in $\mathbb{C}^3$ which is parallel to the directions in p can be viewed as the union of the complex lines in the direction v_1 issued from points in a line having direction v_2, and therefore the f-image of Π is also a complex plane; moreover, parallel planes are mapped to parallel ones. Since this holds for almost all lines p in $\mathbb{C}\mathrm{P}^2$, f takes an arbitrary plane to a plane, and it takes parallel planes to parallel ones. Complex lines can be defined as the intersections of pairs of planes. Thus f takes all complex lines to lines and takes parallel lines to parallel ones. Hence it is an affine map.

In the case when parallel lines in a direction in Γ are taken to nonparallel ones, we cannot apply the same arguments as in Subsection 5.1 directly: a rational map f may have points of indeterminacy, and the images of lines intersecting at such a point at infinity do not necessarily intersect at one point.

Lemma 5. *Under the assumptions of Subsection* 5.2, *f takes the parallel lines going in some direction $v \in \Gamma \subset \mathbb{C}\mathrm{P}^2$ into projective lines intersecting at one point in* $\mathbb{C}\mathrm{P}^3$.

Proof. The set of indeterminacy of f has complex dimension ≤ 1, so for almost all points $v_1 \in \Gamma$ the line connecting v with v_1 does not lie in this set.

It is sufficient to show that the property of intersection at one point holds for the images (or more precisely, the closures in $\mathbb{C}\mathrm{P}^3$ of the images) of all parallel lines in some open subfamily of lines having direction v. For example, let these be the lines that intersect the original domain of definition of f. None of these lies in the set of indeterminacy of f. Specifically, we show that the closures of any two lines p and p' in this subfamily intersect in $\mathbb{C}\mathrm{P}^3$ and their point of intersection is the image of the points at infinity with respect to the restrictions of f to p and p'.

Let v_1 and v_2 be points in Γ such that the projective lines l_1 and l_2 in $\mathbb{C}\mathrm{P}^2$ that connect v with v_1 and v with v_2, respectively, do not lie in the set of indeterminacy of f.

Let p_1 be a line in the direction v_1 which intersects p in the original domain of definition of f. Then $f(p_1)$ is a line intersecting $f(p)$. Now consider the holomorphic family L_1 of lines in the direction v_1 in $\mathbb{C}^3$ that intersect p. The closures of their images in $\mathbb{C}\mathrm{P}^3$ (with the possible exception of a finite number of lines occurring in the set of indeterminacy of f) intersect the projective line equal to the closure of $f(p)$ at the points that are the $f|_p$-images of the points of intersection of p with the preimage lines.

Hence the projective line $f(l_1)$, which is limiting for the lines in $f(L_1)$, also intersects the closure of $f(p)$ in $\mathbb{C}\mathrm{P}^3$.

Proposition 1. *The point of intersection of the projective lines $f(l_1)$ and $f(p)$ is the image of the point at infinity with respect to the restriction of f to p and the image of v_1 with respect to the restriction of f to l_1.*

Proof. In fact the restriction of f to p is a continuous map of p into $\mathbb{C}\mathrm{P}^3$. Consider a sequence of points $z_j \in p$ converging to the point at infinity $z_\infty \in \overline{p}$ and not lying in the set of indeterminacy of f. Then the f-images of the lines l^j in L_1 that pass through z_j converge to $f(l_1)$, so that the points $f(z_j)$ also tend to $f(l_1)$. However, then the $f|_{\overline{p}}$-image of z_∞ lies on $f(l_1)$. We denote it by w.

Now assume that w_1, the $f|_{l_1}$-image of v_1, does not coincide with w. Let $w_2 = f(u)$, where $u \in l_1$ is not a point of indeterminacy of f, be a point on $f(l_1)$ which is distinct from w and w_1.

We choose points u^j on the lines l^j which tend to u. Their images w^j are points on the lines $f(l^j)$ which tend to w_2. By Lemma 3 the restriction of f to the closure of each l^j is injective, so w^j has a unique preimage on l^j.

On l_1 we look at the boundary b of a small ε-neighbourhood of v_1 in some coordinate chart. The image of v is a closed real curve on $f(l_1)$, which is contractible there in the complement of some neighbourhoods of the points w and w_2. Hence for large j the image of a closed real curve on l^j which is close to b (in the topology of $\mathbb{C}\mathrm{P}^3$) is contractible on the corresponding projective curve in the complement of neighbourhoods of the points $f(z_j)$ and w^j, so that the curve itself is contractible in the complement of z_j and u^j. However, if ε is sufficiently small and j is large, then this curve separates z_j and u^j by construction and is not contractible. Hence the points w and w_1 must coincide, which completes the proof. $\square$

Now consider two cases.

1. Apart from p, the line p_1 also intersects p', so that $f(p_1)$ intersects both $f(p')$ and $f(p)$. Then by Proposition 1 the point of intersection of the lines $f(l_1)$ and $f(p)$ is the $f|_{l_1}$-image of v_1 and the $f|_p$-image of the point at infinity of the line p. Similarly, the point of intersection of $f(l_1)$ and $f(p')$ is the $f|_{l_1}$-image of v_1 and the $f|_{p'}$-image of the point at infinity of p'. Thus the lines $f(p)$ and $f(p')$ intersect in $\mathbb{C}\mathrm{P}^3$ and their intersection point is the $f|_p$-image of the point at infinity of the line p and the $f|_{p'}$-image of the point at infinity of p'.

2. If the line p_1 is disjoint from p', then there exists a line p_2 in the direction v_2 that intersects both p_1 and p'. The point of intersection of p_2 and p_1 lies on an affine line p'' which is parallel to p and p'. By what we proved in case 1, the images of the projective lines p and p'' intersect, and their intersection point is the f-image of the points at infinity of the corresponding affine lines. The images of p'' and p' also intersect and their intersection point is the same; moreover, it turns

out to be also the f-image of the point at infinity of the affine line p'. This completes the proof of Lemma 5. □

Arguing now as in Subsection 5.1, we see that if parallel lines having direction in Γ are taken to nonparallel lines, then S' must be a part of a conic surface. Note that in this case S' cannot be covered by case (e), because then f would be affine by what we proved in Subsection 5.1 and then S would also be covered by case (e).

Cases (b) and (c) are also impossible for S': then the curve γ' defined by S' in the same way as γ is defined by S must be an interval of a real straight line or a pair of such intervals, so that the cone S' contains flat pieces and thus $T(S')$ is not a 2-nondegenerate hypersurface.

We see that S' also falls under case (d), so that it is a part of a cone over a real nondegenerate quadric. However, then S' is affinely equivalent to a part of the cone (1), and the map f^{-1} is also rational. Repeating the above arguments, we see that S must also be affinely equivalent to some part of the cone (1).

5.3. Case (c). In this case we can repeat the arguments in Subsection 5.2, with one exception: in the proof of the analogue of Lemma 5 the points v_1 and v_2 must lie in the component of Γ not containing v. As a result we will see that if f is not affine, then, on the one hand, the surface S' must be a part of a cone, while, on the other hand, it cannot be covered by cases (d) and (e). However, as we mentioned in Subsection 5.2, a 2-nondegenerate tube hypersurface covered by case (b) or (c) cannot have a cone as its base.

5.4. Case (b). The case when a locally defined holomorphic diffeomorphism takes all the complex lines parallel to a certain plane in $\mathbb{C}^3$ (for instance, the coordinate plane z_1Oz_2, as in case (b)) to complex lines was in fact left out in [8]. However, in this case f takes triples of pairwise intersecting lines parallel to z_1Oz_2 that do not pass through one point to triples of pairwise intersecting lines, which lie in one plane, so that planes parallel to z_1Oz_2 are mapped into (not necessarily parallel) planes in $\mathbb{C}^3$ and the maps between such pairs of planes are projective.

However, in our case we can say more. In fact, if the surface S falls under case (b), then S' cannot fall under cases (c)–(e), which we considered above. Hence S' is also covered by case (b) and we can assume that it is also a developable surface with generators parallel to the coordinate plane x_1Ox_2.

The map f takes complex lines in the Levi foliation of $T(S)$ to complex lines parallel to the coordinate plane z_1Oz_2. Hence the same holds for all complex lines having direction in γ, and therefore also for complex lines having direction in Γ. That is, the planes parallel to z_1Oz_2 are mapped into planes with the same property.

Let Π be one of such planes that intersects $T(S)$. Their intersection is a (not necessarily connected) tube hypersurface in $\mathbb{C}^2$ with a base equal to an interval of a real straight line (or a union of several disjoint intervals of, generally speaking, different lines), so that it is a subdomain of a real hyperplane P, which is the direct product of a straight strip on a complex line with direction in γ and a real line (or a union of several such subdomains of, generally speaking, different hyperplanes). The same can be said about the intersection of the image of Π with $T(S')$, so we see that the projective transformation f takes the real hyperplane P to another real hyperplane. It is easy to see that if f is not affine, then P must intersect the polar set of f along a complex line, which must be a line in the Levi foliation. The same holds for other complex planes parallel to Π. However, this means that the polar set of f is foliated by leaves of the Levi foliation. In other words, this set lies locally in $T(S)$, so the Levi form of $T(S)$ vanishes identically at the corresponding points, in contradiction to our hypotheses. Thus f must be an affine map.

5.5. The main result. Combining the conclusions from our considerations in the preceding subsections and known results on local CR automorphisms of the boundary $T(F)$ of the future tube (see [5, 6]) yields a fairly exhaustive description of the possible holomorphic maps between hypersurfaces of the form under study.

Theorem 1. *Let $T(S)$ and $T(S')$ be 2-nondegenerate C^3-smooth real tube hypersurfaces in $\mathbb{C}^3(x)$ with Levi form of rank 1, and let f be a local biholomorphic map between them. Then either f is an affine map which is a composition of an affine map with real coefficients taking the bases S and S' of these hypersurfaces into each other and a parallel translation along the imaginary axis, or S and S' are affinely equivalent to the surface* (1), *the boundary of the future cone in $\mathbb{R}^3$, and f can be reduced to the form* (2) *in appropriate coordinates.*

Thus the investigation of holomorphic maps between tube hypersurfaces with degenerate Levi form in $\mathbb{C}^3$ reduces to the investigation of affine maps between developable surfaces in $\mathbb{R}^3$.

As mentioned in Remark 1 in Section 3, in fact a similar result also holds for C^2-smooth tube hypersurfaces.

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