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Multiple Capture of a Given Number of Evaders in a Problem with Fractional Derivatives and a Simple Matrix

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Abstract—A problem of pursuing a group of evaders by a group of pursuers with equal capabilities of all the participants is considered in a finite-dimensional Euclidean space. The system is described by the equation

$$D^{(\alpha)} z_{ij} = a z_{ij} + u_i - v_j, \quad u_i, v_j \in V,$$

where $D^{(\alpha)} f$ is the Caputo fractional derivative of order α of the function f, the set of admissible controls V is strictly convex and compact, and a is a real number. The aim of the group of pursuers is to capture at least q evaders; each evader must be captured by at least r different pursuers, and the capture moments may be different. The terminal set is the origin. Assuming that the evaders use program strategies and each pursuer captures at most one evader, we obtain sufficient conditions for the solvability of the pursuit problem in terms of the initial positions. Using the method of resolving functions as a basic research tool, we derive sufficient conditions for the solvability of the approach problem with one evader at some guaranteed instant. Hall's theorem on a system of distinct representatives is used in the proof of the main theorem.

Keywords: differential game, group pursuit, multiple capture, pursuer, evader, fractional derivative.

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INTRODUCTION

An important direction in the modern theory of differential games is associated with the development of solution methods for game problems of pursuit and evasion with several objects [1–4]. In this area, not only are the classical solution methods deepened, but also new problems are sought to which the existing methods are applicable. In particular, in [5,6], problems of pursuing two objects described by equations with fractional derivatives were considered and sufficient conditions of a capture were obtained. Recently, Gomoyunov [7] proved the existence of the value of a nonlinear differential game with fractional derivatives.

We consider a linear problem of pursuing a group of evaders by a group of pursuers provided that all the participants have equal capabilities. The problem of simple pursuit of a single evader by a group of pursuers was considered by Pshenichnyi [8], who obtained necessary and sufficient

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conditions of a capture. A multiple capture of an evader in a simple group pursuit problem was studied by Grigorenko in [9]. The problem of capturing a given number of evaders in a simple pursuit problem under the conditions that the set of admissible controls is a unit ball centered at zero, the terminal sets are the origins, the evaders use program strategies, and each pursuer captures at most one evader, is presented in [10], where necessary and sufficient conditions for the solvability of the pursuit problem were obtained. The general case of the problem of capturing a given number of evaders in the case of simple pursuit was considered in [11]. The problem on a multiple capture of an evader in Pontryagin's example was presented in [12–14]. A multiple capture in linear differential games was studied in [15]. The problem on a multiple capture of an evader in a differential game with fractional derivatives was investigated in [16] (where a more detailed list of references on these problems is also presented). Sufficient conditions for the capture of a given number of evaders in Pontryagin's stationary example and linear recurrent differential games were obtained in [17, 18].

In the present paper, the problems of multiple capture and capture of a given number of evaders, which were earlier considered separately, are combined into one problem. The aim of the group of pursuers is to capture at least q evaders, and each evader must be captured by at least r pursuers. Under the assumptions that the evaders use program strategies and each of the pursuers captures at most one evader, sufficient conditions for the solvability of the pursuit problem are obtained. Note that, in the case of simple motion, the problem in this statement was studied in [19].

1. PROBLEM STATEMENT

Definition 1 [20]. Suppose that p is a positive integer, $\alpha \in (p-1, p)$, and a function $f : [0,\infty) \to \mathbb{R}^k$ is such that $f^{(p)}$ is absolutely continuous on $[0,\infty)$. The Caputo derivative of f of order α is the function

$$\left(D^{(\alpha)} f\right)(t) = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{f^{(p)}(s)}{(t-s)^{\alpha+1-p}}\, ds, \quad \text{where} \quad \Gamma(\beta) = \int_0^\infty e^{-s} s^{\beta-1}\, ds.$$

In the space $\mathbb{R}^k$ $(k \geq 2)$, we consider an $(n+m)$-person differential game $G(n,m)$ with n pursuers $P_1, \ldots, P_n$ and m evaders $E_1, \ldots, E_m$. Each of the pursuers P_i moves according to a law

$$D^{(\alpha)} x_i = a x_i + u_i, \quad x_i(0) = x_i^0, \quad \ldots, \quad x_i^{(p-1)}(0) = x_i^{p-1}, \quad u_i \in V. \tag{1.1}$$

The motion law of each evader E_j has the form

$$D^{(\alpha)} y_j = a y_j + v_j, \quad y_j(0) = y_j^0, \quad \ldots, \quad y_j^{(p-1)}(0) = y_j^{p-1}, \quad v \in V. \tag{1.2}$$

Here $x_i, y_j, u_i, v_j \in \mathbb{R}^k$, V is a strictly convex compact subset of $\mathbb{R}^k$, $a \in \mathbb{R}^1$, $i \in I = \{1, \ldots, n\}$, and $j \in \{1, \ldots, m\}$. In addition, $x_i^0 \neq y_j^0$ for all i and j.

Instead of systems (1.1) and (1.2), we consider the system

$$D^{(\alpha)} z_{ij} = a z_{ij} + u_i - v_j, \quad u_i, v_j \in V, \tag{1.3}$$

with the initial conditions

$$z_{ij}(0) = z_{ij}^0 = x_i^0 - y_j^0, \quad \ldots, \quad z_{ij}^{(p-1)}(0) = z_{ij}^{p-1} = x_i^{p-1} - y_j^{p-1}. \tag{1.4}$$

Here the solution of system (1.3), (1.4) is understood in the standard way (see, e.g., [21, Sect. 3]).

The aim of the group of pursuers is to capture at least q evaders so that each evader is captured by at least r pursuers ($r \geq 1$ and $1 \leq q \leq m$) provided that first the evaders choose their controls on the whole semiaxis $[0, \infty)$ and then the pursuers, using the information on the choice of the evaders, choose their controls; in addition, each pursuer can capture at most one evader. We assume that $n \geq rq$ and $m \geq q$.

Denote by $z^0 = \{z_{ij}^0, \ldots, z_{ij}^{p-1}, i \in I, j \in J\}$ the vector of initial positions, and let $z_{ij}^{p-1} \neq 0$ for all i and j.

A measurable function $v : [0, \infty) \to \mathbb{R}^k$ is called *admissible* if $v(t) \in V$ for all $t \geq 0$.

Definition 2. An *r-multiple capture* (*capture for* $r = 1$) of an evader E_β occurs in the game $G(n, m)$ if there exists an instant $T > 0$ such that, for any admissible control $v_\beta(t)$, $t \in [0, \infty)$, of the evader E_β, there exist admissible controls $u_i(t)$ ($i \in I$) of the pursuers P_i ($i \in I$), instants $\tau_1, \ldots, \tau_r \in [0, T]$, and pairwise different positive integers $i_1, \ldots, i_r \in I$ such that $z_{i_s\beta}(\tau_s) = 0$ for all $s = 1, \ldots, r$, where $z_{i_s\beta}(t)$ are the solutions of system (1.3), (1.4).

Definition 3. An *r-multiple capture* (*capture for* $r = 1$) *of at least q evaders* occurs in the game $G(n, m)$ if there exists $T > 0$ such that, for any set of admissible controls $v_j(t)$, $t \in [0, \infty)$, of the evaders E_j, $j \in J$, there exist admissible controls $u_i(t) = u_i(t, z_{ij}^0, v_j(s), s \in [0, \infty), j \in J)$ of the pursuers P_i, $i \in I$, that possess the following property: there exist sets

$$M \subset J, \quad |M| = q, \quad \{N_l, l \in M\}, \quad N_l \subset I, \quad |N_l| = r \text{ for all } l \in M, \quad N_l \cap N_s = \varnothing \text{ for all } l \neq s,$$

such that the group of pursuers $\{P_l, l \in N_\beta\}$ performs the r-multiple capture of the evader E_β not later than the instant T; if a pursuer P_l captures an evader E_β, then all the other evaders are considered not caught by P_l.

Introduce the following notation. For a finite set K of positive integers, define

$$\Omega_K(s) = \{(i_1, \ldots, i_s) \mid i_1, \ldots, i_s \in K \text{ are pairwise different}\}, \quad D_\varepsilon(a) = \{z \in \mathbb{R}^k \mid \|z - a\| < \varepsilon\},$$

$$\lambda(h, v) = \sup\{\lambda \geq 0 \mid -\lambda h \in V - v\}, \quad \xi_{ij}(t) = \sum_{l=0}^{p-1} \frac{z_{ij}^l}{\Gamma(l+1)} t^l, \quad \xi_{ij}^1(t) = t^{1-p} \xi_{ij}(t).$$

2. MULTIPLE CAPTURE OF ONE EVADER FOR $a = 0$

In this section, we take $m = 1$. Therefore, the second index can be omitted.

Lemma 1. *Suppose that V is a strictly convex compact set, $b_1, \ldots, b_n \in \mathbb{R}^k$, $b_i \neq 0$ for all $i \in I$, and*

$$\min_{v \in V} \max_{\Lambda \in \Omega_I(r)} \min_{i \in \Lambda} \lambda(b_i, v) > 0.$$

Then there exists $\varepsilon > 0$ such that, for any $z_1, \ldots, z_n \in \mathbb{R}^k$ such that $z_i \in D_\varepsilon(b_i)$ for $i \in I$, the following inequality holds: $\min\limits_{v \in V} \max\limits_{\Lambda \in \Omega_I(r)} \min\limits_{i \in \Lambda} \lambda(z_i, v) > 0$.

Proof. It follows from [2, Lemma 1.3.13] that the function $\lambda(b, v)$ is continuous on the set $B \times V$, where B is an arbitrary compact subset of $\mathbb{R}^k$ not containing zero. Therefore, the functions

$$g_\Lambda(z_1, \ldots, z_n, v) = \min_{\alpha \in \Lambda} \lambda(z_\alpha, v), \; g(z_1, \ldots, z_n, v) = \max_{\Lambda \in \Omega_I(r)} g_\Lambda(z_1, \ldots, z_n, v),$$

$$f(z_1, \ldots, z_n) = \min_{v \in V} g(z_1, \ldots, z_n, v)$$

are continuous. The continuity of f implies the assertion of the lemma. The lemma is proved.

Corollary. *Let $\{z_l^{p-1}\colon l\in I\}$ be such that*

$$\delta_0=\min_{v\in V}\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\lambda\bigl(z_l^{p-1}/\Gamma(p),v\bigr)>0. \tag{2.1}$$

Then there exists $T_0>0$ such that the following inequality holds for all $t>T_0$:

$$\min_{v\in V}\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\lambda\bigl(\xi_l^1(t),v\bigr)\ge 0.5\,\delta_0. \tag{2.2}$$

Proof. This inequality follows from Lemma 1 and the condition $\lim\limits_{t\to+\infty}\xi_i^1(t)=z_i^{p-1}/\Gamma(p)$.

Lemma 2. *Let $a=0$ and $\delta_0>0$, where δ_0 is introduced in (2.1). Then there exists $T_1>0$ such that, for any measurable function $v(\cdot)$ with values $v(t)\in V$, there exists a set $\Lambda\in\Omega_I(r)$ such that the following inequality holds for all $l\in\Lambda$:*

$$T_1^{1-p}\int\limits_0^{T_1}\frac{(T_1-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_l^1(T_1),v(s))ds\ge 1.$$

Proof. It follows from the corollary that there exists $T_0>0$ such that inequality (2.2) holds for all $t>T_0$. Let $T>T_0$. We consider the functions (for $t\in[0,T]$)

$$h_l(t,T,v(\cdot))=t^{1-p}\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_l^1(T),v(s))\,ds.$$

Then

$$\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}h_l(t,T,v(\cdot))\ge\max_{\Lambda\in\Omega_I(r)}t^{1-p}\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\min_{l\in\Lambda}\lambda(\xi_l^1(T),v(s))ds. \tag{2.3}$$

For any nonnegative numbers $\{a_\Lambda\}_{\Lambda\in\Omega_I(r)}$, we have

$$\max_{\Lambda\in\Omega_I(r)}a_\Lambda\ge\frac{1}{C_n^r}\sum_{\Lambda\in\Omega_I(r)}a_\Lambda.$$

Hence, (2.3) implies the inequality $\bigl($for $t\in[0,T]\bigr)$

$$\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}h_l(t,T,v(\cdot))\ge\frac{t^{1-p}}{C_n^r}\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sum_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\lambda(\xi_l^1(T),v(s))\,ds$$

$$\ge\frac{t^{1-p}}{C_n^r}\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\lambda(\xi_l^1(T),v(s))\,ds\ge\frac{t^{1-p}\delta_0}{2C_n^r\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}\,ds=\frac{t^{\alpha-p+1}\delta_0}{2\alpha C_n^r\Gamma(\alpha)}.$$

Consequently,

$$\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}h_l(T,T,v(\cdot))\ge\frac{T^{\alpha-p+1}\delta_0}{2\alpha C_n^r\Gamma(\alpha)}.$$

Since $\alpha-p+1>0$, the lemma is proved.

Define the number

$$\hat{T} = \inf\left\{t \mid \inf_{v(\cdot)} \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} t^{1-p}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_l^1(t), v(s))\,ds \ge 1\right\}.$$

By Lemma 2, we have $\hat{T} < \infty$.

Theorem 1. *Let $a = 0$ and $\delta_0 > 0$, where δ_0 is defined in* (2.1). *Then an r-multiple capture occurs in the game $G(n, 1)$.*

Proof. Let $v(s)$, $s \in [0, \hat{T}]$, be an arbitrary control of the evader. Consider the function

$$H(t) = 1 - \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \hat{T}^{1-p}\int_0^t \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_l^1(\hat{T}), v(s))\,ds.$$

Denote by $T_0 > 0$ the first root of this function. Note that T_0 exists by Lemma 2 and the definition of $\hat{T}$. In addition, there exists a set $\Lambda_0 \in \Omega_I(r)$ such that, for all $j \in \Lambda_0$,

$$1 - \hat{T}^{1-p}\int_0^{T_0} \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_j^1(\hat{T}), v(s))\,ds \le 0.$$

Therefore, there exist instants $\tau_j \le T_0$, $j \in \Lambda_0$, for which

$$1 - \hat{T}^{1-p}\int_0^{\tau_j} \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_j^1(\hat{T}), v(s))\,ds = 0. \tag{2.4}$$

For $j \notin \Lambda_0$, denote by τ_j the instants at which condition (2.4) holds, if such instants exist. Let the controls of the pursuers P_i, $i \in I$, be given by the formula

$$u_i(s) = \begin{cases} v(s) - \lambda(\xi_i^1(\hat{T}), v(s))\xi_i^1(\hat{T}), & s \in [0, \min\{\tau_i, \hat{T}\}], \\ v(s), & s \in [\min\{\tau_i, \hat{T}\}, \hat{T}]. \end{cases}$$

Then the solution of the Cauchy problem (1.3), (1.4) can be presented in the form [21, formula (19)]

$$z_i(t) = \xi_i(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(u_i(s) - v(s))\,ds.$$

Hence,

$$\hat{T}^{1-p} z_i(\hat{T}) = \xi_i^1(\hat{T}) + \hat{T}^{1-p}\int_0^{\hat{T}} \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}(u_i(s) - v(s))\,ds$$

$$= \xi_i^1(\hat{T}) - \hat{T}^{1-p}\int_0^{\hat{T}} \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_i^1(\hat{T}), v(s))\xi_i^1(\hat{T})\,ds$$

$$= \xi_i^1(\hat{T})\Big(1 - \hat{T}^{1-p}\int_0^{\tau_i} \frac{(\hat{T}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\xi_i^1(\hat{T}), v(s))\,ds\Big) = 0$$

for all $i \in \Lambda_0$. Consequently, $z_i(\hat{T}) = 0$ for all $i \in \Lambda_0$. The theorem is proved.

In what follows, let $\operatorname{Int} A$ and $\operatorname{co} A$ be the interior and the convex hull of the set A, respectively.

Lemma 3 [3, Assertion 1.3]**.** *Let V be a strictly convex compact set with smooth boundary, and let*

$$0 \in \bigcap_{\Lambda\in\Omega_I(n-r+1)} \operatorname{Int}\operatorname{co}\{z_j^{p-1}, j \in \Lambda\}. \tag{2.5}$$

Then $\delta_0 > 0$ (see (2.1)).

Theorem 2. *Suppose that $a = 0$, V is a strictly convex compact set with smooth boundary, and condition (2.5) holds. Then an r-multiple capture occurs in the game $G(n,1)$.*

Proof. The validity of this theorem follows from Lemma 3 and Theorem 1.

3. MULTIPLE CAPTURE OF THE EVADERS FOR $a < 0$

Define the generalized Mittag-Leffler function [22, p. 117]

$$E_\rho(z,\mu) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l\rho^{-1}+\mu)}$$

and the number

$$\delta_1 = \min_{v\in V}\max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\lambda(z_l^0, v). \tag{3.1}$$

Lemma 4. *Suppose that $a < 0$, $\alpha \in (0,1)$, and $\delta_1 > 0$, where δ_1 is introduced in (3.1). Then there exists $T_1 > 0$ such that, for any function $v(\cdot)$ with values $v(t) \in V$, there exists a set $\Lambda \in \Omega_I(r)$ such that the following inequality holds for all $l \in \Lambda$:*

$$E_{1/\alpha}(aT_1^\alpha, 1) - \int_0^{T_1}(T_1-s)^{\alpha-1}E_{1/\alpha}(a(T_1-s)^{\alpha-1},\alpha)\lambda(z_l^0, v(s))\,ds \le 0.$$

Proof. Consider the functions

$$h_l(t, v(\cdot)) = E_{1/\alpha}(at^\alpha, 1) - \int_0^{t}(t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha)\lambda(z_l^0, v(s))\,ds.$$

Then

$$H(t, v(\cdot)) = \min_{\Lambda\in\Omega_I(r)}\max_{l\in\Lambda} h_l(t, v(\cdot))$$

$$= E_{1/\alpha}(at^\alpha, 1) - \max_{\Lambda\in\Omega_I(r)}\min_{l\in\Lambda}\int_0^{t}(t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha)\lambda(z_l^0, v(s))\,ds.$$

Since $\alpha \in (0,1)$, it follows from [23, Theorem 4.1.1] that $E_{1/\alpha}(z,\mu)$ has no negative roots for $\mu \in [\alpha,+\infty)$. In addition, $E_{1/\alpha}(z,\mu) \geq 0$ for all $z \geq 0$ and $\mu \geq 0$. Hence, $E_{1/\alpha}(z,\mu) \geq 0$ for all $z \in \mathbb{R}^1$ and $\mu \in [\alpha,+\infty)$. Therefore,

$$\max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha)\lambda(z_l^0, v(s))\,ds$$

$$\geq \max_{\Lambda\in\Omega_I(r)} \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha) \min_{l\in\Lambda} \lambda(z_l^0, v(s))\,ds$$

$$\geq \frac{1}{C_n^r} \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha) \sum_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \lambda(z_l^0, v(s))\,ds$$

$$\geq \frac{1}{C_n^r} \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha) \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \lambda(z_l^0, v(s))\,ds$$

$$\geq \frac{\delta_1}{C_n^r} \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha)\,ds.$$

According to [22, Ch. 3, formula (1.15)],

$$\int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^{\alpha-1},\alpha)\,ds = t^\alpha E_{1/\alpha}(at^\alpha, \alpha+1).$$

Hence,

$$\min_{\Lambda\in\Omega_I(r)} \max_{l\in\Lambda} h_l(t, v(\cdot)) \leq E_{1/\alpha}(at^\alpha, 1) - \frac{\delta_1}{C_n^r} t^\alpha E_{1/\alpha}(at^\alpha, \alpha+1) = H_0(t).$$

Since $a < 0$, the following asymptotic bounds [23, formula (1.2.4)] hold as $t \to +\infty$:

$$E_{1/\alpha}(at^\alpha, 1) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + O\Big(\frac{1}{t^{2\alpha}}\Big), \quad E_{1/\alpha}(at^\alpha, \alpha+1) = -\frac{1}{at^\alpha} + O\Big(\frac{1}{t^{2\alpha}}\Big),$$

where $O(g)$ as $t \to +\infty$ is understood as a specific function G such that the function G/g is bounded on $(A,+\infty)$ for some $A > 0$. Consequently,

$$H_0(t) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + \frac{\delta_1}{aC_n^r} + O\Big(\frac{1}{t^\alpha}\Big).$$

Since $\lim\limits_{t\to+\infty} H_0(t) = \delta_1/aC_n^r < 0$, there exists an instant $T_1 > 0$ such that $H_0(T_1) < 0$. Therefore, $H(T_1, v(\cdot)) < 0$. We have $h_l(0, v(\cdot)) = 1$ for all l and $\min\limits_{\Lambda\in\Omega_I(r)} \max\limits_{l\in\Lambda} h_l(T_1, v(\cdot)) < 0$ for any function $v(\cdot)$. The lemma is proved.

Define the number

$$\hat{T} = \inf\big\{t > 0 \mid \inf_{v(\cdot)} \min_{\Lambda\in\Omega_I(r)} \max_{l\in\Lambda} h_l(t, v(\cdot)) \leq 0\big\}.$$

By Lemma 4, we have $\hat{T} < +\infty$.

Theorem 3. *Suppose that $a < 0$, $\alpha \in (0, 1)$, and $\delta_1 > 0$, where δ_1 is defined in* (3.1)*. Then an r-multiple capture occurs in the game $G(n, 1)$.*

Proof. Let $v(\cdot)$ be an arbitrary admissible control of the evader E. Consider the function

$$H(t) = E_{1/\alpha}(a\hat{T}^\alpha, 1) - \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \int_0^t (\hat{T} - s)^{\alpha-1} E_{1/\alpha}(a(\hat{T} - s)^{\alpha-1}, \alpha)\lambda(z_l^0, v(s))\, ds$$

and denote by T_0 its first root. Note that T_0 exists due to Lemma 4 and the definition of $\hat{T}$. In addition, there exists a set $\Lambda_0 \in \Omega_I(r)$ such that, for all $l \in \Lambda_0$,

$$E_{1/\alpha}(a\hat{T}^\alpha, 1) - \int_0^{T_0} (\hat{T} - s)^{\alpha-1} E_{1/\alpha}(a(\hat{T} - s)^{\alpha-1}, \alpha)\lambda(z_l^0, v(s))\, ds \le 0.$$

Therefore, there exist instants $\tau_l \le T_0$, $l \in \Lambda_0$, for which

$$E_{1/\alpha}(a\hat{T}^\alpha, 1) - \int_0^{\tau_l} (\hat{T} - s)^{\alpha-1} E_{1/\alpha}(a(\hat{T} - s)^{\alpha-1}, \alpha)\lambda(z_l^0, v(s))\, ds = 0. \tag{3.2}$$

For $l \notin \Lambda_0$, denote by τ_j the instants for which condition (3.2) holds, if such instants exist. Let the controls of the pursuers P_i, $i \in I$, be given by the formula

$$u_i(s) = \begin{cases} v(s) - \lambda(z_i^0, v(s))z_i^0, & s \in [0, \min\{\tau_i, \hat{T}\}], \\ v(s), & s \in (\min\{\tau_i, \hat{T}\}, \hat{T}]. \end{cases}$$

The solution of the Cauchy problem (1.3), (1.4) can be presented in the form [21, formula (19)]

$$z_i(t) = E_{1/\alpha}(at^\alpha, 1)z_i^0 - \int_0^t (t - s)^{\alpha-1} E_{1/\alpha}(a(t - s)^{\alpha-1}, \alpha)(u_i(s) - v(s))\, ds.$$

Hence, using (3.2), we obtain

$$\begin{aligned} z_i(\hat{T}) &= E_{1/\alpha}(a\hat{T}^\alpha, 1)z_i^0 - \int_0^{\hat{T}} (\hat{T} - s)^{\alpha-1} E_{1/\alpha}(a(\hat{T} - s)^{\alpha-1}, \alpha)(u_i(s) - v(s))\, ds \\ &= z_i^0\Big(E_{1/\alpha}(a\hat{T}^\alpha, 1) - \int_0^{\tau_i} (\hat{T} - s)^{\alpha-1} E_{1/\alpha}(a(\hat{T} - s)^{\alpha-1}, \alpha)\lambda(z_i^0, v(s))\Big)\, ds = 0 \end{aligned}$$

for all $i \in \Lambda_0$. The theorem is proved.

Theorem 4 [16, Theorem 1]**.** *Suppose that $a < 0$, $\alpha \in (1, 2)$, and*

$$\min\Big\{\min_{v\in V} \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \lambda(z_l^1, v),\ \min_{v\in V} \max_{\Lambda\in\Omega_I(r)} \min_{l\in\Lambda} \lambda(-z_l^1, v)\Big\} > 0.$$

Then an r-multiple capture occurs in the game $G(n, 1)$.

4. MULTIPLE CAPTURE OF A GIVEN NUMBER OF EVADERS

Conjecture 1. *For each $s \in \{0, \ldots, q-1\}$, the following condition is satisfied: for any set $N \subset I$, $|N| = n - sr$, there exists a set $M \subset J$ such that $|M| = q - s$ and*

$$\delta_N(\beta) = \min_{v \in V} \max_{\Lambda \in \Omega_N(r)} \min_{l \in \Lambda} \lambda\Big(\frac{z_{l\beta}^{p-1}}{\Gamma(p)}, v\Big) > 0$$

for all $\beta \in M$.

Theorem 5. *Suppose that $a = 0$ and Conjecture 1 holds. Then an r-multiple capture of at least q evaders occurs in the game $G(n, m)$.*

Proof. Let the assumptions of the theorem hold. Let us prove that any $n - sr$ pursuers perform an r-multiple capture of at least $q - s$ evaders, where $s \in \{0, \ldots, q-1\}$. For $s = 0$, we obtain the assertion of the theorem. The further proof is by induction. Let $s = q - 1$, $N \subset I$, and $|N| = n - (q-1)r$. By the assumptions of the theorem, there exists $\beta \in J$ such that $\delta_N(\beta) > 0$. It follows from Theorem 1 that the pursuers P_l for $l \in N$ perform an r-multiple capture of the evader E_β.

Assume that the statement is proved for all $s \geq p + 1$. Let us prove the statement for $s = p$. Let $N \subset I$ and $|N| = n - pr$. Then there exists a set $M \subset J$ such that $|M| = q - p$ and $\delta_N(\beta) > 0$ for all $\beta \in M$.

Let $v_j(t)$, $t \in [0, \infty)$, be the controls of the evaders E_j, $j \in J$. For each $\beta \in M$, define the set

$$J_\beta = \big\{l \in N \;\big|\; \text{pursuer } P_l \text{ captures evader } E_\beta\big\}.$$

By Theorem 1 and the assumptions of this theorem, the inequality $|J_\beta| \geq r$ holds for all $\beta \in M$. We can assume that $M = \{1, \ldots, q-p\}$. Two cases are possible.

1. $\Big|\bigcup\limits_{\beta=1}^{l} J_\beta\Big| \geq lr$ for all $l = 1, \ldots, q-p$. Then, by the Hall theorem [24, Theorem 5.1.1], there exists a system of distinct representatives for the sets $\{J_\beta, \beta \in M\}$; i.e., there exist sets J'_β, $\beta \in M$, for which

$$J'_\beta \subset J_\beta, \quad |J'_\beta| = r \;\text{ for all }\; \beta \in M, \quad J'_{\beta_1} \cap J'_{\beta_2} = \varnothing \;\text{ for all }\; \beta_1 \neq \beta_2.$$

Consequently, each group of pursuers P_l ($l \in J'_\beta$) performs an r-multiple capture of the evader E_β for all $\beta \in M$. Therefore, the group of pursuers P_l ($l \in N$) performs an r-multiple capture of at least $q - p$ evaders.

2. There exists $l \in \{1, \ldots, q-p\}$ for which $\Big|\bigcup\limits_{\beta=1}^{l}\Big| < lr$. Let l_0 be the smallest positive integer satisfying this inequality. Note that $l_0 > 1$ and $\Big|\bigcup\limits_{\beta=1}^{n_1}\Big| \geq n_1 r$ for all $n_1 \in \{1, \ldots, l_0 - 1\}$. Therefore, for the sets J_β, $\beta = 1, \ldots, l_0 - 1$, there exists a system J'_β of distinct representatives such that

$$J'_\beta \subset J_\beta, \quad |J'_\beta| = r \;\text{ for all }\; \beta = 1, \ldots, l-1, \quad J'_{\beta_1} \cap J'_{\beta_2} = \varnothing \;\text{ for all }\; \beta_1 \neq \beta_2.$$

Consequently, each group of pursuers J'_β performs an r-multiple capture of the evader E_β. Therefore, the pursuers $\bigcup\limits_{\beta=1}^{l_0-1} J'_\beta$ perform an r-multiple capture of $l_0 - 1$ evaders. In what follows, we can assume that $J'_\beta = J_\beta$ for all $\beta = 1, \ldots, l_0 - 1$.

Let $s_0 = p + l_0 - 1$. In this case, $s_0 > p$ and $s_0 \le p + q - p - 1 = q - 1$. Consider the set $N_1 = N \setminus \bigcup\limits_{\beta=1}^{l_0-1} J'_\beta$. For this set, $|N_1| = n - pr - (l_0 - 1)r = n - s_0 r$. By the assumptions of the theorem, there exists a set M_1 such that $M_1 \subset J$, $|M_1| = q - s_0$, and $\delta_{N_1}(\beta) > 0$ for all $\beta \in M_1$. Note that $\{1, \ldots, l_0 - 1\} \cap M_1 = \varnothing$; otherwise, if β belongs to this intersection, then there exists an index $l \in N_1$ for which P_l captures the evader E_β, where $\beta \in \{1, \ldots, l_0 - 1\}$, which contradicts the construction of the set N_1. By the induction assumption, the group of pursuers P_l $(l \in N_1)$ performs an r-multiple capture of at least $q - s_0$ evaders. Consequently, the pursuers P_l $(l \in N)$ perform an r-multiple capture of at least $q - s_0 + l_0 - 1 = q - p$ evaders. This completes the proof.

Theorem 6. *Suppose that $a = 0$, V is a strictly convex compact set with smooth boundary, and the following condition holds for each $s \in \{0, \ldots, q-1\}$: for any set $N \subset I$, $|N| = n - sr$, there exists a set $M \subset J$ such that $|M| = q - s$ and*

$$0 \in \bigcap_{\Lambda \in \Omega_N(n-r+1)} \operatorname{Int} \operatorname{co} \left\{ z_{l\beta}^{p-1}, l \in \Lambda \right\} \quad \textit{for} \quad \textit{all} \quad \beta \in M.$$

Then an r-multiple capture of at least q evaders occurs in the game $G(n, m)$.

Proof. The validity of this theorem follows from Lemma 3 and Theorem 5.

Theorem 7. *Suppose that $a < 0$, $\alpha \in (0, 1)$, and Conjecture 1 holds. Then an r-multiple capture of at least q evaders occurs in the game $G(n, m)$.*

Proof. The theorem is proved similarly to Theorem 5 with the use of Theorem 3.

Theorem 8. *Suppose that $a < 0$, $\alpha \in (1, 2)$, and the following condition holds for each value $s \in \{0, \ldots, q - 1\}$: for any set $N \subset I$ with $|N| = n - sr$, there exists a set $M \subset J$ such that $|M| = q - s$ and, for all $\beta \in M$,*

$$\delta_N(\beta) = \min \left\{ \min_{v \in V} \max_{\Lambda \in \Omega_N(r)} \min_{l \in \Lambda} \lambda(z_{l\beta}^1, v), \min_{v \in V} \max_{\Lambda \in \Omega_N(r)} \min_{l \in \Lambda} \lambda(-z_{l\beta}^1, v) \right\} > 0.$$

Then an r-multiple capture of at least q evaders occurs in the game $G(n, m)$.

Proof. The theorem is proved similarly to Theorem 5 with the use of Theorem 4.

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