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Estimation of Reachable Sets from Above with Respect to Inclusion for Some Nonlinear Control Systems

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Abstract—The study of reachable sets of controlled objects is an important research area in optimal control theory. Such sets describe in a rough form the dynamical possibilities of the objects, which is important for theory and applications. Many optimization problems for controlled objects use the reachable set $D(T)$ in their statements. In the study of properties of controlled objects, it is useful to have some constructive estimates of $D(T)$ from above with respect to inclusion. In particular, such estimates are helpful for the approximate calculation of $D(T)$ by the pixel method. In this paper, we consider two nonlinear models of direct regulation known in the theory of absolute stability with a control term added to the right-hand side of the corresponding system of differential equations. To obtain the required upper estimates with respect to inclusion, we use Lyapunov functions from the theory of absolute stability. Note that the upper estimates for $D(T)$ are obtained in the form of balls in the phase space centered at the origin.

Keywords: reachable set, Lyapunov function, absolute stability, direct regulation.

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INTRODUCTION

The problem of estimating the reachable sets $D(T)$ of controlled objects from above with respect to inclusion is of certain interest for mathematical control theory and its applications. Such estimates are useful in the analysis of dynamic possibilities of controlled objects and in the approximate calculation of $D(T)$ by the pixel method.

In this paper we consider two nonlinear control systems of general form connected with classical models of the theory of absolute stability of direct regulation (see [1, 2]).

The first system (case **1** below) contains one nonlinearity, and the second system (case **2**) contains m nonlinearities, $m \geq 2$. We estimate from above with respect to inclusion the reachable set $D(T)$ (see, for example, [3, 4]) using the techniques of Lyapunov functions, which first appeared in motion stability theory (see, for example, [2, 5, 6] and many other papers). Note that the techniques of Lyapunov functions were used in earlier papers (see, for example, [6]) not only for traditional problems of motion stability theory but also for other qualitative problems of the theory of differential equations.

1. Consider the nonlinear control system

$$\dot{x} = Ax + b\varphi(\sigma(x)) + Mu, \tag{1}$$

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where $x \in \mathbb{R}^n$ $(n \geq 1)$, $b \in \mathbb{R}^n$, A is an $n \times n$ matrix, M is an $n \times r$ matrix $(r \geq 1)$, $\varphi(\sigma)$ is a continuously differentiable scalar function of a variable $\sigma \in \mathbb{R}^1$,

$$\sigma(x) = \langle c, x \rangle \tag{2}$$

for $c \in \mathbb{R}^n$, and u is a control vector from a compact set $U \subset \mathbb{R}^r$. We agree to denote by $\mathbb{R}^k$ $(k \geq 1)$ the arithmetic Euclidean space whose elements are ordered columns of k numbers with the standard scalar product $\langle \cdot, \cdot \rangle$. For a vector $y \in \mathbb{R}^k$, we denote by $|y|$ the standard length of y.

Note that, setting in (1) $u = 0$, we obtain the known model of direct regulation, which has long been studied in the theory of absolute stability of motion (see [1,2]). Thus, the controlled object (1) can be considered as a controlled variant of the known uncontrolled system of direct regulation.

Fix the initial vector of the control system (1)

$$x(0) = x_0. \tag{3}$$

For $t \geq 0$ consider the set $\mathcal{U}$ of all Lebesgue measurable functions $u(t)$ satisfying the condition

$$u(t) \in U, \quad t \geq 0. \tag{4}$$

Fix a control $\tilde{u}(\cdot) \in \mathcal{U}$, substitute it into the system of differential equations (1), and solve the system under the initial condition (3) for $t \geq 0$ in the class of locally absolutely continuous functions. According to [7, pp. 66, 67, Russian transl.], the corresponding unique locally absolutely continuous solution $\tilde{x}(t) = x(t, \tilde{u}(\cdot))$ is defined on some half-open interval $[0, \tau(\tilde{u}(\cdot)))$ maximal with respect to inclusion, where $\tau(\tilde{u}(\cdot))$ is either a finite positive number or $+\infty$. Fix $T > 0$. If $\tau(\tilde{u}(\cdot)) > T$, then the vector $x(T, \tilde{u}(\cdot))$ is defined. If $\tau(\tilde{u}(\cdot)) \leq T$, then the vector $x(T, \tilde{u}(\cdot))$ is not defined, since in this case we can prove by contradiction the existence of a numerical sequence $t_i \in (0, \tau(\tilde{u}(\cdot)))$, $i = 1, 2, \ldots$, such that $t_i \to \tau(\tilde{u}(\cdot))$ and $|x(t_i, \tilde{u}(\cdot))| \to +\infty$ as $i \to +\infty$. For $T > 0$ we define the reachable set $D(T)$ of the controlled object (1)–(4) by the formula

$$D(T) = \{x(T, \tilde{u}(\cdot))\}, \tag{5}$$

where the union of is taken only over $\tilde{u}(\cdot)$ for which $\tau(\tilde{u}(\cdot)) > T$. Note that, in the general case, the set $D(T)$ can be empty for specific $T > 0$.

Our goal is to derive upper estimates with respect to inclusion for the reachable set $D(T)$ of the controlled object (1)–(4). Among the related earlier results, we mention the results of [3, 4]. Note that functions $v(x)$ of Lyapunov type have been useful in this area. The main requirement on the scalar functions $v(x)$ is their continuous differentiability on $\mathbb{R}^n$. The functions must be differentiable along the motions of the control system. That is why we call these functions Lyapunov functions regardless of the fulfilment of other properties of Lyapunov functions from motion stability theory (see, for example, [2, 5, 6] and other papers).

Consider the function (see (1), (2))

$$v(x) = \frac{|x|^2}{2} + \int\limits_0^{\sigma(x)} \varphi(r)\, dr, \tag{6}$$

which will be used in what follows. Such functions are employed in the theory of absolute stability (see, for example, [1, 2]). We will require the fulfilment of the following inequality:

$$\varphi(r) r \geq 0 \quad \forall r \in \mathbb{R}^1. \tag{7}$$

This inequality implies that the integral term in (6) is a nonnegative function for $x \in \mathbb{R}^n$ and, consequently (see (6)), $v(x) > 0$ for $x \neq 0$; note that $v(0) = 0$.

Fix a control $\tilde{u}(\cdot) \in \mathcal{U}$ and consider the functions (see (6))

$$\tilde{x}(t) = x(t, \tilde{u}(\cdot)), \quad \tilde{v}(t) = v(\tilde{x}(t)) \tag{8}$$

on the interval $[0, \tau(\tilde{u}(\cdot)))$. It is easy to see that the function $\tilde{x}(t)$ is locally Lipschitz on $[0, \tau(\tilde{u}(\cdot)))$; hence, $\tilde{v}(t)$ is almost everywhere differentiable on the same interval. The derivative $\dot{\tilde{v}}(t)$ satisfies the equality (see (1), (2), (6))

$$\dot{\tilde{v}}(t) = \langle \nabla v(\tilde{x}(t)), A\tilde{x}(t) + b\varphi(\sigma(\tilde{x}(t))) + M\tilde{u}(t) \rangle \tag{9}$$

for almost all $t \in [0, \tau(\tilde{u}(\cdot)))$. Here $\nabla v(x)$ is the gradient of $v(x)$; we have

$$\nabla v(x) = x + c\varphi(\sigma(x)). \tag{10}$$

In connection with formulas (9) and (10), it is useful to consider the function

$$f(x, \sigma, u) = \langle x + c\varphi(\sigma), Ax + b\varphi(\sigma) + Mu \rangle, \tag{11}$$

where $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^1$, and $u \in \mathbb{R}^r$. This formula can be written in the form

$$f(x, \sigma, u) = g_1(x, u) + \langle c, b \rangle \varphi^2(\sigma) + g_2(x, u)\varphi(\sigma), \tag{12}$$

where

$$g_1(x, u) = \langle x, Ax + Mu \rangle, \tag{13}$$

$$g_2(x, u) = \langle c, Ax + Mu \rangle + \langle x, b \rangle. \tag{14}$$

In what follows, we will assume that

$$\langle c, b \rangle < 0. \tag{15}$$

Forming the perfect square with respect to $\varphi(\sigma)$ in (12) and using condition (15), we obtain the inequality

$$f(x, \sigma, u) \leq g_1(x, u) + \frac{1}{|\langle c, b \rangle|} \left(\frac{g_2(x, u)}{2} \right)^2 \tag{16}$$

for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$. Note that the function $\varphi(\sigma)$ does not enter the right-hand side of (16). Using the Cauchy–Bunyakovskii inequality, the boundedness of U, and formulas (12)–(14) and (16), we can easily derive the inequality

$$f(x, \sigma, u) \leq d_1|x|^2 + d_2|x| + d_3 \tag{17}$$

for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^1$, and $u \in U$, where d_1, d_2, and d_3 are some constructively computable nonnegative constants. Using the inequality $|x| \leq (|x|^2 + 1)/2$ and inequality (17), we find for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^1$, and $u \in U$ that (see (6), (7))

$$f(x, \sigma, u) \leq \alpha v(x) + \beta, \tag{18}$$

where α and β are nonnegative constructively computable constants. Summing the above, we find from formulas (8)–(13) and (16)–(18) that

$$\dot{\tilde{v}}(t) \le \alpha\tilde{v}(t) + \beta \tag{19}$$

for almost all $t \in [0, \tau(\tilde{u}(\cdot)))$, where $\tilde{v}(t) = v(\tilde{x}(t))$. Using the known theorem for differential inequalities (see, for example, [8]), we can prove for $t \in [0, \tau(\tilde{u}(\cdot)))$ that

$$\tilde{v}(t) \le y(t), \tag{20}$$

where $y(t)$ is the solution of the comparison equation

$$\dot{y} = \alpha y + \beta \tag{21}$$

with the initial condition

$$y(0) = v(x_0). \tag{22}$$

Note that, by formulas (6), (7), and (20)–(22), we have

$$|\tilde{x}(t)|^2 \le 2y(t) \tag{23}$$

for $t \in [0, \tau(\tilde{u}(\cdot)))$, where

$$y(t) = e^{\alpha t}v(x_0) + \beta \int\limits_0^t e^{\alpha r}\, dr. \tag{24}$$

Assume that $\tau(\tilde{u}(\cdot))$ is a finite number. In this situation, the finite number $\tau(\tilde{u}(\cdot))$ is greater than zero. Then, as mentioned above, there exists a sequence of numbers $t_i \in (0, \tau(\tilde{u}(\cdot)))$, $i = 1, 2, \ldots$, such that $t_i \to \tau(\tilde{u}(\cdot))$ and $|\tilde{x}(t_i)| \to +\infty$ as $i \to +\infty$. However, this is impossible because of relations (23) and (24). Thus, under the above assumptions (see (7), (15)), we have $\tau(\tilde{u}(\cdot)) = +\infty$. Note that $\tilde{u}(\cdot)$ was an arbitrary admissible control from $\mathcal{U}$ and, consequently, for arbitrary $T > 0$ and $u(\cdot) \in \mathcal{U}$, we have

$$|x(T, u(\cdot))| \le \sqrt{2y(T)}, \tag{25}$$

where the function $y(t)$ is defined by (24). We obtain the following result.

Theorem 1. *For the controlled object* (1)–(4), *under conditions* (7) *and* (15), *for arbitrary* $T > 0$ *and* $u(\cdot) \in \mathcal{U}$, *inequality* (25) *holds, where the function* $y(t)$ *is defined by formula* (24) *with appropriately chosen nonnegative constants* α *and* β *and the value* $v(x_0)$ *is calculated by formula* (6).

Remark. If the linear vector function Ax in equation (1) is replaced by a nonlinear continuously differentiable on $\mathbb{R}^n$ function $g(x)$ with values in $\mathbb{R}^n$ satisfying the inequality

$$|g(x)| \le \mu|x| + \nu, \quad x \in \mathbb{R}^n,$$

with nonnegative constants μ and ν, one can employ an argument similar to the above using the Lyapunov function (6) and obtain in this more general case an upper bound of the form (25) for vectors $x(T, u(\cdot))$ for arbitrary $T > 0$ and $u(\cdot) \in \mathcal{U}$.

2. Here we consider a control system (see [7, 10]) of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i\varphi_i(\sigma_i(x)) + Mu, \tag{26}$$

where $x \in \mathbb{R}^n$, $n \geq 1$; the vectors b_i $(i = 1, \ldots, m,\ m \geq 2)$ belong to $\mathbb{R}^n$; the matrices A and M have size $n \times n$ and $n \times r$ $(r \geq 1)$, respectively; $\varphi_i(\sigma_i)$ $(i = 1, \ldots, m)$ is a continuously differentiable scalar function of a variable $\sigma_i \in \mathbb{R}^1$; and

$$\sigma_i(x) = \langle c_i, x \rangle. \tag{27}$$

Here c_i $(i = 1, \ldots, m)$ is a vector from $\mathbb{R}^n$. Such systems for $u = 0$ are considered in the theory of absolute stability (see, for example, [2,10]). The vector $u \in \mathbb{R}^r$ is subject to a geometric constraint

$$u \in U, \tag{28}$$

where U is a compact set from $\mathbb{R}^r$. Fix an initial condition

$$x(0) = x_0. \tag{29}$$

Substitute a measurable control $\tilde{u}(t) \in U$, $t \geq 0$, into (26) and solve the Cauchy problem for this equation with initial condition (29) for $t \geq 0$ in the class of locally absolutely continuous functions. According to the results from [7, pp. 66, 67, Russian transl.], the corresponding unique solution $x(t, \tilde{u}(\cdot))$ is defined on the maximal (with respect to inclusion) interval $[0, \tau(\tilde{u}(\cdot)))$, where $\tau(\tilde{u}(\cdot))$ is either a finite positive number or $+\infty$. As in case **1**, we define the reachable set $D(T)$ by (5). Note that, in the general case, the set $D(T)$ can be empty for specific $T > 0$. To derive an upper bound with respect to inclusion for the set $D(T)$, we will use an analog of the function (6) (see [10]) of the form (see (27))

$$v(x) = \frac{|x|^2}{2} + \sum_{i=1}^{m} \int\limits_0^{\sigma_i(x)} \varphi_i(r)\, dr. \tag{30}$$

In what follows, we assume that each function $\varphi_i(r)$, $i = 1, 2, \ldots, m$, satisfies the inequality

$$\varphi_i(r) r \geq 0 \quad \forall r \in \mathbb{R}^1. \tag{31}$$

This condition provides the nonnegativity of each integral term for $x \in \mathbb{R}^n$ in formula (30). Define $\tilde{x}(t) = x(t, \tilde{u}(\cdot))$ and $\tilde{v}(t) = v(\tilde{x}(t))$ for $t \in [0, \tau(\tilde{u}(\cdot)))$. It is easy to see that the function $\tilde{v}(t)$ is locally Lipschitz and, consequently, differentiable almost everywhere for $t \in [0, \tau(\tilde{u}(\cdot)))$. Moreover, the derivative $\dot{\tilde{v}}(t)$ satisfies almost everywhere on $[0, \tau(\tilde{u}(\cdot)))$ the formula

$$\dot{\tilde{v}}(t) = \Big\langle \nabla v(\tilde{x}(t)), A\tilde{x}(t) + \sum_{i=1}^{m} b_i \varphi_i(\sigma_i(\tilde{x}(t))) + M\tilde{u}(t) \Big\rangle, \tag{32}$$

where

$$\nabla v(x) = x + \sum_{i=1}^{m} c_i \varphi_i(\sigma_i(x)). \tag{33}$$

By analogy with case **1**, consider the function (compare with (11))

$$f(x, \sigma, u) = \Big\langle x + \sum_{i=1}^{m} c_i \varphi_i(\sigma_i), Ax + \sum_{i=1}^{m} b_i \varphi_i(\sigma_i) + Mu \Big\rangle, \tag{34}$$

where $x \in \mathbb{R}^n$, the vector σ is from $\mathbb{R}^m$, its components are values $\sigma_i \in \mathbb{R}^1$, and $u \in \mathbb{R}^r$. This formula can be written in the form

$$f(x,\sigma,u) = g_1(x,u) + \sum_{i,j=1}^{m} \langle c_i, b_j\rangle \varphi_i(\sigma_i)\varphi_j(\sigma_j) + \sum_{i=1}^{m} h_i(x,u)\varphi_i(\sigma_i), \tag{35}$$

where

$$g_1(x,u) = \langle x, Ax + Mu\rangle, \tag{36}$$

$$h_i(x,u) = \langle c_i, Ax + Mu\rangle + \langle x, b_i\rangle \quad (i = 1,\dots,m). \tag{37}$$

In connection with (35), consider a quadratic form

$$W(\xi) = \langle C\xi, \xi\rangle,$$

where $\xi \in \mathbb{R}^m$ and the symmetric matrix C of order m is constructed from a matrix F of order m with elements $F_{ij} = \langle c_i, b_j\rangle$ by the formula

$$C = \frac{1}{2}(F + F^*). \tag{38}$$

Here $*$ means transposition.

We will assume that the following condition holds.

Condition A. The symmetric matrix C is negative definite; i.e., the matrix $(-1)C$ is positive definite.

It is known (see [11, pp. 210, 211]) that, for the positive definite matrix $(-1)C$, there exists a positive constant γ such that $\forall \xi \in \mathbb{R}^m$

$$\langle (-1)C\xi, \xi\rangle \geq \gamma|\xi|^2; \tag{39}$$

i.e., $\forall \xi \in \mathbb{R}^m$

$$\langle C\xi, \xi\rangle \leq -\gamma|\xi|^2. \tag{40}$$

Note that the largest constant $\gamma > 0$ in (39) is constructively computable. Thus, we find from (34)–(38), (40) that

$$f(x,\sigma,u) \leq |g_1(x,u)| - \gamma|\varphi(\sigma)|^2 + \sum_{i=1}^{m} |h_i(x,u)\varphi_i(\sigma_i)| \tag{41}$$

for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^m$, and $u \in \mathbb{R}^r$, where the vector $\varphi(\sigma)$ with components $\varphi_i(\sigma_i)$, $i = 1,\dots,m$, belongs to R^m.

Denote by $l(x,u,\sigma)$ the sum over i from 1 to m in the right-hand side of inequality (41). From the definition of the function $l(x,u,\sigma)$, formulas (36) and (37), and the Cauchy–Bunyakovskii inequality, it is easy to obtain for $x \in \mathbb{R}^n$, $u \in U$ (U is a compact set in $\mathbb{R}^r$), and $\sigma \in \mathbb{R}^m$ a bound of the form

$$|g_1(x,u)| + l(x,u,\sigma) \leq d_1|x|^2 + d_2|x| + (d_3|x| + d_4)|\varphi(\sigma)|, \tag{42}$$

where d_i are nonnegative constructively computable constants. In connection with inequalities (41) and (42), it is useful to consider the function

$$\xi(x,\sigma) = -\gamma|\varphi(\sigma)|^2 + (d_3|x| + d_4)|\varphi(\sigma)|.$$

Forming here the perfect square with respect to $|\varphi(\sigma)|$, we obtain for $x \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^m$ the inequality

$$\xi(x, \sigma) \leq \frac{1}{\gamma}\Big(\frac{d_3|x| + d_4}{2}\Big)^2.$$

Thus, for the function $f(x, \sigma, u)$ (see (35)), we obtain the inequality

$$f(x, \sigma, u) \leq d_5|x|^2 + d_6|x| + d_7 \tag{43}$$

for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^m$, and $u \in U$, where d_5, d_6, and d_7 are constructively computable nonnegative constants. Note that the right-hand side of (43) is independent of σ. Using the inequality $|x| \leq (|x|^2 + 1)/2$ and (43), we obtain inequality (18) for the function $f(x, \sigma, u)$ (see (35)) for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^m$, and $u \in U$. The further argument follows the scheme of case **1** (see formulas (19)–(24)). Thus, for arbitrary $T > 0$ and $u(\cdot) \in \mathcal{U}$, we prove the inequality

$$|x(T, u(\cdot))| \leq \sqrt{2y(T)}, \tag{44}$$

where the function $y(t)$ is defined in (24). We have established the following result.

Theorem 2. *For the controlled object* (26)–(29), *under condition* (31) *and the condition of negative definiteness of the matrix* C (*see* (38)), *for arbitrary* $T > 0$ *and* $u(\cdot) \in \mathcal{U}$, *inequality* (44) *holds, where the function* $y(t)$ *is defined by* (24) *with appropriately chosen nonnegative constants* α *and* β *and the value* $v(x_0)$ *is calculated by formula* (30).

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