ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2020, Vol. 309, Suppl. 1, pp. S83–S92.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2019, Vol. 25, No. 3, pp. 118–128.

On the Theory of Positional Differential Games for Neutral-Type Systems

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Received April 16, 2019; revised May 14, 2019; accepted May 20, 2019

Abstract—For a dynamical system whose motion is described by neutral-type differential equations in Hale's form, we consider a minimax–maximin differential game with a quality index evaluating the motion history realized up to the terminal time. The control actions of the players are subject to geometric constraints. The game is formalized in classes of pure positional strategies with a memory of the motion history. It is proved that the game has a value and a saddle point. The proof is based on the choice of an appropriate Lyapunov–Krasovskii functional for the construction of control strategies by the method of an extremal shift to accompanying points.

Keywords: neutral-type systems, control theory, differential games.

DOI: 10.1134/S0081543820040100

INTRODUCTION

This paper is devoted to the development of the theory of positional differential games [1–3] for systems of neutral type. We consider a zero-sum differential game in which the motion of a dynamical system is described by differential equations of neutral type in Hale's form [4]. There are geometric constraints on the control actions of the players. The quality of the control process is estimated in terms of the motion history of the system that has formed by the terminal time. The game is formalized in classes of pure positional strategies within the approach of [1–3]. The result of the paper is a theorem on the existence of a value and a saddle point in the differential game under consideration.

Issues of the existence of a value and optimal strategies in positional differential games for neutral-type systems were studied earlier in [5–8]. Linear neutral-type systems were considered in [8]. Differential games for nonlinear systems formalized in classes of control strategies with a guide were the subject of [5,7]. The result closest to the present paper was established in [6], where a differential game in classes of pure positional strategies was considered for nonlinear neutral-type systems of a fairly general form. However, because of a special proof technique based on the constructions of two guides [9, 10], additional considerable constraints were imposed on the game in [6]: it was required that the functional defining the quality index and the functional under the derivative on the left-hand side of the motion equations should satisfy the Lipschitz condition, and

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the Lipschitz constant for the latter functional should be less than 1. These constraints are removed in the present paper, whereas the form of the system is slightly less general as compared to [5–7] but still quite typical. This result was obtained using the classical reasoning scheme from [3], in which an appropriate Lyapunov–Krasovskii functional [11, 12] was chosen.

1. DIFFERENTIAL GAME

Consider a zero-sum differential game in which the motion of the system is described by a differential equation of neutral type in Hale's form [4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t) - g(t, x(t-h))\Big) = f(t, x(t), x(t-h), u(t), v(t)),$$
$$t \in [t_0, \vartheta], \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{U}, \quad v(t) \in \mathbb{V}, \tag{1.1}$$

and the quality index has the form

$$\gamma = \sigma(x_\vartheta(\cdot)). \tag{1.2}$$

Here t is time; $x(t)$ is the state vector at a time t; t_0 and ϑ are fixed initial and terminal times; $h > 0$ is the delay constant; $x_\vartheta(\cdot)$ is the motion history on the interval $[\vartheta - h, \vartheta]$: $x_\vartheta(\xi) = x(\vartheta + \xi)$ for $\xi \in [-h, 0]$; $u(t)$ and $v(t)$ are the current control actions of the first and second players, respectively; and $\mathbb{U} \subset \mathbb{R}^k$ and $\mathbb{V} \subset \mathbb{R}^l$ are compact sets.

The first player aims to minimize the index (1.2); the second, to maximize it.

Throughout the paper, we denote the scalar product of vectors by $\langle \cdot, \cdot \rangle$, the Euclidean norm by $\|\cdot\|$, and the space of Lipschitz functions from $[a, b]$ to $\mathbb{R}^n$ equipped with the uniform norm by $\mathrm{Lip}([a, b], \mathbb{R}^n)$; we also use the notation $\mathrm{Lip} = \mathrm{Lip}([-h, 0], \mathbb{R}^n)$. The uniform norm in Lip is denoted by $\|\cdot\|_\infty$. For $\alpha > 0$, we write $B(\alpha) = \{x \in \mathbb{R}^n \colon \|x\| \le \alpha\}$.

Assume that functions $g \colon [t_0, \vartheta] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $f \colon [t_0, \vartheta] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \mapsto \mathbb{R}^n$ and a functional $\sigma \colon \mathrm{Lip} \mapsto \mathbb{R}$ satisfy the following conditions.

(g) For any $\alpha > 0$, there exists $\lambda_g = \lambda_g(\alpha) > 0$ such that

$$\big\|g(t, x) - g(t', x')\big\| \le \lambda_g\big(|t - t'| + \|x - x'\|\big), \quad t, t' \in [t_0, \vartheta], \quad x, x' \in B(\alpha).$$

(f_1) The function f is continuous.

(f_2) There exists a constant $c_f > 0$ such that

$$\big\|f(t, x, y, u, v)\big\| \le c_f\big(1 + \|x\| + \|y\|\big), \quad (t, x, y, u, v) \in [t_0, \vartheta] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V}.$$

(f_3) For any $\alpha > 0$, there exists $\lambda_f = \lambda_f(\alpha) > 0$ such that

$$\big\|f(t, x, y, u, v) - f\big(t, x', y', u, v)\big\| \le \lambda_f\big(\|x - x'\| + \|y - y'\|\big),$$
$$t \in [t_0, \vartheta], \quad x, y, x', y' \in B(\alpha), \quad u \in \mathbb{U}, \quad v \in \mathbb{V}.$$

(f_4) For any $t \in [t_0, \vartheta]$ and $x, y, s \in \mathbb{R}^n$,

$$\min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \langle f(t, x, y, u, v), s\rangle = \max_{v \in \mathbb{V}} \min_{u \in \mathbb{U}} \langle f(t, x, y, u, v), s\rangle.$$

(σ) The functional σ is continuous.

Fix some numbers $\alpha_0, \lambda_0 > 0$ and define the set of initial positions

$$G_0 = \big\{t_0\big\} \times \big\{w(\cdot) \in \mathrm{Lip} \colon \|w(\xi)\| \le \alpha_0,\ \|w(\xi) - w(\xi')\| \le \lambda_0|\xi - \xi'|,\ \xi, \xi' \in [-h, 0]\big\}.$$

Taking the number c_f from condition (f_2), define the set of admissible positions

$$G = \Big\{(t, x_t(\cdot)) \in [t_0, \vartheta] \times \mathrm{Lip}\colon x(\cdot) \in \mathrm{Lip}([t_0 - h, \vartheta], \mathbb{R}^n),\ (t_0, x_{t_0}(\cdot)) \in G_0,$$

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t) - g(t, x(t-h))\Big)\Big\| \le c_f\big(1 + \|x(t)\| + \|x(t-h)\|\big) \text{ for a.a. } t \in [t_0, \vartheta]\Big\}. \tag{1.3}$$

Hereinafter, $x_t(\cdot)$ is a function such that $x_t(\xi) = x(t+\xi)$, $\xi \in [-h, 0]$.

Let a position $(\tau, w(\cdot)) \in G$, $\tau < \vartheta$, be chosen. Admissible realizations of the control actions $u(t)$ and $v(t)$ on the interval $[\tau, \vartheta]$ are measurable functions $u(\cdot) : [\tau, \vartheta) \mapsto \mathbb{U}$ and $v(\cdot) : [\tau, \vartheta) \mapsto \mathbb{V}$. Proceeding, for example, by the scheme from [13] (see also [14, P1]), one can show that, under conditions (g) and (f_1)–(f_3), any pair of admissible realizations $u(\cdot)$ and $v(\cdot)$ uniquely generates a motion $x(\cdot)$ of system (1.1) from the position $(\tau, w(\cdot))$, which is a function from $\mathrm{Lip}([\tau - h, \vartheta], \mathbb{R}^n)$ satisfying the initial condition $x(\tau + \xi) = w(\xi)$, $\xi \in [-h, 0]$, and, together with $u(t)$ and $v(t)$, equation (1.1) almost everywhere on $[\tau, \vartheta]$. In addition, by the definition (1.3) of the set G, the motion $x(\cdot)$ is such that

$$(t, x_t(\cdot)) \in G, \quad \tau \in [t, \vartheta]. \tag{1.4}$$

The differential game (1.1), (1.2) will be formalized in classes of positional strategies of the players' controls in accordance with the approach [1–3]. In view of condition (f_4), we can restrict ourselves to the class of pure positional strategies [3, Sect. 8].

A control strategy of the first player is understood as a mapping

$$U = U(t, w(\cdot), \varepsilon) \in \mathbb{U}, \quad (t, w(\cdot)) \in G, \quad \varepsilon > 0,$$

where ε is an accuracy parameter [3, p. 68].

Fix a position $(\tau, w(\cdot)) \in G$, a number $\varepsilon > 0$, and a partition of the interval $[\tau, \vartheta]$:

$$\Delta_\delta = \big\{t_j : 0 < t_{j+1} - t_j \le \delta,\ j \in \overline{1, J-1},\ t_1 = \tau,\ t_J = \vartheta\big\}. \tag{1.5}$$

The triple $\{U, \varepsilon, \Delta_\delta\}$ defines a feedback control law of the first player, who forms a piecewise constant (hence, admissible) realization $u(\cdot)$ by the rule

$$u(t) = U(t_j, x_{t_j}(\cdot), \varepsilon), \quad t \in [t_j, t_{j+1}), \quad j \in \overline{1, J-1}. \tag{1.6}$$

This law, together with an admissible realization of the second player's control $v(\cdot)$, uniquely generates a motion $x(\cdot)$ of system (1.1) from the position $(\tau, w(\cdot))$. Denote the corresponding value of the quality index (1.2) by $\gamma(\tau, w(\cdot); U, \varepsilon, \Delta_\delta; v(\cdot))$.

Define the guaranteed result of the strategy U:

$$\rho_u(\tau, w(\cdot), U) = \varlimsup_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup_{\Delta_\delta} \sup_{v(\cdot)} \gamma(\tau, w(\cdot); U, \varepsilon, \Delta_\delta; v(\cdot)). \tag{1.7}$$

Then the optimal guaranteed result of the first player is the value

$$\rho_u^\circ(\tau, w(\cdot)) = \inf_U \rho_u(\tau, w(\cdot), U). \tag{1.8}$$

A strategy U° is called optimal if

$$\rho_u(\tau, w(\cdot), U^\circ) = \rho_u^\circ(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in G.$$

Similarly, with obvious changes, we consider for the second player a control strategy $V = V(t, w(\cdot), \varepsilon) \in \mathbb{V}$, $(t, w(\cdot)) \in G$, $\varepsilon > 0$, the control law $\{V, \varepsilon, \Delta_\delta\}$ defining a piecewise constant realization $v(\cdot)$ by the rule

$$v(t) = V(t_j, x_{t_j}(\cdot), \varepsilon), \quad t \in [t_j, t_{j+1}), \quad j \in \overline{1, J-1},$$

the guaranteed result of the strategy V

$$\rho_v(\tau, w(\cdot), V) = \varliminf_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \inf_{\Delta_\delta} \inf_{u(\cdot)} \gamma(\tau, w(\cdot); u(\cdot); V, \varepsilon, \Delta_\delta), \tag{1.9}$$

and the optimal guaranteed result of the second player

$$\rho_v^\circ(\tau, w(\cdot)) = \sup_V \rho_v(\tau, w(\cdot), V). \tag{1.10}$$

A control strategy of the second player V° is optimal if

$$\rho_v(\tau, w(\cdot), V^\circ) = \rho_v^\circ(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in G.$$

It follows from relations (1.8) and (1.10) that

$$\rho_u^\circ(\tau, w(\cdot)) \geq \rho_v^\circ(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in G. \tag{1.11}$$

In the case where

$$\rho_u^\circ(\tau, w(\cdot)) = \rho_v^\circ(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in G,$$

the differential game (1.1), (1.2) is said to have a *value*, and the pair of optimal strategies $\{U^\circ, V^\circ\}$ is called a *saddle point of the game*.

Theorem. *The differential game* (1.1), (1.2) *has a value and a saddle point* $\{U^\circ, V^\circ\}$.

A key role in the proof of this theorem is played by the following auxiliary Lyapunov–Krasovskii functional [11, 12].

2. LYAPUNOV–KRASOVSKII FUNCTIONAL

Based on relation (1.3), we can show the existence of $\alpha_G, \lambda_G > 0$ such that

$$\|w(\xi)\| \leq \alpha_G, \quad \|w(\xi) - w(\xi')\| \leq \lambda_G |\xi - \xi'|, \quad \xi, \xi' \in [-h, 0], \quad (t, w(\cdot)) \in G. \tag{2.1}$$

Then, by conditions (g) and (f_3), for $\lambda_g = \lambda_g(\alpha_G)$ and $\lambda_f = \lambda_f(\alpha_G)$, we have

$$\begin{gathered} \|g(t, w(-h)) - g(t', w'(-h))\| \leq \lambda_g\big(|t - t'| + \|w(-h) - w'(-h)\|\big), \\ \|f(t, w(0), w(-h), u, v) - f(t, w'(0), w'(-h), u, v)\| \\ \leq \lambda_f\big(\|w(0) - w'(0)\| + \|w(-h) - w'(-h)\|\big), \\ (t, w(\cdot)), (t', w'(\cdot)) \in G, \quad u \in \mathbb{U}, \quad v \in \mathbb{V}. \end{gathered} \tag{2.2}$$

Define the functional

$$V_\varepsilon(t, p, w(\cdot)) = \kappa_\varepsilon(t, p, w(\cdot)) e^{-2(\lambda_f + \lambda_g/h)(t - t_0)}, \quad (t, p, w(\cdot)) \in [t_0, \vartheta] \times \mathbb{R}^n \times \mathrm{Lip}, \tag{2.3}$$

where

$$\kappa_\varepsilon(t,p,w(\cdot)) = \sqrt{\varepsilon^2+\|p\|^2} + \lambda_f \int\limits_{-h}^{0} \Big(1-\frac{2\lambda_g\xi}{h}\Big)\|w(\xi)\|\mathrm{d}\xi, \quad \varepsilon>0. \tag{2.4}$$

Lemma 1. *Assume that* $\tau\in[t_0,\vartheta]$, $\varepsilon>0$, *and functions* $s(\cdot)\in \mathrm{Lip}([\tau,\vartheta],\mathbb{R}^n)$ *and* $z(\cdot)\in \mathrm{Lip}([\tau-h,\vartheta],\mathbb{R}^n)$ *satisfy the bounds*

$$\begin{gathered} \|s(t)-z(t)\|\le\lambda_g\|z(t-h)\|, \quad t\in[\tau,\vartheta],\\ \Big\langle\frac{\mathrm{d}s(t)}{\mathrm{d}t},s(t)\Big\rangle \le \lambda_f\big(\|z(t)\|+\|z(t-h)\|\big)\|s(t)\|+\varepsilon^2 \text{ for a.a. } t\in[\tau,\vartheta]. \end{gathered} \tag{2.5}$$

Then

$$V_\varepsilon(t,s(t),z_t(\cdot)) \le V_\varepsilon(\tau,s(\tau),z_\tau(\cdot)) + (t-\tau)\varepsilon, \quad t\in[\tau,\vartheta]. \tag{2.6}$$

Proof. Based on relations (2.3) and (2.4), since the functions $s(\cdot)$ and $z(\cdot)$ are Lipschitz, we can show that the functions $\omega_1(t)=\kappa_\varepsilon(t,s(t),z_t(\cdot))$ and $\omega_2(t)=V_\varepsilon(t,s(t),z_t(\cdot))$, $t\in[\tau,\vartheta]$, are also Lipschitz. Then, using bounds (2.5), we get

$$\begin{aligned} \frac{\mathrm{d}\omega_1(t)}{\mathrm{d}t} &= \frac{\langle \mathrm{d}s(t)/\mathrm{d}t, s(t)\rangle}{\sqrt{\varepsilon^2+\|s(t)\|^2}} + \lambda_f\|z(t)\| - \lambda_f(1+2\lambda_g)\|z(t-h)\| + \frac{2\lambda_f\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi \\ &\le \varepsilon + 2\lambda_f\|z(t)\| - 2\lambda_f\lambda_g\|z(t-h)\| + \frac{2\lambda_f\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi \le \varepsilon + 2\lambda_f\|s(t)\| + \frac{2\lambda_f\lambda_g}{h}\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi \end{aligned}$$

for almost all $t\in[\tau,\vartheta]$. Hence, we obtain the bound

$$\frac{\mathrm{d}\omega_2(t)}{\mathrm{d}t} = \Big(\varepsilon - 2(\lambda_g/h)\|s(t)\| - 2\lambda_f^2\int\limits_{t-h}^{t}\|z(\xi)\|\mathrm{d}\xi\Big)e^{-2(\lambda_f+\lambda_g/h)(t-t_0)} \le \varepsilon,$$

which implies inequality (2.6).

Lemma 2. *There exists a number* $\lambda_V>0$ *such that, for all* $\varepsilon>0$, $\tau\in[t_0,\vartheta]$, $p\in\mathbb{R}^n$, *and* $w(\cdot)\in\mathrm{Lip}$, *under the condition*

$$\|w(\xi)\|\le 2\alpha_G, \quad \|w(\xi)-w(\xi')\|\le 2\lambda_G|\xi-\xi'|, \quad \xi,\xi'\in[-h,0], \tag{2.7}$$

the following inequality holds:

$$\|w(\cdot)\|_\infty^2 \le \lambda_V V_\varepsilon(t,p,w(\cdot)). \tag{2.8}$$

Proof. Under condition (2.7), there exists $\lambda_*>0$ such that

$$\|w(\cdot)\|_\infty^2 \le \lambda_* \int\limits_{-h}^{0}\|w(\xi)\|\mathrm{d}\xi.$$

Therefore, setting $\lambda_V = e^{2(\lambda_f+\lambda_g/h)(\vartheta-t_0)}\lambda_*/\lambda_f$ and using (2.3), we obtain inequality (2.8).

3. PROOF OF THE THEOREM

For the right-hand side of system (1.1), we define the Hamiltonian

$$H(t,x,y,s)=\min_{u\in\mathbb{U}}\max_{v\in\mathbb{V}}\langle f(t,x,y,u,v),s\rangle,\quad t\in[t_0,\vartheta],\quad x,y,s\in\mathbb{R}^n,\tag{3.1}$$

and the set-valued mappings

$$\begin{aligned}F_+(t,x,y,v)&=\mathrm{co}\big\{f(t,x,y,u,v)\,|\,u\in\mathbb{U}\big\}\subset\mathbb{R}^n,\\ F_-(t,x,y,u)&=\mathrm{co}\big\{f(t,x,y,u,v)\,|\,v\in\mathbb{V}\big\}\subset\mathbb{R}^n,\end{aligned}\quad t\in[t_0,\vartheta],\quad x,y\in\mathbb{R}^n,\quad u\in\mathbb{U},\quad v\in\mathbb{V},\tag{3.2}$$

where co means a convex hull in $\mathbb{R}^n$. By conditions (f_1)–(f_4), we have the following facts.

(H) The following bound holds for the number $\lambda_f>0$ from (2.2) and any $(t,w(\cdot)),(t,w'(\cdot))\in G$ and $s\in\mathbb{R}^n$:

$$|H(t,w(0),w(-h),s)-H(t,w'(0),w'(-h),s)|\le\lambda_f\big(\|w(0)-w'(0)\|+\|w(-h)-w'(-h)\|\big)\|s\|.$$

(F_1) The set-valued mappings F_+ and F_- have convex compact values and are continuous in the Hausdorff metric.

(F_2) The following inequality holds for the number $c_f>0$ from condition (f_2) for any $t\in[t_0,\vartheta]$, $x,y\in\mathbb{R}^n$, $u\in\mathbb{U}$, $v\in\mathbb{V}$, and $l\in F_+(t,x,y,v)\cup F_-(t,x,y,u)$:

$$\|l\|\le c_f\big(1+\|x\|+\|y\|\big).$$

(F_3) For any $t\in[t_0,\vartheta]$ and $x,y,s\in\mathbb{R}^n$,

$$\max_{v\in\mathbb{V}}\min_{l\in F_+(t,x,y,v)}\langle l,s\rangle=H(t,x,y,s)=\min_{u\in\mathbb{U}}\max_{l\in F_-(t,x,y,u)}\langle l,s\rangle.$$

For $(\tau,w(\cdot))\in G$, $u\in\mathbb{U}$, and $v\in\mathbb{V}$, denote by $X_+(\tau,w(\cdot),v)$ and $X_-(\tau,w(\cdot),u)$ the sets of functions $x(\cdot)\in\mathrm{Lip}([\tau-h,\vartheta],\mathbb{R}^n)$ satisfying the condition $x(\tau+\xi)=w(\xi)$ for $\xi\in[-h,0]$ and the following differential inclusions for almost all $t\in[\tau,\vartheta]$, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t)-g(t,x(t-h))\Big)\in F_+(t,x(t),x(t-h),v),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t)-g(t,x(t-h))\Big)\in F_-(t,x(t),x(t-h),u).$$

Proceeding by the scheme from [14, P2], we can show that the sets $X_+(\tau,w(\cdot),v)$ and $X_-(\tau,w(\cdot),u)$ are compact in $\mathrm{Lip}([\tau-h,\vartheta],\mathbb{R}^n)$ and, in view of (1.3),

$$(t,x_t(\cdot))\in G,\quad t\in[\tau,\vartheta],\quad x(\cdot)\in X_+(\tau,w(\cdot),v)\cup X_-(\tau,w(\cdot),u).\tag{3.3}$$

The results of [7] imply the following statement.

Assertion. *There exists a continuous functional* $\varphi\colon G\mapsto\mathbb{R}$ *such that*

$$\varphi(\vartheta,w(\cdot))=\sigma(w(\cdot)),\quad(\vartheta,w(\cdot))\in G;\tag{3.4}$$

$$\begin{gathered}\varphi(\tau,w(\cdot))\ge\min_{x(\cdot)\in X_+(\tau,w(\cdot),v)}\varphi(t,x_t(\cdot)),\qquad\varphi(\tau,w(\cdot))\le\max_{x(\cdot)\in X_-(\tau,w(\cdot),u)}\varphi(t,x_t(\cdot)),\\(\tau,w(\cdot))\in G,\quad t\in[\tau,\vartheta],\quad u\in\mathbb{U},\quad v\in\mathbb{V}.\end{gathered}\tag{3.5}$$

Suppose that $\varepsilon > 0$, $(t, w(\cdot)) \in G$, and a functional V_ε is defined according to (2.3). Denote by $O_\varepsilon(t, w(\cdot))$ the set of positions $(t, r(\cdot)) \in G$ satisfying the inequality

$$V_\varepsilon(t, w(0) - g(t, w(-h)) - r(0) + g(t, r(-h)), w(\cdot) - r(\cdot)) \leq \varepsilon(1 + t - t_0). \tag{3.6}$$

Using the definition (1.3) of the set G, we can show that $O_\varepsilon(t, w(\cdot))$ is compact in $\{t\} \times \mathrm{Lip}$.

Let φ be the functional from the assertion. Define

$$U^*(t, w(\cdot), \varepsilon) \in \operatorname*{argmin}_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \langle f(t, w(0), w(-h), u, v), w(0) - g(t, w(-h)) - r_*(0) + g(t, r_*(-h)) \rangle, \tag{3.7}$$

where

$$(t, r_*(\cdot)) \in \operatorname*{argmin}_{(t, r(\cdot)) \in O_\varepsilon(t, w(\cdot))} \varphi(t, r(\cdot)).$$

Lemma 3. *The inequality $\rho_u(\tau, w(\cdot), U^*) \leq \varphi(\tau, w(\cdot))$ holds for $(\tau, w(\cdot)) \in G$, where ρ_u is defined according to* (1.7).

Proof. By the definition (1.7) of the value $\rho_u(\tau, w(\cdot), U^*)$, it is sufficient to show that, for any $\zeta > 0$, there exist a number $\varepsilon_* = \varepsilon_*(\zeta) > 0$ and a function $\delta_*(\varepsilon) = \delta_*(\zeta, \varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_*)$, such that, for any position $(\tau, w(\cdot)) \in G$, numbers $\varepsilon \in (0, \varepsilon_*)$ and $\delta \in (0, \delta_*(\varepsilon))$, partition Δ_δ (1.5), and admissible realization $v(\cdot)$, the motion $x(\cdot)$ of system (1.1) generated from the position $(\tau, w(\cdot))$ by the control law $\{U^*, \varepsilon, \Delta_\delta\}$ and the realization $v(\cdot)$ satisfies the inequality

$$\gamma(\tau, w(\cdot); U^*, \varepsilon, \Delta_\delta; v(\cdot)) = \sigma(x_\vartheta(\cdot)) \leq \varphi(\tau, w(\cdot)) + \zeta. \tag{3.8}$$

By conditions (f_1) and (F_1) and bounds (2.1), there exists a function $\delta_f(\varepsilon)$, $\varepsilon \in [0, +\infty)$, such that, for any $\varepsilon > 0$, $u \in \mathbb{U}$, $v \in \mathbb{V}$, and $(t, w(\cdot)), (t', w'(\cdot)) \in G$, under the conditions

$$|t - t'| \leq \delta_f(\varepsilon), \quad \|w(0) - w'(0)\| \leq \delta_f(\varepsilon), \quad \|w(-h) - w'(-h)\| \leq \delta_f(\varepsilon),$$

the following bounds hold:

$$\begin{aligned} \|f(t, w(0), w(-h), u, v) - f(t', w'(0), w'(-h), u, v)\| &\leq \varepsilon, \\ \max_{l \in F_+(t, w(0), w(-h), v)} \min_{l' \in F_+(t', w'(0), w'(-h), v)} \|l - l'\| &\leq \varepsilon. \end{aligned} \tag{3.9}$$

Let $\zeta > 0$. By condition (σ) and bounds (2.1), there exists a number $\varepsilon_\sigma > 0$ such that, for any $(\vartheta, w(\cdot)), (\vartheta, w'(\cdot)) \in G$, under the condition $\|w(\cdot) - w'(\cdot)\|_\infty \leq \varepsilon_\sigma$, the following inequality holds:

$$|\sigma(w(\cdot)) - \sigma(w'(\cdot))| \leq \zeta. \tag{3.10}$$

Taking the numbers α_G and λ_G from (2.1) and the numbers λ_g and λ_f from (2.2), define

$$\alpha_s = 2(1 + \lambda_g)\alpha_G, \quad \lambda_s = 2\lambda_g(1 + \lambda_G), \quad c_* = c_f(1 + 2\alpha_G). \tag{3.11}$$

Take the number $\lambda_V > 0$ from Lemma 2 and define

$$\varepsilon_* = \frac{\varepsilon_\sigma^2}{\lambda_V^2(1 + \vartheta - t_0)}, \quad \delta_*(\varepsilon) = \min\left\{\frac{\delta_f(\varepsilon^2/(8\alpha_s))}{1 + \lambda_G}, \frac{\varepsilon^2}{8c_*\lambda_s}, \frac{\varepsilon^2}{16\alpha_s\lambda_f\lambda_G}, \frac{\varepsilon^2}{16\alpha_G\lambda_f\lambda_s}\right\}. \tag{3.12}$$

Let a position $(\tau, w(\cdot)) \in G$, numbers

$$\varepsilon \in (0, \varepsilon_*), \quad \delta \in (0, \delta_*(\varepsilon)), \tag{3.13}$$

a partition Δ_δ (1.5), and an admissible realization $v(\cdot)$ be fixed. Consider the motion $x(\cdot)$ of system (1.1) generated by the control law $\{U^*, \varepsilon, \Delta_\delta\}$ and realization $v(\cdot)$. Let us prove by induction the following bound:

$$\varphi(t_j, r^j(\cdot)) \leq \varphi(\tau, w(\cdot)), \quad j \in \overline{1, J}, \tag{3.14}$$

where

$$(t_j, r^j(\cdot)) \in \underset{(t_j, r(\cdot)) \in O_\varepsilon(t_j, x_{t_j}(\cdot))}{\operatorname{argmin}} \varphi(t_j, r(\cdot)). \tag{3.15}$$

For $j = 1$, the bound holds by the choice (3.15) of the position $(\tau, r^1(\cdot))$. Assume now that (3.14) holds for $j = k$ and prove it for $j = k + 1$. Let

$$v^k \in \underset{v \in \mathbb{V}}{\operatorname{argmax}} \min_{l \in F_+(t_k, r^k(\cdot), v)} \langle l, x(t_k) - g(t_k, x(t_k - h)) - r^k(0) + g(t_k, r^k(-h)) \rangle. \tag{3.16}$$

According to inequalities (3.5), there exists a function $y^k(\cdot) \in X_+(t_k, r^k(\cdot), v^k)$ such that

$$\varphi(t_{k+1}, y^k_{t_{k+1}}(\cdot)) \leq \varphi(t_k, r^k(\cdot)). \tag{3.17}$$

Define the functions

$$z^k(t) = x(t) - y^k(t), \quad s^k(t) = z^k(t) - g(t, x(t-h)) + g(t, y^k(t-h)), \quad t \in [t_k, \vartheta]. \tag{3.18}$$

Then, in view of the choice (3.12), (3.13) of the number δ, notation (3.11), relations (1.4) and (3.3), bounds (2.1) and (2.2), and conditions (f_2) and (F_2), we derive the following inequalities for any $t \in [t_k, t_{k+1}]$, $u \in \mathbb{U}$, and $v \in \mathbb{V}$:

$$\begin{gathered} \max\big\{|t - t_k|, \|x_t(\cdot) - x_{t_k}(\cdot)\|_\infty, \|y^k_t(\cdot) - y^k_{t_k}(\cdot)\|_\infty\big\} \leq \delta_f(\varepsilon^2/(8\alpha_s)), \\ \|z^k_t(\cdot)\|_\infty \leq 2\alpha_G, \quad \|z^k_t(\cdot) - z^k_{t_k}(\cdot)\|_\infty \leq \varepsilon^2/(8\lambda_f\alpha_s), \quad \|s^k(t) - z^k(t)\| \leq \lambda_g\|z^k(t-h)\|, \\ \|s^k(t)\| \leq \alpha_s, \quad \|s^k(t) - s^k(t_k)\| \leq \lambda_s|t - t_k| \leq \min\{\varepsilon^2/(8c_*), \varepsilon^2/(16\alpha_G\lambda_f)\}, \\ \|f(t, x(t), x(t-h), u, v)\| \leq c_*, \quad \sup\{\|l\| \colon l \in F_+(t, y^k(t), y^k(t-h), v)\} \leq c_*. \end{gathered} \tag{3.19}$$

By equation (1.1) and the inclusion $y^k(\cdot) \in X_+(t_k, r^k(\cdot), v^k)$, we have

$$\Big\langle \frac{\mathrm{d}s^k(t)}{\mathrm{d}t}, s^k(t) \Big\rangle = \langle f(t, x(t), x(t-h), u(t), v(t)) - l^k(t), s^k(t) \rangle, \quad l^k(t) \in F(t, y^k(t), y^k(t-h), v_k)$$

for almost all $t \in [t_k, t_{k+1}]$. Hence, using (3.9) and (3.19), we derive

$$\begin{aligned} \Big\langle \frac{\mathrm{d}s^k(t)}{\mathrm{d}t}, s^k(t) \Big\rangle &\leq \langle f(t_k, x(t_k), x(t_k - h), u(t), v(t)) - l^k_*(t), s^k(t) \rangle + \varepsilon^2/4 \\ &\leq \langle f(t_k, x(t_k), x(t_k - h), u(t), v(t)) - l^k_*(t), s^k(t_k) \rangle + \varepsilon^2/2, \end{aligned}$$

where $l^k_*(t) \in F_+(t_k, r^k(0), r^k(-h), v^k)$. Further, using the definition (3.7) of the strategy U^*, the rule (1.6) of forming the realization $u(\cdot)$ corresponding to this strategy, the choice (3.16) of the value v^k, properties (H) and (F_3), the definition (3.18) of the function $z^k(\cdot)$, and bounds (3.19), we obtain

$$\Big\langle \frac{\mathrm{d}s^k(t)}{\mathrm{d}t}, s^k(t)\Big\rangle \le \max_{v\in\mathbb{V}}\langle f(t_k, x(t_k), x(t_k-h), u(t), v), s^k(t_k)\rangle$$

$$-\min_{l\in F(t_k, r^k(0), r^k(-h), v^k)}\langle l, s^k(t_k)\rangle + \varepsilon^2/2$$

$$= H(t_k, x(t_k), x(t_k-h), s^k(t_k)) - H(t_k, y^k(t_k), y^k(t_k-h), s^k(t_k)) + \varepsilon^2/2$$

$$\le \lambda_f\big(\|z^k(t_k)\| + \|z^k(t_k-h)\|\big)\|s^k(t_k)\| + \varepsilon^2/2 \le \lambda_f\big(\|z^k(t)\| + \|z^k(t-h)\|\big)\|s^k(t)\| + \varepsilon^2.$$

Hence, in view of the fourth inequality in (3.19), we conclude that all conditions of Lemma 1 hold for $z(\cdot) = z^k(\cdot)$ and $s(\cdot) = s^k(\cdot)$. Therefore, we have the inequality

$$V_\varepsilon(t_{k+1}, s^k(t_{k+1}), z^k_{t_{k+1}}(\cdot)) \le V_\varepsilon(t_k, s^k(t_k), z^k_{t_k}(\cdot)) + (t_{k+1}-t_k)\varepsilon,$$

which, in view of the inclusion $r^k(\cdot) \in O_\varepsilon(t_k, x_{t_k}(\cdot))$ and inequality (3.6), means that $y^k_{t_{k+1}}(\cdot) \in O_\varepsilon(t_{k+1}, x_{t_{k+1}}(\cdot))$. Thus, according to relations (3.15) and (3.17) and the induction assumption, we derive

$$\varphi(t_{k+1}, r^{k+1}_{t_{k+1}}(\cdot)) \le \varphi(t_{k+1}, y^k_{t_{k+1}}(\cdot)) \le \varphi(t_k, r^k(\cdot)) \le \varphi(\tau, w(\cdot)).$$

Thus, inequality (3.14) is proved for all $j \in \overline{1, J}$.

By Lemma 2, using first the inclusion $(\vartheta, r^J(\cdot)) \in O_\varepsilon(\vartheta, x_\vartheta(\cdot))$ together with inequality (3.6) and then relations (3.12) and (3.13), we obtain

$$\|x_\vartheta(\cdot) - r^J(\cdot)\|^2_\infty \le \lambda_V V_\varepsilon(\vartheta, x(\vartheta) - g(\vartheta, x(\vartheta-h)) - r^J(0) + g(\vartheta, r^J(-h)), x_\vartheta(\cdot) - r^j(\cdot))$$

$$\le \lambda_V(1+\vartheta-t_0)\varepsilon \le \lambda_V(1+\vartheta-t_0)\varepsilon_* = \varepsilon^2_\sigma.$$

Hence, using relations (3.10), (3.14), and (3.4), we conclude that

$$\sigma(x_\vartheta(\cdot)) \le \sigma(r^J(\cdot)) + \zeta = \varphi(\vartheta, r^J(\cdot)) + \zeta \le \varphi(t, w(\cdot)) + \zeta.$$

The lemma is proved.

Similarly, we can prove the following lemma for the second player, setting

$$V^*(t, w(\cdot), \varepsilon) \in \operatorname*{argmax}_{v\in\mathbb{V}}\min_{u\in\mathbb{U}}\big\langle f(t, w(0), w(-h), u, v), r_*(0) - g(t, r_*(-h)) - w(0) + g(t, w(-h))\big\rangle,$$

where

$$(t, r_*(\cdot)) \in \operatorname*{argmax}_{(t, r(\cdot))\in O_\varepsilon(t, w(\cdot))}\varphi(t, r(\cdot)).$$

Lemma 4. *The inequality* $\rho_v(\tau, w(\cdot), V^*) \ge \varphi(\tau, w(\cdot))$ *holds for* $(\tau, w(\cdot)) \in G$*, where* ρ_v *is defined in* (1.9).

Proof of the theorem. From Lemmas 3 and 4 and relations (1.8), (1.10), and (1.11), we obtain the equality

$$\rho^\circ_u(\tau, w(\cdot)) = \rho_u(\tau, w(\cdot), U^*) = \rho_v(\tau, w(\cdot), V^*) = \rho^\circ_v(\tau, w(\cdot)),$$

which shows that the differential game (1.1), (1.2) has a value and the pair of strategies $\{U^*, V^*\}$ forms a saddle point of the game. The theorem is proved.

FUNDING

This work was supported by the Russian President's Grant for Young Russian Scientists no. MK-3566.2019.1.

REFERENCES

1. N. N. Krasovskii and A. I. Subbotin, *Positional Differential Games* (Nauka, Moscow, 1974) [in Russian].
2. Yu. S. Osipov, "On the theory of differential games of systems with aftereffect," J. Appl. Math. Mech. **35** (2), 262–272 (1971).
3. N. N. Krasovskii, *Control of a Dynamical System* (Nauka, Moscow, 1985) [in Russian].
4. J. Hale, *Theory of Functional Differential Equations* (Springer, New York, 1977).
5. N. Yu. Lukoyanov and A. R. Plaksin, "Differential games for neutral-type systems: An approximation model," Proc. Steklov Inst. Math. **291**, 190–202 (2015).
6. M. I. Gomoyunov, N. Yu. Lukoyanov, and A. R. Plaksin, "Existence of a value and a saddle point in positional differential games for neutral-type systems," Proc. Steklov Inst. Math. **299** (Suppl. 1), S37–S48 (2017).
7. M. I. Gomoyunov and A. R. Plaksin, "On the basic equation of differential games for neutral-type systems," J. Appl. Math. Mech. **82** (6), 675–689 (2018). doi 10.31857/S003282350002733-6
8. M. I. Gomoyunov and Yu. N. Lukoyanov, "On the numerical solution of differential games for neutral-type linear systems," Proc. Steklov Inst. Math. **301** (Suppl. 1), S44–S56 (2018). doi 10.1134/S0081543818050048
9. A. V. Kryazhimskii, "On stable position control in differential games," J. Appl. Math. Mech. **42** (6), 1055–1060 (1980).
10. V. I. Maksimov, "A differential game of guidance for systems with a deviating argument of neutral type," in *Problems of Dynamic Control* (IMM UNTs AN SSSR, Sverdlovsk, 1981), pp. 33–45 [in Russian].
11. N. N. Krasovskii, "On the application of A. M. Lyapunov's second method for time-delay equations," Prikl. Mat. Mekh. **20** (3), 315–327 (1956).
12. N. N. Krasovskii, *Some Problems in the Theory of Motion Stability* (Fizmatgiz, Moscow, 1959) [in Russian].
13. A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides* (Nauka, Moscow, 1985; Springer, Berlin, 1988).
14. N. Yu. Lukoyanov, *Functional Hamilton–Jacobi Equations and Control Problems with Hereditary Information* (Izd. Ural. Fed. Univ., Yekaterinburg, 2011) [in Russian].

Translated by E. Vasil'eva