

Hidden Supersymmetry as a Key to Constructing Low-Energy Superfield Effective Actions

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Abstract—In this review paper, we outline and exemplify the general method of constructing the superfield low-energy quantum effective action of supersymmetric Yang–Mills (SYM) theories with extended supersymmetry in the Coulomb phase, grounded upon the requirement of invariance under the non-manifest (hidden) part of the underlying supersymmetry. In this way we restore the $\mathcal{N} = 4$ supersymmetric effective actions in $4D$, $\mathcal{N} = 4$ SYM, $\mathcal{N} = 2$ supersymmetric effective actions in $5D$, $\mathcal{N} = 2$ SYM and $\mathcal{N} = (1, 1)$ supersymmetric effective actions in $6D$, $\mathcal{N} = (1, 1)$ SYM theories. The manifest off-shell fractions of the full supersymmetry are, respectively, $4D$, $\mathcal{N} = 2$, $5D$, $\mathcal{N} = 1$ and $6D$, $\mathcal{N} = (1, 0)$ supersymmetries. In all cases the effective actions depend on the corresponding covariant superfield SYM strengths and the hypermultiplet superfields. The whole construction essentially exploits a power of the harmonic superspace formalism.

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1. INTRODUCTION

Supersymmetry is still a source of various surprises in theoretical and mathematical physics. It is sufficient to say that supersymmetry allows one to construct completely finite models of quantum field theory, to formulate phenomenologically attractive models beyond the Standard Model, to eliminate ghosts in the spectrum of string theory, to obtain exact results in quantum mechanics and in the classical and quantum field theory, etc. Among a lot of works on supersymmetry, we wish to distinguish the papers by A. A. Slavnov [55, 72–74], which are directly or indirectly related to the problem of effective action.

Quantum effective action is a central object of quantum field theory, and it is used in the study of numerous aspects of the latter, such as renormalization, calculation of S -matrix amplitudes, finding the quantum corrections to the classical equations of motion, dynamical symmetry breaking, symmetries of quantum non-abelian gauge theories (they are described by Slavnov–Taylor identities [71, 75]), and many others (see, e.g., [78]).

The low-energy effective action plays an important role in supersymmetric gauge theories, providing a link between superstring/brane theory and quantum field theory. On the one hand, such an effective action can be calculated in the quantum field theory setting and, on the other, it can be derived within the brane stuff. As a result, the low-energy effective action allows one, in principle, to describe the low-energy string effects by methods of quantum field theory and vice versa (see the reviews [10, 22, 23]).

The most elegant way to study the quantum structure of supersymmetric field theories is through their formulations in terms of unconstrained superfields, which secure a manifest supersymmetry at

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all stages of calculations. Such a formulation is well developed for $4D$, $\mathcal{N} = 1$ supersymmetry (see, e.g., [26]). However, for higher dimensional and extended supersymmetric theories the formulation in terms of unconstrained superfields faces some problems, and it has been worked out only for a few special cases. One of these cases is $4D$, $\mathcal{N} = 2$ supersymmetry, where the successful formulation in terms of unconstrained superfields was realized in terms of harmonic superspace [46]. Using this approach allows one to formulate the four-dimensional maximally extended $\mathcal{N} = 4$ gauge theory in such a way that two supersymmetries are manifest and two others are on-shell and hidden (non-manifest). As a result, we arrive at $\mathcal{N} = 4$ supersymmetric formulation of the theory under consideration in terms of $\mathcal{N} = 2$ harmonic superfields.

Extended supersymmetry imposes severe constraints on the classical and effective quantum superfield actions of gauge theories. A good example is the four-derivative term in the low-energy $4D$, $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) effective action (in the Coulomb phase), which, in the sector of $\mathcal{N} = 2$ gauge multiplet, is accommodated by a non-holomorphic superfield potential [39]. An $\mathcal{N} = 4$ supersymmetric completion of this potential by the hypermultiplet terms in $\mathcal{N} = 2$ harmonic superspace was constructed for the first time in our paper [13] on the purely symmetry grounds. It was further reproduced in [21] from the quantum $\mathcal{N} = 2$ supergraph techniques. The origin of non-renormalizability of the $\mathcal{N} = 4$ SYM low-energy effective action against higher-loop quantum corrections was established, and links with the leading terms in the effective action of D3-brane on the $\text{AdS}_5 \times S^5$ background were indicated (see also [1, 32] and [23] for a review).

The key observation of [13] consisted in that the constraints of the second, hidden, $\mathcal{N} = 2$ supersymmetry completing the manifest $\mathcal{N} = 2$ supersymmetry to the total (on-shell) $\mathcal{N} = 4$ one are so strong that they fix the form of the relevant superfield potential up to an overall coefficient, which further has to be calculated from quantum considerations (like in [21]).¹

Later on, a similar approach was applied in $3D$ gauge theories, where it allowed one to determine the leading quantum corrections in $\mathcal{N} = 4$ SYM theory [35, 36] and to construct the $\mathcal{N} = 3$ superfield ABJM action [14]. It also turned out to be useful for revealing the structure of the leading terms of the effective action in $2D$ gauge theories with extended supersymmetry [69]. As the latest application of the method invented in [13], the complete structure of the leading terms in the low-energy effective actions of $5D$, $\mathcal{N} = 2$ and $6D$, $\mathcal{N} = (1, 1)$ SYM theories was established [15, 24].

In this review paper we explain the basics of our method from the single point of view, starting from the original example of [13] and then proceeding to the recent results of [15, 24]. We point out the decisive role of the harmonic superspace approach [43–46] for the derivation of the complete superfield effective actions in the situations when the superfield description with *all* relevant supersymmetries manifest and off-shell is still unknown.

The paper is organized as follows.

In Section 2 we discuss the $\mathcal{N} = 4$ supersymmetric low-energy effective action in $4D$, $\mathcal{N} = 4$ SYM theory. This theory is formulated in harmonic superspace in terms of the gauge multiplet and hypermultiplet superfields. The theory exhibits the manifest off-shell $\mathcal{N} = 2$ supersymmetry, as well as the second additional $\mathcal{N} = 2$ supersymmetry, which is non-manifest (hidden) and forms, together with the manifest $\mathcal{N} = 2$ supersymmetry, the whole $\mathcal{N} = 4$ supersymmetry only on-shell. Then we construct a quantum low-energy effective action in such a theory in the Coulomb phase. We start from the effective potential in the gauge multiplet sector calculated earlier in a series of papers [11, 12, 27, 30, 37, 38, 48, 49, 59, 60, 64, 65, 68] and show how this result can be completed by the hypermultiplet terms to the effective potential depending on all fields of the $\mathcal{N} = 4$ gauge multiplet. This completion is derived algebraically, solely on the basis of the extra on-shell $\mathcal{N} = 2$ supersymmetry, and so demonstrates the power of hidden supersymmetry for such calculations.

¹Earlier, the importance of taking into account the total supersymmetry for studying the effective action of extended supersymmetric gauge theories was pointed out in [30].

Section 3 is devoted to the derivation of the low-energy effective action in $5D$, $\mathcal{N} = 2$ supersymmetric gauge theory. As in Section 2, we begin with the classical formulation of the theory in harmonic superspace, where half of the supersymmetries is realized manifestly and another half in a hidden way. Then we study the structure of the low-energy effective action using both manifest and hidden supersymmetries. Here it is worth discussing an important point. The $5D$, $\mathcal{N} = 1$ gauge theory admits a classical Chern–Simons action; however, its $\mathcal{N} = 2$ supersymmetric generalization does not exist since in the $\mathcal{N} = 1$ case the Chern–Simons action respects the invariance under the $5D$, $\mathcal{N} = 1$ superconformal algebra $F(4)$, which is unique and possesses no higher \mathcal{N} extension. It is possible to show, by direct quantum computations in $5D$, $\mathcal{N} = 1$ superspace [34], that the two-derivative contributions (the Chern–Simons term) to the $\mathcal{N} = 2$ SYM effective action coming from the hypermultiplet and from the ghost superfields precisely cancel each other. This cancelation is analogous to the well-known phenomenon in the $3D$ case [35, 36], where the Chern–Simons term cannot arise as a quantum correction to the effective action in supersymmetric gauge theories with $\mathcal{N} > 2$.

As we demonstrate in this section, the four-derivative term, on the contrary, admits a unique hypermultiplet completion under the requirement of an implicit $5D$, $\mathcal{N} = 1$ on-shell supersymmetry alongside with the manifest off-shell $\mathcal{N} = 1$ one. The procedure of constructing such a hypermultiplet completion is quite analogous to the one in [13].

In Section 4 we address the problem of constructing the low-energy effective action in $6D$, $\mathcal{N} = (1, 1)$ SYM theory. Such a theory is formulated in $6D$, $\mathcal{N} = (1, 0)$ harmonic superspace as the theory of interacting $\mathcal{N} = (1, 0)$ gauge multiplet and hypermultiplet, both in the adjoint representation of the gauge group. The theory possesses a manifest $\mathcal{N} = (1, 0)$ supersymmetry and an additional hidden $\mathcal{N} = (0, 1)$ supersymmetry. On-shell they close on the full $\mathcal{N} = (1, 1)$ supersymmetry. Exploiting the hidden supersymmetry, we find the effective action in the gauge multiplet sector basically on the symmetry grounds.

In Section 5 we list the basic results and discuss some further possible developments.

2. LOW-ENERGY EFFECTIVE ACTION OF $4D$, $\mathcal{N} = 4$ SYM THEORY

It is known that D3-branes are related to $4D$, $\mathcal{N} = 4$ SYM theory (see, e.g., [5, 47]). The interaction of D3-branes is described in the abelian bosonic sector by the Born–Infeld action, the leading low-energy correction being of the form $\sim F^4/X^4$, where F^4 denotes a structure of the fourth degree in an abelian field strength F_{mn} and X stands for the scalar fields of $4D$, $\mathcal{N} = 4$ gauge (vector) multiplet. The one-loop calculation of such an effective action in the Coulomb branch of $\mathcal{N} = 4$ SYM theory, both in the component approach and in terms of $\mathcal{N} = 1, 2$ superfields, has been accomplished in [11, 12, 27, 30, 37, 48, 49, 59, 60, 65, 68]. The complete $\mathcal{N} = 4$ structure of the one-loop low-energy effective action has been found in [13, 21]. The two-loop contributions to the low-energy effective actions of $\mathcal{N} = 4$ SYM theory have been considered in [31, 56, 61]. The structure of the low-energy effective action in the mixed Coulomb–Higgs branch was a subject of [33]. A review of the results related to the calculations of low-energy effective actions in $4D$ extended supersymmetric gauge theories can be found, for example, in [10, 22].²

Studying the low-energy effective action³ of $\mathcal{N} = 4$ SYM models was initiated in [39]. In the $\mathcal{N} = 2$ superfield formulation, the full $\mathcal{N} = 4$ gauge multiplet is constituted by the $\mathcal{N} = 2$ gauge multiplet and hypermultiplet. The authors of [39] studied the effective action of $\mathcal{N} = 4$ SYM theory with the gauge group $SU(2)$ spontaneously broken to $U(1)$ and considered that part of this action which depends only on the fields of massless $U(1)$ $\mathcal{N} = 2$ gauge multiplet. The requirements of scale invariance and R-invariance specify this part of the effective action up to a numerical coefficient.

²Various aspects of $\mathcal{N} = 2$ harmonic superfield models were also discussed in [2, 3, 25, 28, 29].

³By the low-energy effective action we always mean the leading (in the external momenta) piece of the full quantum effective action.

The result can be given in terms of the non-holomorphic effective potential

$$\mathcal{H}(W, \bar{W}) = c \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda}, \quad (2.1)$$

where W and \bar{W} are the $\mathcal{N} = 2$ superfield strengths, Λ is an arbitrary scale, and c is an arbitrary real constant. The effective action defined as an integral of $\mathcal{H}(W, \bar{W})$ over the full $\mathcal{N} = 2$ superspace with the coordinates $z = (x^m, \theta_{\alpha i}, \bar{\theta}_{\dot{\alpha}}^i)$ is independent of the scale Λ . It is worth pointing out that the result (2.1) was obtained in $\mathcal{N} = 4$ SYM theory entirely on the symmetry grounds.⁴

Equation (2.1) provides the *exact* form of the low-energy effective action of $\mathcal{N} = 4$ SYM theory in the $\mathcal{N} = 2$ gauge superfield sector. Any quantum corrections can be absorbed into the coefficient c . One can show [39, 65] that the non-holomorphic effective potential (2.1) receives neither perturbative nor non-perturbative contributions beyond one loop. As a result, the construction of the exact low-energy effective action for SU(2) SYM theory in the Coulomb branch (i.e., with SU(2) broken down to U(1)) is reduced to computing the coefficient c in the one-loop approximation.

The direct derivation of the potential (2.1), computation of the coefficient c , and hence the final reconstruction of the full exact low-energy U(1) effective action in the gauge field sector from the quantum $\mathcal{N} = 4$ SYM theory were undertaken in [27, 49, 68]. In particular, it was found that $c = (4\pi)^{-2}$. Further studies showed that the result (2.1) for the gauge group SU(2) spontaneously broken to U(1) can be generalized to the group SU(N) broken to its maximal abelian subgroup [11, 37, 48, 65]. The relevant one-loop effective potential is given by

$$\mathcal{H}(W, \bar{W}) = c \sum_{I < J} \ln \frac{W^I - W^J}{\Lambda} \ln \frac{\bar{W}^I - \bar{W}^J}{\Lambda}, \quad (2.2)$$

with the same coefficient c as in (2.1) for the SU(2) group. Here $I, J = 1, 2, \dots, N$, $W = \sum_I W^I e_{II}$ belongs to the Cartan subalgebra of the algebra $\mathfrak{su}(N)$, $\sum_I W^I = 0$, and e_{IJ} is the Weyl basis in the $\mathfrak{su}(N)$ algebra (for details see [11]).

2.1. Action of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace. The “microscopic” action of $\mathcal{N} = 4$ SYM theory in the formulation through $\mathcal{N} = 2$ harmonic superfields can be written as

$$S[V^{++}, q^+] = \frac{1}{8} \left(\int d^8 \zeta_L \operatorname{tr} W^2 + \int d^8 \zeta_R \operatorname{tr} \bar{W}^2 \right) - \frac{1}{2} \int d\zeta^{(-4)} \operatorname{tr} q^{+a} (D^{++} + igV^{++}) q_a^+. \quad (2.3)$$

The real analytic superfield V^{++} is the harmonic gauge potential of $\mathcal{N} = 2$ SYM theory, and the analytic superfields q_a^+ , $a = 1, 2$, describe the hypermultiplets (they satisfy the pseudo-reality condition $q^{+a} \equiv \tilde{q}_a^+ = \varepsilon^{ab} q_b^+$, with the generalized conjugation \sim defined in [43]). The $\mathcal{N} = 2$ superfield strengths W and \bar{W} are expressed in terms of V^{++} . The superfields V^{++} and q_a^+ belong to the adjoint representation of the gauge group, g is a coupling constant, $d^8 \zeta_L = d^4 x d^2 \theta^+ d^2 \theta^- du$, $d^8 \zeta_R = d^4 x d^2 \bar{\theta}^+ d^2 \bar{\theta}^- du$, and $d\zeta^{(-4)} = d^4 x d^2 \theta^+ d^2 \bar{\theta}^- du$ are the measures of integration over chiral, anti-chiral, and harmonic analytic $\mathcal{N} = 2$ superspaces, and du is the measure of integration over the harmonic variables $u^{\pm i}$, $u^{+i} u_i^- = 1$. Any further details regarding the action (2.3), including the precise form of the analyticity-preserving harmonic derivative D^{++} , can be found in [43–46]. We will basically follow the notation of the book [46].

Either term in (2.3) is manifestly $\mathcal{N} = 2$ supersymmetric. Moreover, the action (2.3) possesses an extra hidden $\mathcal{N} = 2$ supersymmetry which mixes up W and \bar{W} with q_a^+ (see [11, 44–46]). As a result, the model under consideration is actually $\mathcal{N} = 4$ supersymmetric. Our aim is to examine the

⁴Non-holomorphic potentials of the form (2.1) as possible contributions to the effective action in $\mathcal{N} = 2$ SYM theories were earlier considered in [38].

possibility of constructing $\mathcal{N} = 4$ supersymmetric functionals whose q^+ -independent parts would have the form of (2.1), (2.2).

The effective potentials (2.1) and (2.2) involve the chiral and anti-chiral abelian strengths W and \bar{W} satisfying the free classical equations of motion $(D^+)^2 W = (\bar{D}^+)^2 \bar{W} = 0$, where the harmonic projections of the spinor $\mathcal{N} = 2$ derivatives D_α^i and \bar{D}_α^i are defined as $D_\alpha^\pm = D_\alpha^i u_i^\pm$ and $\bar{D}_\alpha^\pm = \bar{D}_\alpha^i u_i^\pm$. So, in order to construct the above functionals, we need to know the hidden $\mathcal{N} = 2$ supersymmetry transformations only for on-shell W and \bar{W} and, respectively, for on-shell q^{+a} ($D^{++} q^{+a} = 0$). For further use, it is instructive to write down the complete set of equations for the involved quantities, both on- and off-shell. *Off-shell* we have

$$\bar{D}_\alpha^\pm W = D_\alpha^\pm \bar{W} = 0, \quad (D^\pm)^2 W = (\bar{D}^\pm)^2 \bar{W}, \quad D_\alpha^+ q^{+a} = \bar{D}_\alpha^+ q^{+a} = 0.$$

On-shell we have

$$(D^\pm)^2 W = (\bar{D}^\pm)^2 \bar{W} = 0, \quad D^{++} q^{+a} = D^{--} q^{-a} = 0, \\ q^{-a} \equiv D^{--} q^{+a}, \quad D^{++} q^{-a} = q^{+a}, \quad D_\alpha^- q^{-a} = \bar{D}_\alpha^- q^{-a} = 0.$$

In checking the on-shell relations for the hypermultiplet superfield an essential use of the commutation relation $[D^{++}, D^{--}] = D^0$ should be made, with D^0 being the operator which counts harmonic U(1) charges, $D^0 q^{\pm a} = \pm q^{\pm a}$.

It is known that, in the central basis of the harmonic superspace,

$$q^{\pm a} = q^{ia}(z) u_i^\pm, \tag{2.4}$$

where $q^{ia}(z)$ is the on-shell hypermultiplet superfield independent of harmonic variables and defined on the standard $\mathcal{N} = 2$ superspace with the coordinates $z = (x^m, \theta_{\alpha i}, \bar{\theta}_\alpha^i)$. Note that in this on-shell description, harmonic variables are to some extent redundant, and everything can be formulated in terms of ordinary $\mathcal{N} = 2$ superfields $W(z)$, $\bar{W}(z)$, and $q^{ia}(z)$. The use of the harmonic superspace language is still convenient, e.g., because of the possibility to integrate by parts with respect to the harmonic derivatives in the effective action.

Taking into account these remarks, we can write the on-shell form of the hidden $\mathcal{N} = 2$ transformations as [46]

$$\delta W = \frac{1}{2} \bar{\epsilon}^{\dot{\alpha} a} \bar{D}_\alpha^- q_a^+, \quad \delta \bar{W} = \frac{1}{2} \epsilon^{\alpha a} D_\alpha^- q_a^+, \tag{2.5} \\ \delta q_a^+ = \frac{1}{4} (\epsilon_a^\beta D_\beta^+ W + \bar{\epsilon}_a^{\dot{\alpha}} \bar{D}_\alpha^+ \bar{W}), \quad \delta q_a^- = \frac{1}{4} (\epsilon_a^\beta D_\beta^- W + \bar{\epsilon}_a^{\dot{\alpha}} \bar{D}_\alpha^- \bar{W}),$$

where $\epsilon^{\alpha a}$ and $\bar{\epsilon}^{\dot{\alpha} a}$ are the Grassmann transformation parameters.

2.2. The Coulomb phase effective action. We start with the calculation of the $\mathcal{N} = 4$ supersymmetric low-energy effective action extending the non-holomorphic $\mathcal{N} = 2$ superfield potential (2.1). This action is assumed to have the following general form:

$$\Gamma[W, \bar{W}, q^+] = \int d^{12}z du [\mathcal{H}(W, \bar{W}) + \mathcal{L}_q(W, \bar{W}, q^+)] = \int d^{12}z du \mathcal{L}_{\text{eff}}(W, \bar{W}, q^+). \tag{2.6}$$

Here $d^{12}z$ is the full $\mathcal{N} = 2$ superspace integration measure, $\mathcal{H}(W, \bar{W})$ is given by (2.1), and $\mathcal{L}_q(W, \bar{W}, q^+)$ is some (for the moment unknown) function which should ensure, together with $\mathcal{H}(W, \bar{W})$, the invariance of the functional (2.6) with respect to the transformations (2.5). Note that the Lagrangian $\mathcal{L}_q(W, \bar{W}, q^+)$, being a function of on-shell superfields, must be in fact independent of the harmonics u_i^\pm .

The first term in (2.6) is transformed under (2.5) as

$$\delta \int d^{12}z du \mathcal{H}(W, \bar{W}) = \frac{1}{2}c \int d^{12}z du \frac{q^{+a}}{\bar{W}W} (\epsilon_a^\alpha D_\alpha^- W + \bar{\epsilon}_a^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}^- \bar{W}). \quad (2.7)$$

Then $\mathcal{L}_q(W, \bar{W}, q^+)$ is to be determined from the condition that its variation cancels the variation (2.7).

We introduce the quantity

$$\mathcal{L}_q^{(1)} \equiv -c \frac{q^{+a} q_a^-}{\bar{W}W} \quad (2.8)$$

and notice that it transforms as

$$\delta \frac{q^{+a} q_a^-}{\bar{W}W} = \frac{q^{+a}}{2\bar{W}W} (\epsilon_a^\alpha D_\alpha^- W + \bar{\epsilon}_a^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}^- \bar{W}) + (q^{+a} q_a^-) \delta \left(\frac{1}{\bar{W}W} \right) + D^{--} \left(\frac{\delta q^{+a} q_a^+}{\bar{W}W} \right). \quad (2.9)$$

Then we consider

$$\mathcal{L}_{\text{eff}}^{(1)} = \mathcal{H}(W, \bar{W}) - c \frac{q^{+a} q_a^-}{\bar{W}W} = \mathcal{H}(W, \bar{W}) + \mathcal{L}_q^{(1)}. \quad (2.10)$$

Under the full harmonic $\mathcal{N} = 2$ superspace integral, the variation (2.7) in $\mathcal{L}_{\text{eff}}^{(1)}$ is canceled by the first term in (2.9). The variation of (2.10) generated by the second term in (2.9) remains non-canceled. After some algebra, it can be brought into the form

$$\begin{aligned} \delta \int d^{12}z du \mathcal{L}_{\text{eff}}^{(1)} &= \frac{c}{2} \int d^{12}z du \frac{q^{+b} q_b^-}{(\bar{W}W)^2} (\bar{W} \bar{\epsilon}^{\dot{\alpha}a} \bar{D}_{\dot{\alpha}}^- q_a^+ + W \epsilon^{\alpha a} D_\alpha^- q_a^+) \\ &= -\frac{c}{3} \int d^{12}z du \frac{q^{+b} q_b^-}{(\bar{W}W)^2} q^{+a} (\bar{\epsilon}_a^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}^- \bar{W} + \epsilon_a^\alpha D_\alpha^- W), \end{aligned} \quad (2.11)$$

where we have integrated by parts and used the off- and on-shell relations for W , \bar{W} , and q_a^\pm together with cyclic identities for the $SU(2)$ doublet indices.

Now let us consider the quantity

$$\mathcal{L}_{\text{eff}}^{(2)} = \mathcal{L}_{\text{eff}}^{(1)} + \frac{c}{3} \left(\frac{q^{+a} q_a^-}{\bar{W}W} \right)^2 \equiv \mathcal{L}_{\text{eff}}^{(1)} + \mathcal{L}_q^{(2)}, \quad (2.12)$$

where $\mathcal{L}_{\text{eff}}^{(1)}$ is given by (2.10). The coefficient in the new term $\mathcal{L}_q^{(2)}$ has been picked up so that the variation of the numerator of this term cancel (2.11). The rest of the full variation of $\mathcal{L}_q^{(2)}$ once again survives, and in order to cancel it, one must add an appropriate term $\mathcal{L}_q^{(3)}$ to $\mathcal{L}_q^{(1)} + \mathcal{L}_q^{(2)}$, and so forth.

The above consideration implies that the q^{+a} -dependent part of the full effective action (2.6), $\mathcal{L}_q = \mathcal{L}_q(W, \bar{W}, q^+)$, should be of the form

$$\mathcal{L}_q = \sum_{n=1}^{\infty} \mathcal{L}_q^{(n)} = c \sum_{n=1}^{\infty} c_n \left(\frac{q^{+a} q_a^-}{\bar{W}W} \right)^n \quad (2.13)$$

with some initially unknown coefficients c_n . We have already specified $c_1 = -1$ and $c_2 = 1/3$. The further analysis proceeds by induction.

Let us consider two adjacent terms in the general expansion (2.13),

$$c_{n-1} \left(\frac{q^{+a} q_a^-}{\bar{W}W} \right)^{n-1} + c_n \left(\frac{q^{+a} q_a^-}{\bar{W}W} \right)^n, \quad (2.14)$$

and assume that the variation of the numerator of the first term has been already used to cancel the remaining part of the variation of the preceding term (under the integral over the total harmonic

superspace, as in (2.6)). Then we rearrange the rest of the full variation of the first term as in (2.11) and require this part to be canceled by the variation of the numerator of the second term in (2.14). This results in the following recursive relation:

$$c_n = -2 \frac{(n-1)^2}{n(n+1)} c_{n-1} \tag{2.15}$$

and $c_1 = -1$. This immediately yields

$$c_n = \frac{(-2)^n}{n^2(n+1)}. \tag{2.16}$$

As a result, the full structure of \mathcal{L}_q is determined to be

$$\mathcal{L}_q(W, \bar{W}, q^+) \equiv \mathcal{L}_q(X) = c \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} X^n = c \left\{ (X-1) \frac{\ln(1-X)}{X} + [\text{Li}_2(X) - 1] \right\}, \tag{2.17}$$

where $X = -2q^{+a}q_a^- / (\bar{W}W)$ and $\text{Li}_2(X)$ is the Euler dilogarithm. Let us point out that the expression for X does not depend on harmonics due to the on-shell representation (2.4),

$$X = -\frac{q^{ia}q_{ia}}{\bar{W}W}. \tag{2.18}$$

Therefore, $\mathcal{L}_q(X)$ does not depend on harmonics on-shell either, and the integral over harmonics in the effective action (2.6) can be omitted.

Thus, the full $\mathcal{N} = 4$ supersymmetric low-energy effective action for the $\mathcal{N} = 4$ SYM model with gauge group $\text{SU}(2)$ spontaneously broken down to $\text{U}(1)$ is given by

$$\Gamma[W, \bar{W}, q^+] = \int d^{12}z \mathcal{L}_{\text{eff}}(W, \bar{W}, q^+), \tag{2.19}$$

$$\mathcal{L}_{\text{eff}}(W, \bar{W}, q^+) = \mathcal{H}(W, \bar{W}) + \mathcal{L}_q(X), \tag{2.20}$$

where $\mathcal{H}(W, \bar{W})$ and $\mathcal{L}_q(X)$ are given by (2.1) and (2.17), respectively, and X , by (2.18).⁵

The expression (2.17) is the exact low-energy result. Indeed, the non-holomorphic effective potential $\mathcal{H}(W, \bar{W})$ (2.1) is exact, as was argued in [39]. The Lagrangian $\mathcal{L}_q(X)$ (2.17) was uniquely restored from (2.1) by $\mathcal{N} = 4$ supersymmetry, and it is the only one forming, together with $\mathcal{H}(W, \bar{W})$, an invariant of $\mathcal{N} = 4$ supersymmetry. Therefore, the functional (2.19), (2.20) is the exact low-energy effective action for the theory under consideration.

Let us elaborate on the component structure of the full effective action (2.19), (2.20). We consider only its bosonic part, so that

$$\begin{aligned} W &= \varphi(x) + 4i\theta_{(\alpha}^+\theta_{\beta)}^- F^{(\alpha\beta)}(x), & \bar{W} &= \bar{\varphi}(x) + 4i\bar{\theta}_{(\dot{\alpha}}^+\bar{\theta}_{\dot{\beta})}^- \bar{F}^{(\dot{\alpha}\dot{\beta})}(x), & q^{ia} &= f^{ia}(x), \\ D_{\alpha}^+ D_{\beta}^- W &= -4iF_{(\alpha\beta)}, & \bar{D}_{\dot{\alpha}}^+ \bar{D}_{\dot{\beta}}^- \bar{W} &= 4i\bar{F}_{(\dot{\alpha}\dot{\beta})}. \end{aligned} \tag{2.21}$$

Here $\varphi(x)$ is the complex scalar field of the $\mathcal{N} = 2$ gauge multiplet, $F^{\alpha\beta}(x)$ and $\bar{F}^{\dot{\alpha}\dot{\beta}}(x)$ are the self-dual and anti-self-dual components of the abelian field strength F_{mn} , and $f^{ia}(x)$ collects four scalar fields of the hypermultiplet $q^{ia}(z)$. In this bosonic approximation, the functional argument X (2.18) becomes

$$X|_{\theta=0} = -\frac{f^{ia}f_{ia}}{|\varphi|^2} \equiv X_0. \tag{2.22}$$

⁵The functional (2.19) contains only quantum corrections. To write the whole effective action, we have to add the classical action to the functional (2.19).

We ignore all x -derivatives of the involved fields, since we are interested only in the leading part of the expansion of the full effective action in the external momenta.

The component form of the effective action (2.19) can be straightforwardly computed by performing integration over the θ 's. After some computations we obtain a remarkably simple result in the bosonic sector,

$$\Gamma_{\text{bos}} = 4c \int d^4x \frac{F^2 \bar{F}^2}{(|\varphi|^2 + f^{ia} f_{ia})^2}, \quad (2.23)$$

where $F^2 = F^{\alpha\beta} F_{\alpha\beta}$ and $\bar{F}^2 = \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}}$. The expression in the denominator is nothing other than the SU(4)-invariant square of six scalar fields of the $\mathcal{N} = 4$ vector multiplet. After proper redefinitions, it can be cast in the manifestly SU(4)-invariant form

$$|\varphi|^2 + f^{ia} f_{ia} \sim \phi^{AB} \bar{\phi}_{AB}, \quad \phi^{AB} = -\phi^{BA}, \quad \bar{\phi}_{AB} = \frac{1}{2} \varepsilon_{ABCD} \phi^{CD}, \quad A, B, C, D = 1, \dots, 4.$$

This indicates that the effective action (2.19), besides being $\mathcal{N} = 4$ supersymmetric, also possesses hidden invariance under the R-symmetry group SU(4)_R of $\mathcal{N} = 4$ supersymmetry.

The result (2.19) can be generalized to the theory with gauge group SU(N) spontaneously broken down to [U(1)] ^{$N-1$} . In this case the effective action is given by the general expression (2.17), where $\mathcal{H}(W, \bar{W})$ has the form (2.2) and

$$\mathcal{L}_q(W, \bar{W}, q^+) = \sum_{I < J} \mathcal{L}_q^{IJ}(W, \bar{W}, q^+), \quad (2.24)$$

with each \mathcal{L}_q^{IJ} being of the form (2.17), in which X should be replaced by

$$X_{IJ} = -2 \frac{q_{IJ}^{+a} q_{aIJ}^-}{W_{IJ} \bar{W}_{IJ}} = - \frac{q_{IJ}^{ia} q_{iaIJ}}{W_{IJ} \bar{W}_{IJ}}, \quad (2.25)$$

$$W_{IJ} = W^I - W^J, \quad \bar{W}_{IJ} = \bar{W}^I - \bar{W}^J, \quad q_{IJ}^{+a} = q_I^{+a} - q_J^{+a}. \quad (2.26)$$

The hypermultiplet superfields are $q^{+a} = \sum_I q_I^{+a} e_{II}$, $\sum_I q_I^{+a} = 0$, and e_{IJ} is the Weyl basis in the algebra su(N). These hypermultiplet superfields belong to the Cartan subalgebra of su(N). In the SU(N) case the bosonic effective action is represented by a sum of terms (2.23).

As a final remark we note that the functional arguments X (2.18), (2.25) have the zero dilatation weight and are scalars of the U(1) R-symmetry, since $q^{\pm a}$ and W have the same dilatation weights [46] and $q^{\pm a}$ behave as scalars under the R-symmetry group. So, the full effective action (2.19) and its su(N) analog are expected to be invariant under $\mathcal{N} = 2$ superconformal symmetry like their pure W, \bar{W} part (2.1) or (2.2) (see [30]). Being also $\mathcal{N} = 4$ supersymmetric, these actions respect the whole (on-shell) $\mathcal{N} = 4$ superconformal symmetry.

3. LOW-ENERGY EFFECTIVE ACTION OF 5D, $\mathcal{N} = 2$ SYM THEORY

In this section, we study the implications of extended supersymmetry for the low-energy effective action of 5D SYM theory. This theory is of interest from several points of view. It is non-renormalizable by power counting because of the dimensionful coupling constant g , $[g] = -1/2$. Nevertheless, it was argued that a non-perturbative quantum completion of this model describes 6D, $\mathcal{N} = (2, 0)$ superconformal field theory compactified on a circle [40, 62, 63]. An additional support to this conjecture came from the exact computations of the partition function in this theory by the localization technique (see, e.g., the review [23] and the references therein).

In spite of the non-renormalizability of 5D, $\mathcal{N} = 1$ SYM, it is still reasonable to study one-loop quantum corrections in it, keeping in mind that in the odd-dimensional field theories divergences can appear (within the dimensional regularization) only at even loops. One-loop contributions to

the effective action of $5D, \mathcal{N} = 1$ SYM theory were calculated in [34, 57] for the gauge group $SU(2)$ spontaneously broken to $U(1)$. The leading contribution is given by the $5D$ supersymmetric Chern–Simons term [57], while the next-to-leading one reads [34]

$$c_0 \int d^5|8z du W \ln \frac{W}{\Lambda}, \tag{3.1}$$

where W is the $5D, \mathcal{N} = 1$ abelian gauge superfield strength, Λ is a scale parameter, $[\Lambda] = 1$, and the integration is over the full $\mathcal{N} = 1$ harmonic superspace with the measure $d^5|8z du \equiv d^5x d^8\theta du$. It is easy to check that the action (3.1) is Λ -independent. The Chern–Simons term incorporates two-derivative quantum corrections to the effective action, while (3.1) is an $\mathcal{N} = 1$ superfield extension of the four-derivative “ F^4/ϕ^3 ”-terms.

Our purpose is to study the leading terms in the low-energy effective action of $5D, \mathcal{N} = 2$ SYM theory in harmonic superspace. Although such terms might be found by direct quantum computations in $5D, \mathcal{N} = 1$ superspace, we determine them here on the symmetry grounds, just by constructing an $\mathcal{N} = 2$ completion of the $5D, \mathcal{N} = 1$ SYM effective action by the proper hypermultiplet terms. The effective action constructed corresponds to the Coulomb branch of $5D, \mathcal{N} = 2$ SYM theory, with the gauge group being broken to some abelian subgroup (for example, the maximal torus), and, in general, involves the massless abelian $\mathcal{N} = 2$ gauge multiplets valued in the algebra of this subgroup. For simplicity, we focus on the case of the gauge group $SU(2)$ and only briefly address (in Subsection 3.3) the case of $SU(N)$ gauge symmetry.

An additional motivation for studying the quantum effective action of $5D, \mathcal{N} = 2$ SYM theory comes from the D-brane stuff, as in the previous $4D, \mathcal{N} = 4$ example. The classical action of $5D, \mathcal{N} = 1$ SYM theory with $U(N)$ gauge group can be interpreted as an action of a stack of N D4-branes in flat space–time. Then the $\mathcal{N} = 2$ supersymmetric completion of the $5D, \mathcal{N} = 1$ SYM effective action can be identified with that of the four-derivative term in the low-energy effective action of a single D4-brane on the $AdS_6 \times S^4$ background.

3.1. Classical action. We start our consideration with a brief account of the $\mathcal{N} = 1$ SYM and hypermultiplet models in $5D$ harmonic superspace. We follow the notation and conventions of [34, 58].

The $\mathcal{N} = 2$ gauge multiplet in $5D, \mathcal{N} = 1$ harmonic superspace is described by a pair of analytic superfields (V^{++}, q_a^+) , where V^{++} is the $\mathcal{N} = 1$ gauge multiplet and $q_a^+ \equiv (q^+, -\bar{q}^+)$ is the hypermultiplet. The classical action of V^{++} is written as an integral over the full harmonic superspace [80]:

$$S_{\text{YM}} = \frac{1}{2g^2} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \text{tr} \int d^5|8z du_1 \dots du_n \frac{V^{++}(z, u_1) V^{++}(z, u_2) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) (u_2^+ u_3^+) \dots (u_n^+ u_1^+)}, \tag{3.2}$$

g being a coupling constant of dimension $-1/2$. The V^{++} equation of motion reads

$$(\mathcal{D}^+)^2 W = 0, \tag{3.3}$$

where $(\mathcal{D}^+)^2 \equiv \mathcal{D}^{+\hat{\alpha}} \mathcal{D}_{\hat{\alpha}}^+$ and W is a superfield strength of the gauge $\mathcal{N} = 1$ multiplet:

$$W = \frac{i}{8} (\mathcal{D}^+)^2 V^{--}. \tag{3.4}$$

Here, the connection V^{--} is related to V^{++} through the harmonic flatness condition

$$D^{++} V^{--} - D^{--} V^{++} + i[V^{++}, V^{--}] = 0. \tag{3.5}$$

The classical action of the hypermultiplet in the adjoint representation of the gauge group is written as [43–45]

$$S_q = \frac{1}{2g^2} \text{tr} \int d\zeta^{(-4)} q_a^+ \mathcal{D}^{++} q^{+a}, \tag{3.6}$$

where $d\zeta^{(-4)}$ is the measure of integration over the analytic superspace and $\mathcal{D}^{++} = D^{++} + iV^{++}$ is the gauge-covariant harmonic derivative. The equation of motion for q_a^+ is

$$\mathcal{D}^{++}q_a^+ = 0. \quad (3.7)$$

The action of the $\mathcal{N} = 2$ gauge multiplet in the $\mathcal{N} = 1$ harmonic superspace formulation is just the sum of (3.2) and (3.6),

$$S^{\mathcal{N}=2} = S_{\text{YM}} + S_q. \quad (3.8)$$

This action is invariant under an implicit $\mathcal{N} = 1$ supersymmetry

$$\delta q_a^+ = -\frac{1}{2}(D^+)^4[\epsilon_{a\hat{\alpha}}\theta^{-\hat{\alpha}}V^{--}], \quad \delta V^{++} = \epsilon_{\hat{\alpha}}^a\theta^{+\hat{\alpha}}q_a^+, \quad (3.9)$$

where $\epsilon_{\hat{\alpha}}^a$ is the relevant anticommuting parameter. Although equation (3.3) is modified for the total action (3.8) by the hypermultiplet source term on the right-hand side, this is not the case for the massless Cartan-subalgebra valued abelian superfields, which we will be interested in. In the abelian case, the equations of motion for the $\mathcal{N} = 1$ gauge multiplet (3.3) and hypermultiplet (3.7) are simplified to the form

$$(D^+)^2W = 0, \quad D^{++}q_a^+ = 0. \quad (3.10)$$

It is straightforward to show that on these equations the implicit supersymmetry transformations (3.9) are reduced to

$$\delta q_a^\pm = \frac{i}{2}\epsilon_{\hat{\alpha}}^a(D_{\hat{\alpha}}^\pm W), \quad \delta W = -\frac{i}{8}\epsilon_{\hat{\alpha}}^a D^{-\hat{\alpha}}q_a^+ + \frac{i}{8}\epsilon_{\hat{\alpha}}^a D^{+\hat{\alpha}}q_a^-. \quad (3.11)$$

3.2. $\mathcal{N} = 2$ effective action. In this subsection, we construct the complete low-energy effective action of 5D, $\mathcal{N} = 2$ SYM theory with the gauge group $\text{SU}(2)$ and both the gauge $\mathcal{N} = 1$ SYM and the hypermultiplet sectors taken into account.

The part of the superfield $\mathcal{N} = 1$ SYM effective action containing the component four-derivative term reads [34]

$$S_0 = c_0 \int d^{5|8}z du W \ln \frac{W}{\Lambda}, \quad (3.12)$$

where W is the abelian gauge superfield strength, Λ is a scale parameter, c_0 is a dimensionless constant, and the integration is performed over the full $\mathcal{N} = 1$ harmonic superspace with the measure $d^{5|8}z du \equiv d^5x d^8\theta du$. The representation (3.4) implies $\int d^{5|8}z du W = 0$, so the action (3.12) is independent of the scale Λ , $dS_0/d\Lambda = 0$.

The precise value of the constant c_0 in the effective action (3.12) depends on the gauge group representation content of the hypermultiplet matter [34]. Here, we do not fix this constant and construct an $\mathcal{N} = 2$ supersymmetric generalization of (4.1), keeping c_0 arbitrary. This construction follows the same steps as in [13] (and in the previous Section 2).

The variation of the action (3.12) under the hidden supersymmetry transformations (3.11) may be cast in the form

$$\delta S_0 = \frac{ic_0}{4} \int d^{5|8}z du \epsilon_{\hat{\alpha}}^a q_a^+ \frac{D^{-\hat{\alpha}}W}{W}. \quad (3.13)$$

In deriving this equation we employed the abelian counterparts of relations (3.4) and (3.5), the equations of motion (3.10), and integration by parts with respect to the harmonic and covariant spinor derivatives.

The variation (3.13) may be partly canceled by the variation of the action

$$S_1 = c_1 \int d^{5|8}z du \frac{q^{+a}q_a^-}{W}, \quad (3.14)$$

where the coefficient c_1 will be defined below. The variation of this action under (3.11) reads

$$\delta S_1 = ic_1 \int d^5|8 z du \frac{q^{+a} \epsilon_a^{\hat{\alpha}} (D_{\hat{\alpha}}^- W)}{W} - \frac{i}{8} c_1 \int d^5|8 z du \frac{q^{+a} q_a^-}{W^2} (\epsilon_{\hat{\alpha}}^b D^{+\hat{\alpha}} q_b^- - \epsilon_{\hat{\alpha}}^b D^{-\hat{\alpha}} q_b^+). \quad (3.15)$$

The first term on the right-hand side of (3.15) cancels the variation (3.13) if

$$c_1 = -\frac{c_0}{4}, \quad (3.16)$$

while the last term in (3.15) may be cast in the form

$$\delta(S_0 + S_1) = -\frac{ic_0}{12} \int d^5|8 z du \frac{q^{+a} q_a^-}{W^3} \epsilon_{\hat{\alpha}}^b q_b^+ D^{-\hat{\alpha}} W. \quad (3.17)$$

To cancel this expression, we are led to add the new term

$$S_2 = c_2 \int d^5|8 z du \frac{(q^{+a} q_a^-)^2}{W^3}, \quad c_2 = \frac{c_0}{24}. \quad (3.18)$$

Instead of evaluating the variation of the term (3.18), we proceed to the general case and look for the full $\mathcal{N} = 2$ effective action in the form

$$S_{\text{eff}}^{\mathcal{N}=2} = \int d^5|8 z du \left[c_0 W \ln \frac{W}{\Lambda} + \sum_{n=1}^{\infty} c_n \frac{(q^{+a} q_a^-)^n}{W^{2n-1}} \right] \quad (3.19)$$

with some coefficients c_n . Let us select two adjacent terms in the sum in (3.19):

$$c_n \frac{(q^{+a} q_a^-)^n}{W^{2n-1}} + c_{n+1} \frac{(q^{+a} q_a^-)^{n+1}}{W^{2n+1}}. \quad (3.20)$$

It is possible to show that the variation of the denominator in the first term cancels the variation of the numerator in the second term if the coefficients are related as

$$(n+1)c_{n+1} = -c_n \frac{n(2n-1)}{n+2}. \quad (3.21)$$

Taking into account equation (3.16), we find from this recurrence relation the generic coefficient

$$c_n = \frac{(-1)^n (2n-2)!}{2^n n! (n+1)!} c_0. \quad (3.22)$$

This allows us to sum up the series in (3.19) and to represent the effective action in the closed form

$$S_{\text{eff}}^{\mathcal{N}=2} = c_0 \int d^5|8 z du W \left[\ln \frac{W}{\Lambda} + \frac{1}{2} H(Z) \right], \quad (3.23)$$

where

$$Z \equiv \frac{q^{+a} q_a^-}{W^2} \quad (3.24)$$

and

$$H(Z) = 1 + 2 \ln \frac{1 + \sqrt{1+2Z}}{2} + \frac{2}{3} \frac{1}{1 + \sqrt{1+2Z}} - \frac{4}{3} \sqrt{1+2Z}. \quad (3.25)$$

It is easy to check that $H(0) = 0$ and $H'(0) = -1/2$, in agreement with (3.22). The result (3.23) was derived in the work [24].

The action (3.23) is an $\mathcal{N} = 2$ supersymmetric extension of the effective action (3.12). It would be interesting to reproduce this result from the perturbative quantum computations in $5D$ harmonic superspace, as it was done in the $4D$, $\mathcal{N} = 4$ case in [1, 21, 32].

It is worth pointing out that the term (3.1) we have started with (as well as its analogs for the higher-rank gauge groups) may arise in quantum theory only as a one-loop quantum correction to the effective action. Indeed, it is scale-invariant and so is independent of the gauge coupling constant g . On the other hand, within the background field method in harmonic superspace [12, 34], all higher-loop Feynman graphs involve a gauge superfield vertex with the coupling constant g . Thus, all higher-loop quantum contributions to the effective action are not scale-invariant and for this reason cannot give rise to a renormalization of the coefficient c_0 in equation (3.1). However, in contrast to the $4D$ case, this coefficient is not protected against non-perturbative corrections. Such corrections will be discussed elsewhere.

It is straightforward to generalize this result to a higher-rank gauge group. For instance, for the $SU(N)$ gauge group spontaneously broken to the maximal torus $[U(1)]^{N-1}$ we obtain

$$S_{\text{eff}}^{\mathcal{N}=2} = c_0 \sum_{I < J}^N \int d^5|8 z du W_{IJ} \left[\ln \frac{W_{IJ}}{\Lambda} + \frac{1}{2} H(Z_{IJ}) \right], \quad (3.26)$$

where $Z_{IJ} = (q^{+a})_{IJ}(q_a^-)_{IJ}/W_{IJ}^2$ and $W_{IJ} = W_I - W_J$, $(q^{\pm a})_{IJ} = q_I^{\pm a} - q_J^{\pm a}$. The superfields W_I and $q_I^{\pm a}$ obey the constraints $\sum_I W_I = 0$ and $\sum_I q_I^{\pm a} = 0$ and span the Cartan directions in the algebra $\mathfrak{su}(N)$. The function $H(Z_{IJ})$ for each argument Z_{IJ} is given by the expression (3.25).

3.3. Component structure. We will be interested in deriving the term F^4/ϕ^3 from the effective action (3.23). To this end, it is enough to leave only the following component fields in the involved superfields:

$$q^+ = f^i(x)u_i^+, \quad \bar{q}^+ = -\bar{f}^i(x)u_i^+, \quad (3.27)$$

$$W = \sqrt{2}\phi(x) - 2i\theta^{+\hat{\alpha}}\theta^{-\hat{\beta}}F_{\hat{\alpha}\hat{\beta}}(x). \quad (3.28)$$

Here $\bar{\phi} = \phi$ and $\bar{f}^i = \bar{f}_i$ are scalar fields and $F_{\hat{\alpha}\hat{\beta}} = F_{\hat{\beta}\hat{\alpha}}$ is the Maxwell field strength.

Substituting (3.28) into the first term in (3.23), we find

$$S_0 = c_0 \frac{\sqrt{2}}{3} \int d^5|8 z \frac{(\theta^{+\hat{\alpha}}\theta^{-\hat{\beta}}F_{\hat{\alpha}\hat{\beta}})^4}{\phi^3} = \frac{c_0}{4\sqrt{2}} \int d^5|8 z \frac{\det F}{\phi^3} (\theta^+)^2(\theta^+)^2(\theta^-)^2(\theta^-)^2, \quad (3.29)$$

where $\det F = (1/4!)\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}F_{\hat{\alpha}\hat{\mu}}F_{\hat{\beta}\hat{\nu}}F_{\hat{\gamma}\hat{\rho}}F_{\hat{\delta}\hat{\sigma}}$ and $(\theta^\pm)^2 = \theta^{\pm\hat{\alpha}}\theta_{\hat{\alpha}}^\pm$. We integrate over the Grassmann variables according to the rule

$$\int d^5|8 z (\theta^+)^2(\theta^+)^2(\theta^-)^2(\theta^-)^2 f(x) = 4 \int d^5 x f(x) \quad (3.30)$$

for some $f(x)$. Thus the action (3.29) yields the component term

$$S_0 = \frac{c_0}{\sqrt{2}} \int d^5 x \frac{\det F}{\phi^3}. \quad (3.31)$$

In a similar way one can find the contribution of the last term in (3.23):

$$\int d^5|8 z W H(Z) = \sqrt{2} \int d^5 x \frac{\det F}{\phi^3} [4z^4 H^{(4)}(z) + 28z^3 H'''(z) + 39z^2 H''(z) + 6z H'(z)], \quad (3.32)$$

where

$$z \equiv Z|_{\theta=0} = \frac{f^i \bar{f}_i}{\phi^2}. \quad (3.33)$$

Substituting the function (3.25) into (3.32), we find

$$\frac{c_0}{2} \int d^5x {}^z WH(Z) = \frac{c_0}{\sqrt{2}} \int d^5x \frac{\det F}{(\phi^2 + f^i \bar{f}_i)^{3/2}} - \frac{c_0}{\sqrt{2}} \int d^5x \frac{\det F}{\phi^3}. \quad (3.34)$$

The last term exactly cancels (3.31). As a result, the total F^4/ϕ^3 term in the component form of the effective action (3.23) is given by the expression

$$S_{\text{eff}}^{\mathcal{N}=2} = \frac{c_0}{\sqrt{2}} \int d^5x \frac{\det F}{(\phi^2 + f^i \bar{f}_i)^{3/2}} + \dots, \quad (3.35)$$

where the dots stand for the remaining terms. It is remarkable that the scalar fields appear in the denominator in (3.35) just in the SO(5)-invariant combination. This is non-trivial, since the field ϕ comes from the gauge $\mathcal{N} = 1$ multiplet, while f^i and \bar{f}_i , from the hypermultiplet. In the SU(N) case (3.26), $S_{\text{eff}}^{\mathcal{N}=2}$ is a sum of the appropriate terms (3.35).

4. LOW-ENERGY EFFECTIVE ACTION OF 6D, $\mathcal{N} = (1, 1)$ SYM THEORY

Another interesting class of extended objects in superstring/brane theory is presented by D5-branes (see, e.g., [5, 47]). These objects are related to 6D, $\mathcal{N} = (1, 1)$ SYM theory likewise D3-branes are related to 4D, $\mathcal{N} = 4$ SYM theory. Similarly to the D3-brane case, the interaction of D5-branes is described by the 6D Born–Infeld action [76] (see, e.g., [4, 41, 50] and the references therein for aspects of the Born–Infeld action in diverse dimensions). Since the D5-brane is related to 6D, $\mathcal{N} = (1, 1)$ SYM theory, it is natural to expect that the D5-brane interaction in the low-energy limit can be calculated proceeding from the low-energy quantum effective action of this theory.

In this section we consider quantum aspects of 6D, $\mathcal{N} = (1, 1)$ SYM theory. It is the maximally extended supersymmetric gauge theory in six dimensions, with eight left-handed and eight right-handed supercharges. An equal number of spinors with mutually opposite chiralities guarantees the absence of chiral anomaly in this theory. From the point of view of 6D, $\mathcal{N} = (1, 0)$ supersymmetry, the model is built on a gauge (vector) multiplet and a hypermultiplet. Accordingly, its bosonic sector involves a real 6D gauge field and two complex (or four real) scalar fields.

Although 6D, $\mathcal{N} = (1, 1)$ non-abelian SYM theory is non-renormalizable by power counting, it is on-shell finite at one and two loops [6–8, 42, 51, 52, 66, 67]. Moreover, it was recently shown that this theory is one-loop finite even off-shell [16–18] and that the two-loop diagrams with hypermultiplet legs are also off-shell finite [19]. A review of our approach was presented in [20].

To preserve as many manifest supersymmetries as possible, we use the harmonic superspace approach [43, 46]. The theory under consideration is formulated in terms of $\mathcal{N} = (1, 0)$ harmonic superfields describing the gauge multiplet and the hypermultiplet. Therefore, it possesses the manifest $\mathcal{N} = (1, 0)$ supersymmetry and, in addition, a non-manifest (hidden) $\mathcal{N} = (0, 1)$ supersymmetry mixing the $\mathcal{N} = (1, 0)$ gauge multiplet and hypermultiplet. These supersymmetries close on-shell on the total on-shell $\mathcal{N} = (1, 1)$ supersymmetry. Such a formulation of $\mathcal{N} = (1, 1)$ SYM theory was described in detail in [9] (see also [53, 79]). An essential difference of our consideration here is the use of the so-called “ ω -form” of the hypermultiplet (see below).

We consider the case when the gauge symmetry SU(N) is broken to SU($N - 1$) \times U(1) \subset SU(N). Technically, this means that background superfields align through the fixed generator of the Cartan subalgebra of su(N), which corresponds to an abelian subgroup U(1). In this case the effective action of the theory depends only on the abelian vector multiplet and hypermultiplet. In the bosonic sector

we find out the effective action for the single real U(1) gauge field and four real scalar fields. The same number of bosonic world-volume degrees of freedom is exhibited by a single D5-brane in six dimensions [70].

4.1. 6D, $\mathcal{N} = (1, 1)$ SYM in the $\mathcal{N} = (1, 0)$ harmonic formulation with the ω hypermultiplet. We start with the formulation of 6D, $\mathcal{N} = (1, 1)$ SYM theory in terms of 6D, $\mathcal{N} = (1, 0)$ harmonic analytic superfields V^{++} and ω , which represent the gauge multiplet and the hypermultiplet. The action of $\mathcal{N} = (1, 1)$ SYM theory is written as

$$S_0[V^{++}, q^+] = \frac{1}{f^2} \left\{ \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \operatorname{tr} \int d^{14}z du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)} - \frac{1}{2} \operatorname{tr} \int d\zeta^{(-4)} \nabla^{++} \omega \nabla^{++} \omega \right\}, \quad (4.1)$$

where f is a dimensionful coupling constant ($[f] = -1$) and the measure of integration over the analytic subspace $d\zeta^{(-4)}$ includes the integration over harmonics, $d\zeta^{(-4)} = d^6 x_{(\text{an})} du (D^-)^4$. Both V^{++} and ω superfields take values in the adjoint representation of the gauge group. The covariant harmonic derivative ∇^{++} acts on the hypermultiplet ω as

$$\nabla^{++} \omega = D^{++} \omega + i[V^{++}, \omega]. \quad (4.2)$$

The action (4.1) is invariant under the infinitesimal gauge transformations

$$\delta V^{++} = -\nabla^{++} \Lambda, \quad \delta \omega = i[\Lambda, \omega], \quad (4.3)$$

where $\Lambda(\zeta, u) = \tilde{\Lambda}(\zeta, u)$ is a real analytic gauge parameter.

Besides the analytic gauge connection V^{++} , we introduce a non-analytic one V^{--} (see [46]) which is a solution of the zero curvature condition (3.5). Using V^{--} , we define one more covariant harmonic derivative $\nabla^{--} = D^{--} + iV^{--}$ and the $\mathcal{N} = (1, 0)$ gauge superfield strength

$$W^{+a} = -\frac{i}{6} \varepsilon^{abcd} D_b^+ D_c^+ D_d^+ V^{--} \quad (4.4)$$

possessing the useful off-shell properties

$$\nabla^{++} W^{+a} = \nabla^{--} W^{-a} = 0, \quad W^{-a} = \nabla^{--} W^{+a}. \quad (4.5)$$

Introducing an analytic superfield F^{++} ,

$$F^{++} = \frac{1}{4} D_a^+ W^{+a} = i(D^+)^4 V^{--}, \quad D_a^+ F^{++} = \nabla^{++} F^{++} = 0, \quad (4.6)$$

we can write the classical equations of motion corresponding to the action (4.1) as

$$F^{++} + [\omega, \nabla^{++} \omega] = 0, \quad (\nabla^{++})^2 \omega = 0. \quad (4.7)$$

The $\mathcal{N} = (1, 0)$ superfield action (4.1) enjoys the additional $\mathcal{N} = (0, 1)$ supersymmetry

$$\delta V^{++} = (\varepsilon^{+A} u_A^+) \omega - (\varepsilon^{+A} u_A^-) \nabla^{++} \omega = 2(\varepsilon^{+A} u_A^+) \omega - \nabla^{++} ((\varepsilon^{+A} u_A^-) \omega), \quad (4.8)$$

$$\delta \omega = -(D^+)^4 ((\varepsilon^{-A} u_A^-) V^{--}) = i(\varepsilon^{-A} u_A^-) F^{++} - i(\varepsilon_a^A u_A^-) W^{+a}, \quad (4.9)$$

where $A = 1, 2$ is the Pauli–Gürsey $SU(2)$ index. To check this, one derives, using (4.8) and (4.9), the $\mathcal{N} = (0, 1)$ transformation law of $\nabla^{++}\omega$,

$$\delta(\nabla^{++}\omega) = i((\epsilon^{-A}u_A^+) + (\epsilon^{+A}u_A^-))F^{++} - i(\epsilon_a^A u_A^+)W^{+a} + i(\epsilon^{+A}u_A^-)[\omega, \nabla^{++}\omega]. \tag{4.10}$$

Then one varies the classical action (4.1) with respect to (4.8) and (4.10):

$$\delta S = \frac{1}{f^2} \left\{ \text{tr} \int d^{14}z du V^{--} \delta V^{++} - \text{tr} \int d\zeta^{(-4)} \nabla^{++}\omega \delta(\nabla^{++}\omega) \right\}. \tag{4.11}$$

In the first integral, we pass to the integration over the analytic subspace and use the explicit form of the variations (4.8) and (4.10):

$$\begin{aligned} \delta S = & -\frac{i}{f^2} \text{tr} \int d\zeta^{(-4)} \left\{ 2F^{++}(\epsilon^{+A}u_A^+)\omega + \nabla^{++}\omega((\epsilon^{-A}u_A^+) + (\epsilon^{+A}u_A^-))F^{++} \right. \\ & \left. - F^{++}\nabla^{++}((\epsilon^{+A}u_A^-)\omega) - \epsilon_a^A u_A^+ \nabla^{++}\omega W^{+a} \right\} = 0. \end{aligned} \tag{4.12}$$

The last two terms in (4.12) are the total harmonic derivative ∇^{++} due to the properties of F^{++} and W^{+a} , and so they vanish under the analytic integration measure $d\zeta^{(-4)}$. The first two terms cancel each other after integration by parts with respect to the harmonic derivative ∇^{++} in view of the properties $\nabla^{++}\epsilon^{-A} = \epsilon^{-A}$ and $\nabla^{++}u_A^- = u_A^+$. Finally, the term $\text{tr}(\nabla^{++}\omega[\omega, \nabla^{++}\omega])$ vanishes due to the cyclic property of trace.

The zero curvature condition (3.5) allows one to express the transformation of the non-analytic gauge connection δV^{--} through δV^{++} ,

$$\nabla^{++}\delta V^{--} - \nabla^{--}\delta V^{++} = 0, \tag{4.13}$$

and to find the transformation law of the strength W^{+a} under the hidden supersymmetry

$$\delta W^{+a} = \varepsilon^{abcd} \epsilon_d^A \nabla_{bc}(u_A^+\omega - u_A^- \nabla^{++}\omega) + i\epsilon^{-A}[W^{+a}, u_A^+\omega - u_A^- \nabla^{++}\omega], \tag{4.14}$$

where

$$\nabla_{bc} = \partial_{bc} - \frac{1}{2} D_b^+ D_c^+ V^{--}. \tag{4.15}$$

As usual, we make use of the background superfield method. The gauge group of the theory (4.1) is assumed to be $SU(N)$. For the further consideration, we will also assume that the background superfields \mathbf{V}^{++} and $\mathbf{\Omega}$ align in a fixed direction in the Cartan subalgebra of $\mathfrak{su}(N)$:

$$\mathbf{V}^{++} = V^{++}(\zeta, u)H, \quad \mathbf{\Omega} = \Omega(\zeta, u)H, \tag{4.16}$$

where H is a fixed generator in the Cartan subalgebra generating some abelian subgroup $U(1)$. Our choice of the background corresponds to the spontaneous symmetry breaking $SU(N) \rightarrow SU(N-1) \times U(1)$. We should note that the pair of background superfields (V^{++}, Ω) forms an abelian vector $\mathcal{N} = (1, 1)$ multiplet, which, in the bosonic sector, contains a single real gauge vector field $A_M(x)$ and four real scalars $\phi(x)$ and $\phi^{(ij)}(x)$, $i, j = 1, 2$, where ϕ and $\phi^{(ij)}$ are the scalar components of the Ω hypermultiplet [46]. The abelian vector field and four scalars in six-dimensional space–time constitute just the bosonic world-volume degrees of freedom of a single D5-brane [5, 47].

The classical equations of motions (4.7) for the background superfields V^{++} and Ω are

$$F^{++} = 0, \quad (D^{++})^2 \Omega = 0. \tag{4.17}$$

In that follows we assume that the background superfields solve the classical equation of motion (4.17). We will also consider the background slowly varying in space–time, i.e.,

$$\partial_M W^{+a} = 0, \quad \partial_M \Omega = 0. \quad (4.18)$$

Finally, we are left with an abelian background analytic superfields V^{++} and Ω , which satisfy the classical equation of motion (4.17) and conditions (4.18). Under these assumptions the gauge superfield strength W^{+a} is analytic,⁶ $D_a^+ W^{+b} = \delta_a^b F^{++} = 0$. In the further analysis we will use the $\mathcal{N} = (0, 1)$ transformation for the gauge superfield strength W^{+a} (4.14). For the slowly varying abelian on-shell background superfields, the hidden $\mathcal{N} = (0, 1)$ supersymmetry transformations (4.9) and (4.14) acquire the very simple form

$$\delta\Omega = -i(\epsilon_a^A u_A^-) W^{+a}, \quad \delta W^{+a} = 0. \quad (4.19)$$

These transformation rules follow from the abelian version of the transformations (4.9) and (4.14), in which one should take into account conditions (4.18) and (4.17). It is worth pointing out that these conditions are covariant under $\mathcal{N} = (0, 1)$ supersymmetry.

4.2. Effective action with hidden $\mathcal{N} = (0, 1)$ supersymmetry. Let us now consider the simplest $\mathcal{N} = (1, 1)$ invariants which can be constructed out of the abelian analytic superfields W^{+a} and Ω under the assumptions (4.17) and (4.18). It is evident that the gauge-invariant action

$$I = f^2 \int d\zeta^{(-4)} (W^+)^4 \mathcal{F}(f\Omega), \quad (4.20)$$

where $(W^+)^4 = -(1/24)\varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d}$ and $\mathcal{F}(f\Omega)$ is an arbitrary function of Ω , is invariant under the transformation (4.19) due to the nilpotency condition $(W^+)^5 \equiv 0$. For our further consideration, of the main interest is the choice

$$I_1 = c \int d\zeta^{(-4)} \frac{(W^+)^4}{\Omega^2}, \quad (4.21)$$

which corresponds to $\mathcal{F} = 1/(f^2\Omega^2)$ in (4.20). The coefficient c in (4.21) cannot be fixed only on the symmetry grounds and should be calculated in the framework of the quantum field theory [15]. In fact, the same concerns the specific choice of the function $\mathcal{F}(f\Omega)$.

As a result of the exact calculations [15], we arrived at the following expression for the one-loop effective action:

$$\Gamma_{\text{lead}}^{(1)} = \frac{N-1}{(4\pi)^3} \int d\zeta^{(-4)} \frac{(W^+)^4}{\Omega^2}. \quad (4.22)$$

As expected, the leading low-energy contribution (4.22) to the effective action in the model (4.1) is just the $\mathcal{N} = (1, 1)$ invariant I_1 (4.21). The coefficient c now takes the value [15]

$$c = \frac{N-1}{(4\pi)^3}. \quad (4.23)$$

The expression for c is similar to that in the four-dimensional $\mathcal{N} = 4$ SYM theory (see, e.g., [56] and references therein). In the bosonic sector the effective action (4.22) has the structure

$$\Gamma_{\text{bos}}^{(1)} \sim \int d^6x \frac{F^4}{\phi^2} \left(1 + \frac{\phi^{(ij)}\phi_{(ij)}}{\phi^2} + \dots \right), \quad (4.24)$$

where $F^4 = F_{MN}F^{MN}F_{PQ}F^{PQ} - 4F^{NM}F_{MR}F^{RS}F_{SN}$ and F_{MN} is the gauge field strength.

We should note that even in the one-loop approximation there might exist more complicated contributions to the effective action, which are beyond the scope of our consideration. We hope to come back to these issues elsewhere.

⁶In general this is not true and $F^{++} \neq 0$.

5. SUMMARY AND OUTLOOK

In this paper we have reviewed a hidden-supersymmetry based approach to constructing the low-energy effective actions for extended and higher dimensional supersymmetric gauge theories. We considered the $4D$, $\mathcal{N} = 4$ SYM, $5D$, $\mathcal{N} = 2$ SYM, and $6D$, $\mathcal{N} = (1, 1)$ SYM theories formulated in harmonic superspace. All these theories are characterized by some number of manifest off-shell supersymmetries and some number of hidden on-shell supersymmetries. The complete supersymmetry of the theories under consideration is due to a combination of the manifest and hidden ones. The low-energy effective actions were analyzed on the purely algebraic grounds and can be obtained in a general form up to numerical coefficients. To fix these coefficients, one needs to carry out concrete quantum field-theoretical calculations. This approach has been completely accomplished for all three theories under consideration.

In $4D$, $\mathcal{N} = 4$, $SU(N)$ SYM theory we begun with the known manifestly $\mathcal{N} = 2$ supersymmetric non-holomorphic effective potential in the gauge multiplet sector and, using the hidden supersymmetry transformations, completed it by hypermultiplet terms to the full $\mathcal{N} = 4$ supersymmetric low-energy effective potential in the Coulomb phase [13]. The result was later confirmed by the one-loop supergraph calculations [1, 21, 32].

Generalizing the approach of [13] to the $5D$ case, we constructed the leading term in the low-energy effective action of $5D$, $\mathcal{N} = 2$ SYM theory as the appropriate sum of the effective action of $5D$, $\mathcal{N} = 1$ SYM theory and the interaction with the hypermultiplet. This interaction is fixed, up to an overall coupling constant c_0 , by the requirement of the implicit on-shell $5D$, $\mathcal{N} = 1$ supersymmetry extending the manifest off-shell $\mathcal{N} = 1$ supersymmetry to an on-shell $5D$, $\mathcal{N} = 2$ one. We discussed in detail the case of the gauge group $SU(2)$ spontaneously broken to $U(1)$, in which case the effective action depends on a single pair of abelian $5D$, $\mathcal{N} = 1$ gauge multiplet and hypermultiplet, and then considered a more general situation with the $SU(N)$ gauge group broken to its maximal torus, with $N - 1$ pairs of such abelian multiplets [24].

The next obvious problem is to reproduce the $5D$ effective actions found in [24] from the appropriate set of quantum $5D$, $\mathcal{N} = 1$ supergraphs involving the interacting hypermultiplet and $\mathcal{N} = 1$ gauge superfields. Also, it would be interesting to establish manifest links with the relevant D-brane dynamics and the $4D$ and $6D$ cousins of the $5D$ effective action constructed. Finding out an explicit form of the next-to-leading corrections to this effective action, also based on the demand of implicit $5D$, $\mathcal{N} = 1$ supersymmetry, is another interesting task.

In $6D$, $\mathcal{N} = (1, 1)$ SYM theory we dealt with its formulation in terms of $\mathcal{N} = (1, 0)$ harmonic superfields as the theory of interacting $\mathcal{N} = (1, 0)$ gauge multiplet and hypermultiplet, both in the adjoint representation of the gauge group. This theory is characterized by manifest $\mathcal{N} = (1, 0)$ supersymmetry and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetry. The low-energy effective action has been constructed by combining the gauge multiplet superfield strengths and hypermultiplet, so as to achieve invariance under the hidden supersymmetry. The result has been actually confirmed by a direct quantum computation [15].

The results concerning the low-energy effective actions and the roles of hidden supersymmetries in the above three theories can be generalized along many directions. First, in all cases we found only the leading low-energy effective actions, omitting all the superspace derivative-dependent terms. However, such terms could in principle be essential for establishing the precise links with superstring/brane low-energy effective actions. Second, it would be interesting to sum up all superfield strength-dependent terms without derivatives and to obtain in this way the Heisenberg–Euler or Born–Infeld type effective actions possessing both manifest and hidden supersymmetries. The third interesting direction is related to exploring the quantum structure of the higher-derivative supersymmetric theories, for example, of $6D$ renormalizable higher-derivative supersymmetric gauge theory [54, 77].

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