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Best One-Sided Approximation in the Mean of the Characteristic Function of an Interval by Algebraic Polynomials

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Abstract—Let υ be a weight on $(-1,1)$, i.e., a measurable integrable nonnegative function nonzero almost everywhere on $(-1,1)$. Denote by $L^\upsilon(-1,1)$ the space of real-valued functions f integrable with weight υ on $(-1,1)$ with the norm $\|f\| = \int_{-1}^1 |f(x)|\upsilon(x)\,dx$. We consider the problems of the best one-sided approximation (from below and from above) in the space $L^\upsilon(-1,1)$ to the characteristic function of an interval (a,b), $-1<a<b<1$, by the set of algebraic polynomials of degree not exceeding a given number. We solve the problems in the case where a and b are nodes of a positive quadrature formula under some conditions on the degree of its precision as well as in the case of a symmetric interval $(-h,h)$, $0<h<1$, for an even weight υ.

Keywords: one-sided approximation, characteristic function of an interval, algebraic polynomials.

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1. DISCUSSION OF THE PROBLEM. A LOWER ESTIMATE FOR THE BEST ONE-SIDED APPROXIMATION

Let υ be a measurable integrable nonnegative function nonzero almost everywhere on $(-1,1)$; such a function is called a weight (on $(-1,1)$). Denote by $L^\upsilon(-1,1)$ the space of real-valued functions f integrable with weight υ on $(-1,1)$; this space is equipped with the norm

$$\|f\| = \|f\|_{L^\upsilon(-1,1)} = \int_{-1}^{1} |f(x)|\upsilon(x)\,dx.$$

For a nonnegative integer n, denote by $\mathscr{P}_n$ the set of algebraic polynomials $p(x) = \sum_{k=0}^n a_k x^k$ of degree at most n with real coefficients.

In the present paper, for a pair of measurable functions f and g on the interval $(-1,1)$, the inequality $f \le g$ means that $f(x) \le g(x)$ for almost all $x \in (-1,1)$. For a function f defined and measurable on the interval $[-1,1]$, we consider the sets

$$\mathscr{P}_n^-(f) = \{p \in \mathscr{P}_n \colon p \le f\}, \quad \mathscr{P}_n^+(f) = \{p \in \mathscr{P}_n \colon p \ge f\} \tag{1.1}$$

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of polynomials from $\mathscr{P}_n$ whose graphs lie under or over the graph of f, respectively. The function f is assumed to be bounded from below in the former case and from above in the latter case. We are interested in the values

$$E^{\mp}_{n,\upsilon}(f) = \inf\{\|f - p\|: p \in \mathscr{P}^{\mp}_n(f)\} \tag{1.2}$$

of the best approximation to the function f from below and from above by the set $\mathscr{P}_n$ in the space $L^{\upsilon}(-1, 1)$ as well as in extremal polynomials at which the infima in (1.2) are attained.

An important tool for studying problems (1.2) is a result by Bojanic and DeVore [1, proof of Theorem 2] presented in Theorem A below. However, in Theorem A, it is assumed that the inequality in (1.1) holds not almost everywhere but everywhere on $[-1, 1]$. In view of this, let us impose some constraints on the approximated function f.

For a function f measurable and lower bounded on the interval $[-1, 1]$, consider a function $\underline{f}$ defined on $[-1, 1]$ by the relation

$$\underline{f}\,(x) = \lim_{\varepsilon\to+0} \operatorname{ess\,inf}\{f(t) : t \in (x - \varepsilon,\, x + \varepsilon) \cap [-1, 1]\}, \quad x \in [-1, 1].$$

Let $\mathscr{R}^-$ be the set of functions defined and lower bounded on $[-1, 1]$, belonging to the space $L^{\upsilon}(-1, 1)$, and such that $\underline{f}\,(x) \le f(x)$ for all $x \in [-1, 1]$. Functions $f \in \mathscr{R}^-$ have the property that if the inequality $\varphi \le f$ holds for some continuous function φ almost everywhere on $[-1, 1]$, then it holds everywhere on this interval. As a consequence, for functions $f \in \mathscr{R}^-$ and all $n \ge 0$, we have

$$\mathscr{P}^-_n(f) = \bigl\{p \in \mathscr{P}_n : p(x) \le f(x),\ x \in [-1, 1]\bigr\}.$$

Define $\mathscr{R}^+ = -\mathscr{R}^- = \{f :\, -f \in \mathscr{R}^-\}$. The sets $\mathscr{R}^{\mp}$ contain, for example, functions continuous on the interval $[-1, 1]$ as well as discontinuous functions having on $[-1, 1]$ only discontinuities of the first kind at interior points of the interval under the condition that the value at a discontinuity point is between the right and left limits.

Consider a quadrature formula

$$\int_{-1}^{1} \upsilon(x)p(x)\,dx = \sum_{k=1}^{M} \lambda_k p(x_k), \quad p \in \mathscr{P}_n, \tag{1.3}$$

with nodes $-1 \le x_1 < x_2 < \cdots < x_M \le 1$ and positive weights $\lambda_k > 0$, $1 \le k \le M$. Such formulas are called positive. An important role in studying the problems of one-sided approximation of functions by polynomials is played by positive quadrature formulas exact on the set of polynomials $\mathscr{P}_n$. The highest degree n of polynomials for which formula (1.3) holds is called its *algebraic degree of precision*. Depending on the situation, some nodes in (1.3) can be fixed while the others are assumed to be free; more exactly, the latter are chosen so that the formula has the highest degree of precision (see, e.g., [2, Sect. 7.1]). One of the most known positive quadrature formulas is the Gauss quadrature formula (1866), in which all M nodes are free; its degree of precision is $2M - 1$ (see, e.g., [2, Sect. 7.1]).

The following assertion is a special case of a more general result contained in [1, proof of Theorem 2] (see also [3, Theorem 1.7.5]).

Theorem A. *Assume that a positive quadrature formula* (1.3) *holds on the set* $\mathscr{P}_n$. *Then the following estimates are valid for functions* $f \in \mathscr{R}^{\mp}$, *respectively*:

$$E^-_n(f) \ge \int_{-1}^{1} \upsilon(x)f(x)\,dx - \sum_{k=1}^{M} \lambda_k f(x_k), \quad E^+_n(f) \ge \sum_{k=1}^{M} \lambda_k f(x_k) - \int_{-1}^{1} \upsilon(x)f(x)\,dx. \tag{1.4}$$

If an inequality in (1.4) turns into an equality, then the quadrature formula (1.3) is *extremal* in the corresponding problem (1.2).

2. ONE-SIDED APPROXIMATION FROM BELOW TO THE CHARACTERISTIC FUNCTION OF A HALF-OPEN INTERVAL $(a, 1]$

For $-1 \le a < b \le 1$, we introduce the universal notation for (half-)open intervals

$$J = J(a,b) = \begin{cases} (a,b), & -1 < a < b < 1, \\ (a,1], & -1 < a < b = 1, \\ [-1,b), & a = -1 < b < 1. \end{cases} \tag{2.1}$$

Consider the problem on the one-sided approximation from below to the characteristic function

$$\mathbf{1}_J(t) = \begin{cases} 1, & t \in J, \\ 0, & t \in [-1,1] \setminus J, \end{cases} \tag{2.2}$$

of an interval (2.1) by algebraic polynomials of a given degree $n \ge 0$ in the space $L^{\upsilon}(-1,1)$. The problem consists in finding the value

$$E_n^-(\mathbf{1}_J) = E_{n,\upsilon}^-(\mathbf{1}_J) = \inf\big\{\|\mathbf{1}_J - p_n\|_{L^{\upsilon}(-1,1)} \colon p_n \in \mathscr{P}_n^-(\mathbf{1}_J)\big\}. \tag{2.3}$$

The characteristic functions (2.2) of intervals (2.1) belong to the sets $\mathscr{R}^{\mp}$; hence, the first inequality in (1.4) holds for (2.3).

Problems of weighted one-sided integral approximation to the characteristic functions of intervals and related functions by algebraic or trigonometric polynomials arise in various branches of mathematics and have a rich history. In this subject area, there are exact results (some of which will be discussed later), order results, studies of asymptotic behavior (see [4–6] and references therein), and various applications (see [3, 5, 7, 8] and references therein).

Let us outline only a few exact results on problem (2.3) closely related to the present paper; for a more complete presentation of the topic, see [9]. In [5], the problem of one-sided approximation to the periodic extension of the characteristic function of an interval (a, b) by trigonometric polynomials in the integral metric with the Jacobi weight on the period was studied. An exact solution was found in [5, Theorem 3] for some values a and b satisfying special conditions. In the case of the unit weight, the problem was solved in [8] for an arbitrary interval located on the period; after the cosine change, this result gives a solution to problem (2.3) for $J = (a, 1]$ for all $a \in (-1, 1)$ with the Chebyshev weight of the first kind $\upsilon(t) = (1 - t^2)^{-1/2}$. In [7], problem (2.3) of one-sided integral approximation to the characteristic function of an arbitrary half-open interval $(a, 1]$, $-1 < a < 1$, by algebraic polynomials on $[-1, 1]$ with the unit weight was solved and the whole class of extremal polynomials was described. This problem in the space $L^{\upsilon}(-1, 1)$ with an arbitrary weight was solved in [9]. Let us describe the main result of [9] in a form convenient for us.

In the study of problems (2.3) of one-sided approximation to the characteristic function of an interval by polynomials, M-point quadrature formulas (1.3) are used, in which the set $\mathfrak{u}$ of fixed nodes either is empty or contains one, two, or three nodes of a specific form:

$$\varnothing, \quad \{-1\}, \quad \{1\}, \quad \{-1, 1\}, \tag{2.4}$$

$$\{\theta\}, \quad \{-1,\theta\}, \quad \{\theta,1\}, \quad \{-1,\theta,1\}, \quad \text{where} \quad \theta\in(-1,1), \tag{2.5}$$

and the other $M-|\mathfrak{u}|$ nodes are chosen so as to maximize the algebraic degree of precision; here, $|\mathfrak{u}|$ is the cardinality of the set $\mathfrak{u}$, i.e., the number of points in it. It is known (see, e.g., [2, Sect. 7.1]) that the degree of precision of this formula is $n=2M-1-|\mathfrak{u}|$. Formulas (1.3) take the form

$$\int_{-1}^{1} \upsilon(x)p(x)dx = \sum_{k=1}^{M}\lambda_k p\big(x_k\big), \quad p\in\mathscr{P}_{2M-1-|\mathfrak{u}|}; \tag{2.6}$$

in what follows, we sometimes use more precise (in comparison with (1.3)) notation for the nodes $\{x_k = x_k^{\mathfrak{u}} = x_k(\mathfrak{u},\upsilon,M)\}_{k=1}^M$ and weights (coefficients) $\{\lambda_k=\lambda_k^{\mathfrak{u}}=\lambda_k(\mathfrak{u},\upsilon,M)\}_{k=1}^M$ in formula (2.6).

In the case of the empty set $\mathfrak{u}$ (there are no fixed nodes), formula (2.6) is the classical Gauss quadrature formula (see [2, Sect. 7.1]). In the case of one fixed node coinciding with one of the end points of the interval $[-1,1]$, i.e., in the case where $\mathfrak{u}=\{-1\}$ or $\mathfrak{u}=\{1\}$, formula (2.6) is the left and right Radau quadrature formula, respectively; in the case $\mathfrak{u}=\{-1,1\}$, (2.6) is the Lobatto quadrature formula. It is known (see the references in [7,9,10]) that formula (2.6) is positive in all these cases.

For each of the sets $\mathfrak{u}$ of fixed nodes from (2.5), the set $\Theta_M^{\mathfrak{u}}$ of values of the parameter $\theta\in(-1,1)$ for which quadrature formula (2.6) has positive weights was described in [7,10]. Such formulas are called quasi Gauss, quasi (left and right) Radau, and quasi Lobatto positive quadrature formulas. In what follows, we consider formula (2.6) with fixed nodes (2.5) only for $\theta\in\Theta_M^{\mathfrak{u}}$.

Thus, a quadrature formula of the form (2.6) with fixed nodes (2.4) and with fixed nodes (2.5) is positive. The degree of precision of formula (2.6) with fixed nodes (2.4) and with fixed nodes (2.5) is $N=2M-1-|\mathfrak{u}|$. Moreover, for each $n\in\mathbb{N}$ and $a\in(-1,1)$, there exists [10, Theorem 1.1, Corollary 1.2, Remark 1.3] a specific positive quadrature formula of the form (2.6).

The value of the best approximation from below

$$E_{n,\upsilon}^{-}(\mathbf{1}_{(a,1]}) = \min\{\|\mathbf{1}_{(a,1]}-p_n\|_{L^{\upsilon}(-1,1)} : p_n\in\mathscr{P}_n^{-}(\mathbf{1}_{(a,1]})\} \tag{2.7}$$

and an extremal polynomial $p_n^a = p_{n,a}^{\upsilon}$ at which the minimum in (2.7) is attained were found for all values $a\in(-1,1)$ and $n\ge 1$ in the case of the unit weight $\upsilon\equiv 1$ in [7] and in the case of an arbitrary weight υ in [9]. The results of several statements from [9, Sect. 3] containing the solution of problem (2.7) are collected in the following theorem in a form convenient for us.

Theorem B [9, Sect. 3]. *The following statements hold for $M\in\mathbb{N}$, $M\ge 3$.*

(1) *If the number $a\in(-1,1)$ coincides with one of the nodes of an M-point positive quadrature formula* (2.6) *different from the maximum node, i.e., $a=x_\nu^{\mathfrak{u}}$, $1\le\nu\le M-1$, then*

$$E_{n,\upsilon}^{-}(\mathbf{1}_{(a,1]}) = \int_{(a,1]} \upsilon(x)dx - \sum_{k=\nu+1}^{M}\lambda_k^{\mathfrak{u}}$$

for $n=2M-2-|\mathfrak{u}|$ and $n=2M-1-|\mathfrak{u}|$ in the case of fixed nodes (2.4) *and for $n=2M-1-|\mathfrak{u}|$ in the case of fixed nodes* (2.5). *Moreover, the corresponding quadrature formula is extremal, and the polynomial of the best approximation from below is the polynomial $p_n^a\in\mathscr{P}_n^{-}(\mathbf{1}_{(a,1]})$ that interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes of the quadrature formula*; *the degree of this polynomial is $n=2M-2-|\mathfrak{u}|$ for $\mathfrak{u}$ from* (2.4) *and $n=2M-1-|\mathfrak{u}|$ for $\mathfrak{u}$ from* (2.5).

(2) *If the maximum node* $x^{\mathfrak{u}}_M$ *of formula* (2.6) *is less than* 1, *then*

$$E^-_{n,\upsilon}(\mathbf{1}_{(a,1]}) = \int\limits_{(a,1]} \upsilon(x)\,dx$$

for $x^{\mathfrak{u}}_M \le a < 1$ *and all* $0 \le n \le 2M-1-|\mathfrak{u}|$, *and* $p^* \equiv 0$ *is the polynomial of the best approximation from below.*

3. ONE-SIDED APPROXIMATION TO THE CHARACTERISTIC FUNCTION OF AN INTERVAL INTERIOR FOR $[-1,1]$

In this section, we discuss the problem of one-sided approximation from below and from above to the characteristic function $\mathbf{1}_{(a,b)}$ of an interval (a,b) whose end points a and b are nodes of a positive quadrature formula with some properties specified below. In particular, Theorem B makes it possible to find solutions of problems (3.1) for intervals (a,b) whose end points are nodes of any quadrature formulas (2.6). The problem of one-sided approximation from below

$$E^-_n(\mathbf{1}_{(a,b)}) = E^-_{n,\upsilon}(\mathbf{1}_{(a,b)}) = \inf\left\{\|\mathbf{1}_{(a,b)} - p_n\|_{L^\upsilon(-1,1)} \colon p_n \in \mathscr{P}^-_n(\mathbf{1}_{(a,b)})\right\} \tag{3.1}$$

will be studied most thoroughly.

3.1. Extremal polynomials in problem (2.7). Let us first discuss the construction and properties of extremal polynomials in problem (2.7) under the assumption that its solution is obtained by means of a certain positive quadrature formula.

Let

$$\int\limits_{-1}^{1} \upsilon(x)p(x)\,dx = \sum_{k=1}^{M} \lambda_k p(x_k), \quad p \in \mathscr{P}_N, \tag{3.2}$$

be a positive quadrature formula with the nodes $\{x_k\}_{k=1}^M$ indexed in ascending order, and let N be its degree of precision. We will assume that the parameter a coincides with one of the nodes of this formula different from the largest node: $a = x_{k(a)}$, $1 \le k(a) < M$; thus, $M \ge 2$. We are interested in the situation when (3.2) is an extremal quadrature formula of problem (2.7), i.e., the first inequality in (1.4) turns into an equality for the function $f = \mathbf{1}_{(a,1]}$ and the corresponding degree n. This equality takes the form

$$E^-_{n,\upsilon}(\mathbf{1}_{(a,1]}) = \int\limits_a^1 \upsilon(t)dt - \sum_{k(a)<k\le M} \lambda_k.$$

The degree of precision N of formula (3.2) may differ form the degree n in problem (2.7); i.e., in general, $N \ge n$.

The degree of precision of formula (3.2), the degree n in (2.7), and the exact degree of an extremal polynomial as well as its other properties depend on certain characteristics of the nodes of the formula. In particular, it is important whether the points ∓ 1 are the nodes. Following [9], we introduce parameters s and r, each of which can take only two values $\{0,1\}$ in accordance with the following rule. We set $s = 1$ if the point -1 is a node of formula (3.2) and $s = 0$ otherwise. Similarly, $r = 1$ if the point 1 is a node of formula (3.2) and $r = 0$ otherwise.

Denote by ρ a Hermite polynomial that interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes $\{x_k\}_{k=1}^M$ with different multiplicity. More precisely, at the point a and at the points ∓ 1, if they are nodes of the quadrature formula, the polynomial ρ interpolates only the values of the function $\mathbf{1}_{(a,1]}$; the number of such nodes is $1+s+r$. At the other $M-(1+s+r)$ nodes of formula (2.7), the polynomial ρ interpolates both the values of the function $\mathbf{1}_{(a,1]}$ and the values of its derivative: $\rho(x_k)=\mathbf{1}_{(a,1]}(x_k)$ and $\rho'(x_k)=0$. The total number of interpolation conditions is

$$K = 2(M-(1+s+r)) + (1+s+r) = 2M-1-s-r.$$

A polynomial with such interpolation properties exists; its degree is $n_0 = K-1 = 2M-2-s-r$ (see, for example, [11, Ch. 2, Sect. 11] or [12, Lecture 4, Sect. 4.3]). In what follows, we denote this polynomial by $p_{n_0}^a$ and call it the *Hermite interpolation polynomial of the function* $\mathbf{1}_{(a,1]}$ *at the nodes of the quadrature formula* (3.2).

Lemma 1. *Assume that the parameter* $a \in (-1,1)$ *is not the largest node of a positive quadrature formula* (3.2); *more precisely,* $a = x_{k(a)}$, $1 \le k(a) \le M-1$. *Formula* (3.2) *is extremal in problem* (3.1) *if and only if the degree of precision* N *of formula* (3.2) *satisfies the condition*

$$N \ge n_0 = 2M-2-s-r \tag{3.3}$$

and $n_0 \le n \le N$. *The following statements hold in this situation.*

(1) *The value* (2.7) *is the same for all* $n_0 \le n \le N$:

$$E_n^-(\mathbf{1}_{(a,1]}) = \int_a^1 \upsilon(t)dt - \sum_{k(a)<k\le M} \lambda_k. \tag{3.4}$$

(2) *The Hermite interpolation polynomial* $p_{n_0}^a$ *of the function* $\mathbf{1}_{(a,1]}$ *at the nodes* $\{x_k\}_{k=1}^M$ *of the quadrature formula* (3.2) *belongs to the set* $\mathscr{P}_{n_0}^-(\mathbf{1}_{(a,1]})$ *and is an extremal polynomial of problem* (2.7) *for all* n *with the property* $n_0 \le n \le N$.

Proof. The lemma is proved by methods known in this subject area (see, for example, [9]). However, in view of some specific aspects, we consider it necessary to present a complete proof. Assume that formula (3.2) is extremal in problem (2.7). This means that $n \le N$ and the first inequality in (1.4) turns into an equality for the function $f = \mathbf{1}_{(a,1]}$. Hence, the polynomial $p_n \in \mathscr{P}_n^-(\mathbf{1}_{(a,1]})$ is extremal in problem (2.7) if and only if this polynomial interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes $\{x_k\}_{k=1}^M$ of the quadrature formula (3.2). The condition $p_n(x) \le \mathbf{1}_{(a,1]}(x)$, $x \in [-1,1]$, implies that if $x_k \in (-1,1)$ and $x_k \ne a$, then, along with the property of the Lagrange interpolation $p_n(x_k) = \mathbf{1}_{(a,1]}(x_k)$, the condition $p_n'(x_k) = 0$ must also hold. As noted above, the degree of such polynomial is $n_0 = 2M-2-s-r$. Therefore, $n_0 \le n \le N$. Property (3.3) is verified. Let us check that statements (1) and (2) of the lemma also hold.

The polynomial $p_{n_0}^a$ interpolating the function $\mathbf{1}_{(a,1]}$ at the nodes of the quadrature formula (3.2) has the property $p_{n_0}^a(x) \le \mathbf{1}_{(a,1]}(x)$, $x \in [-1,1]$; i.e., $p_{n_0}^a \in \mathscr{P}_{n_0}^-(\mathbf{1}_{(a,1]})$. The proof of this and similar properties of Hermite interpolation polynomials goes back to A.A. Markov and T.I. Stieltjes and has a rich history (see [9] and references therein).

For convenience, define $\rho = p_{n_0}^a$. Consider zeros of the derivative ρ' of the polynomial ρ. Let $a = x_{k(a)}$, $1 < k(a) < M$. By Rolle's theorem, the derivative ρ' has a zero in each of the intervals (x_k, x_{k+1}), $k = 1, 2, \dots, k(a)-1, k(a)+1, \dots, M$; the number of such zeros is $M-2$. In addition, the derivative has $M-(1+s+r)$ zeros at the nodes of the quadrature formula. As a result, ρ'

has at least $M-2+M-(1+s+r)=2M-3-s-r=n_0-1$ zeros on the interval $(-1,1)$. The derivative ρ' is a polynomial of degree n_0-1 and, therefore, ρ' has no other zeros. Hence, the polynomial ρ increases from 0 to 1 on $[x_{k(a)},\, x_{k(a)+1}]$. It is easy to see that the graph of ρ does not exceed the graph of $\mathbf{1}_{(a,1]}$ on each of the intervals $[x_k, x_{k+1}]$, $k \neq k(a)$, and on the intervals $[-1, x_1]$ and $[x_M, 1]$. Thus, indeed, $\rho(x) \le \mathbf{1}_{(a,1]}(x)$ for $x \in [-1,1]$; i.e., $\rho = p^a_{n_0} \in \mathscr{P}^-_{n_0}(\mathbf{1}_{(a,1]})$.

Consequently, the polynomial $\rho = p^a_{n_0}$ is extremal in problem (2.7) for $n = n_0$. In the case when the degree of precision N of formula (3.2) is greater than n_0, the polynomial $p^a_{n_0}$ is extremal in problem (2.7) for all n such that $n_0 \le n \le N$.

Conversely, assume that condition (3.3) holds. The Hermite interpolation polynomial $p^a_{n_0}$ of the function $\mathbf{1}_{(a,1]}$ at the nodes of the quadrature formula (3.2) has degree n_0 and, as shown above, belongs to $\mathscr{P}^-_{n_0}(\mathbf{1}_{(a,1]})$. Hence, the quadrature formula (3.2) and the polynomial $p^a_{n_0}$ are extremal in problem (2.7) for all n such that $n_0 \le n \le N$. Indeed, for an arbitrary polynomial $p_n \in \mathscr{P}^-_n(\mathbf{1}_{(a,1]})$ with $n_0 \le n \le N$, we have

$$\|\mathbf{1}_{(a,1]} - p_n\|_{L^\upsilon(-1,1)} = \int_{-1}^{1} \upsilon(t)(\mathbf{1}_{(a,1]}(t) - p_n(t))dt = \int_a^1 \upsilon(t)dt - \int_{-1}^{1} \upsilon(t)p_n(t)dt.$$

Applying formula (3.2) and the property $p_n \le \mathbf{1}_{(a,1]}$ on $[-1,1]$, we obtain

$$\int_{-1}^{1} \upsilon(t)p_n(t)dt = \sum_{k=1}^{M} \lambda_k p(x_k) \le \sum_{k=1}^{M} \lambda_k\ \mathbf{1}_{(a,1]}(x_k) = \sum_{k(a)<k\le M} \lambda_k.$$

Thus,

$$\|\mathbf{1}_{(a,1]} - p_n\|_{L^\upsilon(-1,1)} \ge \int_a^1 \upsilon(t)dt - \sum_{k(a)<k\le M} \lambda_k. \tag{3.5}$$

The right-hand side of the latter inequality is $\|\mathbf{1}_{(a,1]} - p^a_{n_0}\|_{L^\upsilon(-1,1)}$. Therefore, (3.5) implies that

$$\|\mathbf{1}_{(a,1]} - p_{n_0}\|_{L^\upsilon(-1,1)} \ge E^-_n(\mathbf{1}_{(a,1]}) \ge \int_a^1 \upsilon(t)dt - \sum_{k(a)<k\le M} \lambda_k = \|\mathbf{1}_{(a,1]} - p_{n_0}\|_{L^\upsilon(-1,1)}.$$

Consequently, the quadrature formula (3.2) for all $n_0 \le n \le N$ and the polynomial $p^a_{n_0}$ are extremal in problem (2.7). Lemma 1 is proved completely. $\square$

The following statement is well known (see, e.g., [9, Proposition 2]); we present it here without proof.

Proposition A. *If the largest node x_M of a positive quadrature formula* (3.2) *is less than* 1, *then value* (2.7) *for all $0 \le n \le N$ satisfies the following equality for $x_M \le a \le 1$:*

$$E^-_n(\mathbf{1}_{(a,1]}) = \int_a^1 \upsilon(t)dt,$$

and the polynomial $p^a_n \equiv 0$ is extremal.

Consider the problem on the best approximation from below

$$E^-_n(\mathbf{1}_{[-1,b)}) = E^-_{n,\upsilon}(\mathbf{1}_{[-1,b)}) = \min\left\{\|\mathbf{1}_{[-1,b)} - p_n\|_{L^\upsilon(-1,1)} : p_n \in \mathscr{P}^-_n(\mathbf{1}_{[-1,b)})\right\} \tag{3.6}$$

to the characteristic function $\mathbf{1}_{[-1,b)}$ of the half-open interval $[-1,b)$, $-1 < b < 1$, by algebraic polynomials of a given degree. This problem is related to (2.7).

A statement similar to Lemma 1 holds for problem (3.6) and formula (3.2). Denote by $q^b_{n_0}$ the Hermite polynomial that interpolates the function $\mathbf{1}_{[-1,b)}$ at the nodes $\{x_k\}_{k=1}^M$ in the same sense as earlier. Specifically, the polynomial $q^b_{n_0}$ interpolates only the values of $\mathbf{1}_{[-1,b)}$ at the point a and at the points ∓ 1 if they are nodes of the quadrature formula. The polynomial $q^b_{n_0}$ interpolates both the values of the function $\mathbf{1}_{[-1,b)}$ and the values of its derivative at the other nodes of formula (3.2): $q^b_{n_0}(x_k) = \mathbf{1}_{[-1,b)}(x_k)$ and $(q^b_{n_0})'(x_k) = 0$. The degree of this polynomial is $n_0 = 2M - 2 - s - r$ again. We will call this polynomial the *Hermite interpolation polynomial of the function* $\mathbf{1}_{[-1,b)}$ *at the nodes of the quadrature formula* (3.2). The following statement is proved by the same scheme as Lemma 1; we omit its proof.

Lemma 2. *Assume that a positive quadrature formula* (3.2) *has the property* $N \geq n_0 = 2M - 2 - s - r$. *Then the following statements hold for any node* b *of this formula lying on* $(-1,1)$ *and different from the first node, i.e., for* $b = x_{k(b)}$, $1 < k(b) \leq M$.

(1) *The value* (3.6) *is the same for all* $n_0 \leq n \leq N$:

$$E^-_n(\mathbf{1}_{[-1,b)}) = E^-_{n_0}(\mathbf{1}_{[-1,b)}) = \int_{-1}^{b} \upsilon(t)dt - \sum_{1\leq k<k(b)} \lambda_k. \tag{3.7}$$

In particular, this means that formula (3.2) *is extremal in problem* (3.6).

(2) *The Hermite interpolation polynomial* $q^b_{n_0}$ *of the function* $\mathbf{1}_{[-1,b)}$ *at the nodes* $\{x_k\}_{k=1}^M$ *of the quadrature formula* (3.2) *belongs to the set* $\mathscr{P}^-_{n_0}(\mathbf{1}_{[-1,b)})$ *and is an extremal polynomial in problem* (3.6) *for all* n *such that* $n_0 \leq n \leq N$.

The following statement is an analog of Proposition A.

Proposition B. *If the smallest node* x_1 *of a positive quadrature formula* (3.2) *is greater than* -1, *then value* (2.7) *satisfies the following equality for all* $0 \leq n \leq N$ *and* $-1 < b \leq x_1$:

$$E_n(\mathbf{1}_{[-1,b)}) = \int_{-1}^{b} \upsilon(t)dt,$$

and the polynomial $q^b_n \equiv 0$ *is extremal.*

Remark 1. Let us agree that if the set of summation indices in a sum is empty, then the sum is zero. Then, according to Proposition A and Lemma 1, formula (3.4) is also valid when a is the largest node of the quadrature formula (3.2). Similarly, according to Proposition B and Lemma 2, formula (3.7) is also valid when b is the smallest node of the quadrature formula (3.2).

3.2. One-sided approximation from below to the characteristic function of an interval $(a,b) \subset [-1,1]$. The following statement for problem (3.1) can be proved with the use of Lemmas 1 and 2.

Theorem 1. *Assume that a positive quadrature formula* (3.2) *has the property* $N \geq n_0 = 2M - 2 - s - r$. *Then, for any two nodes* $a, b \in (-1,1)$ *of this formula, specifically,* $a = x_{k(a)}$ *and* $b = x_{k(b)}$, *where* $1 \leq k(a) < k(b) \leq M$, *the following statements hold.*

(1) *The value* (3.1) *is the same for all* $n_0 \le n \le N$:

$$E_n(\mathbf{1}_{(a,b)}) = E_{n_0}(\mathbf{1}_{(a,b)}) = \int\limits_a^b \upsilon(t)dt - \sum_{k(a)<k<k(b)} \lambda_k.$$

In particular, this means that formula (3.2) *is extremal in problem* (3.1).

(2) *The polynomial*

$$\varrho_{n_0}^{ab} = p_{n_0}^a + q_{n_0}^b - 1 \tag{3.8}$$

of degree n_0 *has the property*

$$\varrho_{n_0}^{ab}(x) \le \mathbf{1}_{(a,b)}(x), \quad x \in [-1,1]; \tag{3.9}$$

i.e., $\varrho_{n_0}^{ab} \in \mathscr{P}_{n_0}^-(\mathbf{1}_{(a,b)})$; *it is extremal in problem* (3.1) *for all* n *such that* $n_0 \le n \le N$.

Proof. Obviously, $\mathbf{1}_{(a,1]} + \mathbf{1}_{[-1,b)} - 1 = \mathbf{1}_{(a,b)}$. This implies property (3.9). Let us now use the standard argument that we have already applied above to prove Lemma 1. For an arbitrary polynomial $p_n \in \mathscr{P}_n^-(\mathbf{1}_{(a,b)})$ with $n_0 \le n \le N$, we have

$$\|\mathbf{1}_{(a,b)} - p_n\|_{L^\upsilon(-1,1)} = \int\limits_{-1}^1 \upsilon(t)(\mathbf{1}_{(a,b)}(t) - p_n(t))dt = \int\limits_a^b \upsilon(t)dt - \int\limits_{-1}^1 \upsilon(t)p_n(t)dt.$$

Applying formula (3.2) and the property $p_n(x) \le \mathbf{1}_{(a,b)}(x)$, $x \in [-1,1]$, we obtain the estimate

$$\|\mathbf{1}_{(a,b)} - p_n\|_{L^\upsilon(-1,1)} \ge \int\limits_a^b \upsilon(t)dt - \sum_{k(a)<k<k(b)} \lambda_k. \tag{3.10}$$

Inequality (3.10) turns into an equality at the polynomial $\varrho_{n_0}^{ab} = p_{n_0}^a + q_{n_0}^b - 1$. Indeed,

$$\|\mathbf{1}_{(a,b)} - \varrho_{n_0}^{ab}\|_{L^\upsilon(-1,1)} = \int\limits_{-1}^1 \upsilon(t)\left(\mathbf{1}_{(a,1]}(t) - p_{n_0}^a(t) + \mathbf{1}_{[-1,b)}(t) - q_{n_0}^b(t)\right)dt.$$

Using formulas (3.4) and (3.7), we obtain

$$\|\mathbf{1}_{(a,b)} - \varrho_{n_0}^{ab}\|_{L^\upsilon(-1,1)} = \int\limits_a^1 \upsilon(t)dt + \int\limits_{-1}^b \upsilon(t)dt - \Bigg(\sum_{k(a)<k\le M} \lambda_k + \sum_{1\le k<k(b)} \lambda_k\Bigg)$$

$$= \int\limits_{-1}^1 \upsilon(t)dt - \sum_{1\le k\le M} \lambda_k - \int\limits_a^b \upsilon(t)dt + \sum_{k(a)<k<k(b)} \lambda_k.$$

Formula (3.2) for the polynomial $p \equiv 1$ takes the form $\displaystyle\int_{-1}^1 \upsilon(t)\,dt = \sum_{k=1}^M \lambda_k$. Thus, we indeed have the equality

$$\|\mathbf{1}_{(a,b)} - \varrho_{n_0}^{ab}\|_{L^\upsilon(-1,1)} = \int\limits_a^b \upsilon(t)dt - \sum_{k(a)<k<k(b)} \lambda_k.$$

Using this equality, we obtain

$$\|\mathbf{1}_{(a,b)} - \varrho^{ab}_{n_0}\|_{L^\upsilon(-1,1)} \geq E_n(\mathbf{1}_{(a,b)}) \geq \int\limits_a^b \upsilon(t)dt - \sum_{k(a)<k<k(b)} \lambda_k = \|\mathbf{1}_{(a,b)} - \varrho^{ab}_{n_0}\|_{L^\upsilon(-1,1)}.$$

Consequently, the quadrature formula (3.2) and the polynomial $\varrho^{ab}_{n_0}$ are extremal in problem (3.1) for all n such that $n_0 \leq n \leq N$. Theorem 1 is proved completely. □

Remark 2. The extremal polynomial (3.8) has degree $n_0 = 2M - 2 - s - r$ and has property (3.9), as seen from its construction. Consider the Hermite polynomial ρ that interpolates the function $\mathbf{1}_{(a,b)}$ at the nodes $\{x_k\}_{k=1}^M$; more precisely, the polynomial ρ interpolates only the values of $\mathbf{1}_{(a,b)}$ at the points a and b and at the points ∓ 1 if they are nodes of the quadrature formula, while, at the other nodes x_k, it interpolates the values of the function $\mathbf{1}_{(a,b)}$ and of its derivative, which, in this case, means the property $\rho'(x_k) = 0$. The degree of such polynomial is $\underline{n} = n_0 - 1 = 2M - 3 - s - r$. As in the proof of Lemma 1, it is easy to see that $\rho(x) \leq \mathbf{1}_{(a,b)}(x)$, $x \in [x_1, x_M]$. However, the inequality

$$\rho(x) \leq \mathbf{1}_{(a,b)}(x), \quad x \in [-1,1], \tag{3.11}$$

may not hold on the whole interval $[-1,1]$. An example of such situation is the case of the 5-point Gauss quadrature formula with the nodes

$$x_1 = -x_5 = -\frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}, \quad x_2 = -x_4 = -\frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}, \quad x_3 = 0$$

for $a = x_1$ and $b = x_3$. In this case, the polynomial ρ constructed by the described method has degree 7. Calculations with the Maple package give an approximate value $\rho(-1) = 0.1650513613\ldots$; it only matters that this value is positive. Thus, in some neighborhood of the point -1, the graph of ρ lies above the graph of $\mathbf{1}_{(a,b)}$, so that property (3.11) is violated in this case.

If property (3.11) holds, then we have $E^-_{n_0-1}(\mathbf{1}_{(a,b)}) = E^-_{n_0}(\mathbf{1}_{(a,b)})$ for value (3.1). This is exactly the situation in the example considered in Section 3.3.

We now apply Theorem 1 for problem (3.1) under the assumptions of Theorem B. The degree of precision of (2.6) is $N = 2(M - (s+r)) - 1 + (s+r) = 2M - 1 - (s+r))$ in the case of fixed nodes (2.4) and $N = 2(M - (1+s+r)) - 1 + (1+s+r) = 2M - 1 - (1+s+r)$ in the case (2.5). The parameter n_0 of formula (2.6) is $n_0 = 2M - 2 - (s+r)$ in both cases; thus, the condition $n_0 \leq N$ holds in both cases. Recall that, for the nodes a and b of the quadrature formula (2.6) different from the largest and smallest nodes, respectively, the polynomial $\varrho^{ab}_{n_0} = p^a_{n_0} + q^b_{n_0} - 1$ of degree n_0 is defined, in which $p^a_{n_0}$ and $q^b_{n_0}$ are the polynomials (of degree n_0 each) that perform the corresponding Hermitian interpolation of the functions $\mathbf{1}_{(a,1]}$ and $\mathbf{1}_{[-1,b)}$ at the nodes of the quadrature formula. The following statement follows from the above argument.

Theorem 2. *If numbers a and b, $-1 < a < b < 1$, are nodes of an M-point positive quadrature formula* (2.6), *more precisely,*

$$a = x^{\mathrm{u}}_{k(a)}, \quad b = x^{\mathrm{u}}_{k(b)}, \quad k(a) < k(b),$$

then the following equality holds for $n = 2M - 2 - |\mathfrak{u}|$ *and* $n = 2M - 1 - |\mathfrak{u}|$ *in the case of fixed nodes of the form* (2.4) *and for* $n = 2M - 1 - |\mathfrak{u}|$ *in the case of fixed nodes of the form* (2.5):

$$E_n^-(\mathbf{1}_{(a,b)}) = \int_a^b \upsilon(x)dx - \sum_{k(a)<k<k(b)} \lambda_k^{\mathfrak{u}}.$$

The corresponding quadrature formula is extremal and the polynomial of the best approximation from below is polynomial (3.8) *of degree* $n = 2M - 2 - |\mathfrak{u}|$ *in the case of fixed nodes of the form* (2.4) *and of degree* $n = 2M - 1 - |\mathfrak{u}|$ *in the case of fixed nodes of the form* (2.5).

3.3. A specific example of one-sided approximation to the characteristic function of an interval. Consider problem (3.1) in the case of the unit weight $\upsilon \equiv 1$ for the nodes $a = x_{1,4}^*$ and $b = x_{3,4}^*$ of the 4-point Gauss quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{\ell=1}^{4} \lambda_{\ell,4}^* f(x_{\ell,4}^*), \quad f \in \mathscr{P}_7, \tag{3.12}$$

whose degree of precision is $N = 7$. In this case, $M = 4$ and $s = r = 0$; hence, $n_0 = 6$. Theorems 1 and 2 can be applied in this situation. However, our aim is to show that, based on Remark 2, we can also obtain a solution of the problem for $n = n_0 - 1 = 5$. The construction of (3.12) and justification of Theorem 3 below are carried out with the help of elementary, though cumbersome, calculations; we will give them here only schematically.

The nodes of formula (3.12) are zeros of the Legendre polynomial (see, for example, [13, Ch. IV])

$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3);$$

specifically,

$$x_{1,4}^* = -x_{4,4}^* = -\frac{1}{35}\sqrt{525 + 70\sqrt{30}}, \quad x_{2,4}^* = -x_{3,4}^* = -\frac{1}{35}\sqrt{525 - 70\sqrt{30}}.$$

The quadrature formula (3.12) is interpolating, and its coefficients are found by the formula

$$\lambda_{\ell,4}^* = \int_{-1}^{1} \frac{\omega(x)}{\omega'(x_{\ell,4}^*)(x - x_{\ell,4}^*)}\,dx, \quad \omega(x) = \prod_{\ell=1}^{4}(x - x_{\ell,4}^*);$$

they are positive and have the following values:

$$\lambda_{1,4}^* = \lambda_{4,4}^* = -\frac{1}{36}\sqrt{30} + \frac{1}{2}, \qquad \lambda_{2,4}^* = \lambda_{3,4}^* = \frac{1}{36}\sqrt{30} + \frac{1}{2}.$$

Theorems 1 and 2 contain a solution of problem (3.1) for the interval $J = (x_{1,4}^*, x_{3,4}^*)$ for $n = 6$ and 7. Now we will give a solution of the problem for $n = 5$. To construct an extremal polynomial for the problem, we start with the fifth degree polynomial

$$\rho(t) = (t - \xi)(t - x_{1,4}^*)(t - x_{3,4}^*)(t - x_{4,4}^*)^2. \tag{3.13}$$

We choose the root ξ of polynomial (3.13) from the condition $\rho'(x^*_{2,4}) = 0$. An elementary calculation gives the value

$$\xi = -\frac{-45\sqrt{525-70\sqrt{30}}+12\sqrt{525-70\sqrt{30}}\,\sqrt{30}+15\sqrt{525+70\sqrt{30}}-2\sqrt{525+70\sqrt{30}}\,\sqrt{30}}{-1575+280\sqrt{30}+\sqrt{525-70\sqrt{30}}\,\sqrt{525+70\sqrt{30}}}; \tag{3.14}$$

this value is approximately $-1.161692293\ldots$. What we need next is the fact that $\xi < -1$, which one can see with the help of elementary transformations, based on (3.14).

Theorem 3. *For the unit weight $\upsilon \equiv 1$, the value $E^-_n(\mathbf{1}_J)$ of one-sided approximation from below to the characteristic function of the interval $J = (x^*_{1,4}, x^*_{3,4})$ by polynomials of degree $n = 5$, 6, and 7 is the same*:

$$E^-_n(\mathbf{1}_J) = x^*_{3,4} - x^*_{1,4} - \lambda^*_{2,4}.$$

In this case, the fifth degree polynomial

$$p^*(t) = \frac{\rho(t)}{\rho(x^*_{2,4})}, \quad \rho(t) = (t-\xi)(t-x^*_{1,4})(t-x^*_{3,4})(t-x^*_{4,4})^2, \tag{3.15}$$

in which the point ξ is defined by formula (3.14), *is a polynomial of the best approximation of the function $\mathbf{1}_J$ from below*; *this polynomial interpolates the function $\mathbf{1}_J$ at the nodes of formula* (3.12)*.*

Proof. It is seen from (3.13) that the polynomial ρ has the following signs on $[-1,1]$:

$$\rho(t) \ge 0 \ \text{ for } \ t \in [-1, x^*_{1,4}] \ \text{ and } \ t \in [x^*_{3,4}, 1]; \qquad \rho(t) < 0 \ \text{ for } \ t \in (x^*_{1,4}, x^*_{3,4}).$$

The derivative ρ' of the polynomial ρ can vanish on $[x^*_{1,4}, x^*_{3,4}]$ at only one point; by the construction of ρ, this is the point $x^*_{2,4}$. Hence, $\rho(x^*_{2,4}) < 0$, and this is the absolute minimum of ρ on $[-1,1]$. Therefore, polynomial (3.15) interpolates the function $\mathbf{1}_J$ at the nodes of formula (3.12) and has the property $p^* \le \mathbf{1}_J$. This polynomial gives an upper estimate for $E^-_{n,1}(\mathbf{1}_J)$ for all $n \ge 5$.

By Theorem A, the Gauss quadrature formula (3.12) gives a lower estimate for $E^-_{n,1}(\mathbf{1}_J)$ for all $n \le 7$, which coincides with the upper estimate for $5 \le n \le 7$. Theorem 3 is proved. $\square$

3.4. One-sided approximation from above to the characteristic function of an interval $(a,b) \subset [-1,1]$. Let us now discuss the problem of one-sided approximation

$$E^+_n(\mathbf{1}_{(a,b)}) = E^+_{n,\upsilon}(\mathbf{1}_{(a,b)}) = \inf\big\{\|\mathbf{1}_{(a,b)} - p_n\|_{L^\upsilon(-1,1)} \colon p_n \in \mathscr{P}^+_n(\mathbf{1}_{(a,b)})\big\} \tag{3.16}$$

to the characteristic function $\mathbf{1}_{(a,b)}$ of an interval (a,b) from above. For this problem, analogs of all statements presented above in Sections 3.1 and 3.2 for problem (3.1) of one-sided approximation from below are valid.

We start with the positive quadrature formula (3.2), whose degree of precision N satisfies the condition $N \ge n_0 = 2M - 2 - s - r$. Let $a \in (-1,1)$ be a node of this formula. If it is not the smallest node, then denote by $\overline{\rho} = \overline{p}^a_{n_0}$ the Hermite polynomial of degree n_0 that interpolates the characteristic function $\mathbf{1}_{[a,1]}$ of the interval $[a,1]$ at the nodes of formula (3.2); more precisely, it interpolates only the values of this function at the points ∓ 1 if they are nodes of the quadrature formula and at the point a; in particular, $\overline{\rho}(a) = 1$. At the other nodes of formula (3.2), the polynomial $\overline{\rho}$ interpolates both the values of the function $\mathbf{1}_{[a,1]}$ and the values of its derivative: $\overline{\rho}(x_k) = \mathbf{1}_{[a,1]}(x_k)$ and $\overline{\rho}'(x_k) = 0$. A polynomial with these interpolation properties exists (see, for example, [11, Ch. 2, Sect. 11] or [12, Lect. 4, Sect. 4.3]). If $a \in (-1,1)$ is the smallest node of formula (3.2), then we define $\overline{p}^a_{n_0} \equiv 1$; this polynomial again performs the Hermitian interpolation of the function $\mathbf{1}_{[a,1]}$ at the nodes of formula (3.2).

The polynomial $\overline{p}^a_{n_0}$ belongs to the set $\mathscr{P}^+_{n_0}(\mathbf{1}_{[a,1]})$ and is extremal in problem (3.6) of approximation to the characteristic function from above for all n with the property $n_0 \le n \le N$. This can be verified in various ways, for example, using the following considerations. Obviously, the equality $E^+_n(f) = E^-_n(1-f)$ holds for every measurable bounded function f for all $n \ge 1$ together with the corresponding relation between extremal polynomials. For the function $\mathbf{1}_{[a,1]}$, we have $1 - \mathbf{1}_{[a,1]} = \mathbf{1}_{[-1,a)}$. The polynomial $1 - \overline{p}^a_{n_0}$ performs the Hermitian interpolation of the function $\mathbf{1}_{[-1,a)}$ at the nodes of the quadrature formula (3.2). Hence, it coincides with the extremal polynomial $q^a_{n_0}$ of the problem on studying the value $E^-_{n_0}(\mathbf{1}_{[-1,a)})$ defined by formula (3.6). Thus, $\overline{p}^a_{n_0} = 1 - q^a_{n_0}$. Lemma 2 and Proposition B guarantee all extremal properties of $\overline{p}^a_{n_0}$.

For a node $b \in (-1,1)$ of formula (3.2) different from the largest one, denote by $\overline{q}^a_{n_0}$ the Hermite polynomial of degree n_0 that interpolates the characteristic function $\mathbf{1}_{[-1,b]}$ of the interval $[-1,b]$ at the nodes of formula (3.2). Note that, at the points ∓ 1 if they are nodes of the quadrature formula and at the point a, the polynomial $\overline{q}^b_{n_0}$ interpolates only the values of the function $\mathbf{1}_{[a,1]}$; in particular, $\overline{q}^b_{n_0}(b) = 1$. At the other nodes of formula (3.2), the polynomial $\overline{q}^b_{n_0}$ interpolates both the values of the function $\mathbf{1}_{[a,1]}$ and the values of its derivative. If $b \in (-1,1)$ is the largest node of formula (3.2), then we define $\overline{q}^b_{n_0} \equiv 1$; this polynomial also performs the Hermitian interpolation of the function $\mathbf{1}_{[-1,b]}$ at the nodes of formula (3.2).

The polynomial $\overline{q}^b_{n_0}$ belongs to the set $\mathscr{P}^+_{n_0}(\mathbf{1}_{[-1,b]})$ and is extremal in the problem of approximation to the characteristic function $\mathbf{1}_{[-1,b]}$ of the interval $[-1,b]$ from above for all n with the property $n_0 \le n \le N$.

The following statement is proved by the same scheme as Theorem 1.

Theorem 4. *Let a quadrature formula* (3.2) *be positive, and let* $N \ge n_0 = 2M - 2 - s - r$. *Then the following statements hold for every two nodes* $a, b \in (-1,1)$ *of this formula,* $a = x_{k(a)}$ *and* $b = x_{k(b)}$, *where* $1 \le k(a) < k(b) \le M$.

(1) *The value* (3.16) *is the same for all* $n_0 \le n \le N$:

$$E^+_n(\mathbf{1}_{(a,b)}) = E^+_{n_0}(\mathbf{1}_{(a,b)}) = \sum_{k=k(a)}^{k(b)} \lambda_k - \int_a^b \upsilon(t)dt.$$

(2) *The polynomial* $\overline{\varrho}^{ab}_{n_0} = \overline{p}^a_{n_0} + \overline{q}^b_{n_0} - 1$ *of degree* n_0 *has the property*

$$\overline{\varrho}^{ab}_{n_0}(x) \ge \mathbf{1}_{(a,b)}(x), \quad x \in [-1,1];$$

i.e., $\overline{\varrho}^{ab}_{n_0} \in \mathscr{P}^+_{n_0}(\mathbf{1}_{(a,b)})$, *and it is extremal in problem* (3.16) *for all* n *such that* $n_0 \le n \le N$.

As a consequence of Theorem 4, an analog of Theorem 2 is also valid for problem (3.16).

4. ONE-SIDED APPROXIMATION TO THE CHARACTERISTIC FUNCTION OF A SYMMETRIC INTERVAL IN THE CASE OF AN EVEN WEIGHT

The results of [9] related to problem (2.7) and described in Theorem B allow us to write a solution of the problem of best one-sided integral approximation from below and from above to the characteristic function of a symmetric interval $J = (-h,h)$, $0 < h < 1$, in the case of an even weight.

We restrict ourselves to the problem (2.3) of approximation from below and start with the following problem. Let v be a weight on the interval $(0,1)$. Consider the problem on the best approximation from below

$$E_n^-(\mathbf{1}_{[0,h^2)}) = E_n^-(\mathbf{1}_{[0,h^2)})_{L^v(0,1)} = \min\left\{\|\mathbf{1}_{[0,h^2)} - p_n\|_{L^v(0,1)} \colon p_n \in \mathscr{P}_n^-(\mathbf{1}_{[0,h^2)})\right\} \tag{4.1}$$

to the characteristic function $\mathbf{1}_{[0,h^2)}$ of a half-open interval $J=[0,h^2)$ in the space $L^v(0,1)$ on the interval $(0,1)$ by the set of algebraic polynomials of degree $n \ge 1$. By a linear change of variable, this problem reduces to a problem of type (2.3) on $(-1,1)$ whose solution, as noted above, was given in [9]. In the following statement, by means of fairly simple considerations, it is shown that problem (4.1) is equivalent to the problem

$$E_{2n+1}^-(\mathbf{1}_{(-h,h)})_{L^w(-1,1)} = \min\left\{\|\mathbf{1}_{(-h,h)} - q_{2n+1}\|_{L^w(-1,1)} \colon q_{2n+1} \in \mathscr{P}_{2n+1}^-(\mathbf{1}_{(-h,h)})\right\} \tag{4.2}$$

for the weight $w(t) = v(t^2)|t|$, $t \in (-1,1)$.

Theorem 5. *For $0 < h < 1$ and $n \ge 1$, problems (4.2) and (4.1) are equivalent; more precisely, the following statements hold:*

(1)
$$E_{2n+1}^-(\mathbf{1}_{(-h,h)})_{L^w(-1,1)} = E_n^-(\mathbf{1}_{[0,h^2)})_{L^v(0,1)}; \tag{4.3}$$

(2) *a polynomial p_n^* is extremal in problem (4.1) if and only if the polynomial $p_n^*(t^2)$ is extremal in problem (4.2).*

Proof. Assume that $p_n \in \mathscr{P}_n^-(\mathbf{1}_{[0,h^2)})$ on $(0,1)$, i.e., p_n is a polynomial of degree at most n with the property $p_n(t) \le \mathbf{1}_{[0,h^2)}(t)$, $t \in (0,1)$. Then the polynomial $q_{2n}(t) = p_n(t^2)$ has the property $q_{2n}(t) \le \mathbf{1}_{(-h,h)}(t)$, $t \in (-1,1)$. Making the change $\eta = t^2$ in the integral

$$\|\mathbf{1}_{[0,h^2)} - p_n\|_{L^v(0,1)} = \int_0^1 v(\eta)(\mathbf{1}_{[0,h^2)}(\eta) - p_n(\eta))d\eta,$$

we get

$$\|\mathbf{1}_{[0,h^2)} - p_n\|_{L^v(0,1)} = 2\int_0^1 v(t^2)t(\mathbf{1}_{[0,h^2)}(t^2) - p_n(t^2))dt$$

$$= 2\int_0^1 v(t^2)t(\mathbf{1}_{[0,h)}(t) - q_{2n}(t))dt = \int_{-1}^1 v(t^2)|t|(\mathbf{1}_{(-h,h)}(t) - q_{2n}(t))dt.$$

Hence, $E_{2n+1}^-(\mathbf{1}_{(-h,h)})_{L^w(-1,1)} \le \|\mathbf{1}_{[0,h^2)} - p_n\|_{L^v(0,1)}$, which yields

$$E_{2n+1}^-(\mathbf{1}_{(-h,h)})_{L^w(-1,1)} \le E_n^-(\mathbf{1}_{[0,h^2)})_{L^v(0,1)}. \tag{4.4}$$

Conversely, let a polynomial q_{2n+1} of degree $2n+1$ be such that $q_{2n+1}(t) \le \mathbf{1}_{(-h,h)}(t)$ for $t \in (-1,1)$. The polynomial $q_{2n}(t) = (q_{2n+1}(t) + q_{2n+1}(-t))/2$ also satisfies the inequality $q_{2n}(t) \le \mathbf{1}_{(-h,h)}(t)$, $t \in (-1,1)$. The polynomial q_{2n} is even, has degree $2n$, and, hence, is representable in the form $q_{2n}(t) = p_n(t^2)$, where p_n is a polynomial of degree n with the property $p_n(t) \le \mathbf{1}_{[0,h^2)}(t)$, $t \in (0,1)$. Hence,

$$E_n^-(\mathbf{1}_{[0,h^2)})_{L^v(0,1)} \le \|\mathbf{1}_{(-h,h)} - q_{2n}\|_{L^w(-1,1)} = \|\mathbf{1}_{(-h,h)} - q_{2n+1}\|_{L^w(-1,1)}.$$

Consequently,

$$E_n^-(\mathbf{1}_{[0,h^2)})_{L^v(0,1)} \le E_{2n+1}^-(\mathbf{1}_{(-h,h)})_{L^w(-1,1)}. \tag{4.5}$$

Inequalities (4.4) and (4.5) imply equality (4.3). The first statement of the theorem is proved. The second statement is easy to obtain by analyzing the above proof. Theorem 5 is proved. □

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REFERENCES

1. R. Bojanic and R. DeVore, "On polynomials of best one-sided approximation," Enseign. Math. **2** (12), 139–164 (1966).
2. V. I. Krylov, *Approximate Calculation of Integrals* (Fizmatgiz, Moscow, 1959; Dover, New York, 2006).
3. N. P. Korneichuk, A. A. Ligun, and V. G. Doronin, *Constrained Approximation* (Naukova Dumka, Kiev, 1982) [in Russian].
4. J. Bustamante, J. M. Quesada, and R. Martínez-Cruz, "Best one-sided L_1 approximation to the Heaviside and sign functions," J. Approx. Theory **164**, 791–802 (2012). doi 10.1016/j.jat.2012.02.006
5. X.-J. Li and J. D. Vaaler, "Some trigonometric extremal functions and the Erdös–Turán type inequalities," Indiana Univ. Math. J. **48** (1), 183–236 (1999). doi 10.1512/iumj.1999.48.1508
6. V. P. Motornyi, O. V. Motornaya, and P. K. Nitiema, "One-sided approximation of a step by algebraic polynomials in the mean," Ukr. Math. J. **62** (3), 467–482 (2010).
7. J. Bustamante, R. Martínez-Cruz, and J. M. Quesada, "Quasi orthogonal Jacobi polynomials and best one-sided L_1 approximation to step functions," J. Approx. Theory **198**, 10–23 (2015). doi 10.1016/j.jat.2015.05.001
8. A. G. Babenko, Yu. V. Kryakin, and V. A. Yudin, "One-sided approximation in l of the characteristic function of an interval by trigonometric polynomials," Proc. Steklov Inst. Math. **280** (Suppl. 1), S39–S52 (2013).
9. A. G. Babenko, M. V. Deikalova, and Sz. G. Revesz, "Weighted one-sided integral approximations to characteristic functions of intervals by polynomials on a closed interval," Proc. Steklov Inst. Math. **297** (Suppl. 1), S11–S18 (2017). doi 10.1134/S0081543817050029
10. B. Beckermann, J. Bustamante, R. Martínez-Cruz, and J. M. Quesada, "Gaussian, Lobatto and Radau positive quadrature rules with a prescribed abscissa," Calcolo **51** (2), 319–328 (2014). doi 10.1007/s10092-013-0087-3
11. I. S. Berezin and N. P. Zhidkov, *Computing Methods* (Fizmatgiz, Moscow, 1962; Pergamon, Oxford, 1965), Vol. 1.
12. V. V. Arestov, V. I. Berdyshev, N. I. Chernykh, T. V. Demina, N. N. Kholschevnikova, S. V. Konyagin, Yu. N. Subbotin, S. A. Telyakovskii, I. G. Tsar'kov, and V. A. Yudin, "Exposition of the lectures by S. B. Stechkin on approximation theory," Eurasian Math. J. **2** (4), 5–155 (2011).
13. P. K. Suetin, *Classical Orthogonal Polynomials* (Nauka, Moscow, 1976) [in Russian].

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