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On the Solvability of a Class of Nonlinear Hammerstein–Stieltjes Integral Equations on the Whole Line

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Abstract—We consider a nonlinear integral equation on the whole line with a Hammerstein–Stieltjes integral operator whose pre-kernel is a continuous distribution function. Under certain conditions imposed on the nonlinearity, we prove constructive existence and uniqueness theorems for nonnegative monotone bounded solutions. Some qualitative properties of the constructed solution are also studied. In particular, the results proved in the paper contain a theorem of O. Diekmann as a special case.

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1. INTRODUCTION

In this paper we study the following class of nonlinear integral equations of the Hammerstein–Stieltjes type:

$$f(x)=\int_{-\infty}^{\infty} G(f(x-t))\,dF(t), \qquad x\in\mathbb{R}, \tag{1.1}$$

with respect to a sought nonnegative bounded function $f(x)$. In equation (1.1) the pre-kernel F satisfies the following conditions:

(P1) $F\in C(\mathbb{R})$, $F(-\infty)=0$, $F(+\infty)=1$, and F increases on $\mathbb{R}$;

(P2) $\int_0^\infty (1-F(x))\,dx<+\infty$ and $\int_{-\infty}^\infty x^2\,dF(x)<+\infty$;

(P3) $\nu(F):=\int_{-\infty}^\infty x\,dF(x)>0$.

Concerning the function G, we assume that the following conditions hold (Fig. 1):

(N1) there exists a number $\eta>0$ such that $G(u)$ increases in u on the interval $[0,\eta]$;

(N2) $G(u)$ is concave on the interval $[0,\eta]$, with $G(0)=0$ and $G(\eta)=\eta$;

(N3) there exists a finite derivative $G'(0)>1$ such that

$$G(u)\le G'(0)u, \qquad u\in[0,\eta];$$

(N4) there exist numbers $c>0$ and $\varepsilon>0$ such that

$$G(u)\ge G'(0)u-cu^{1+\varepsilon}, \qquad u\in[0,\eta].$$

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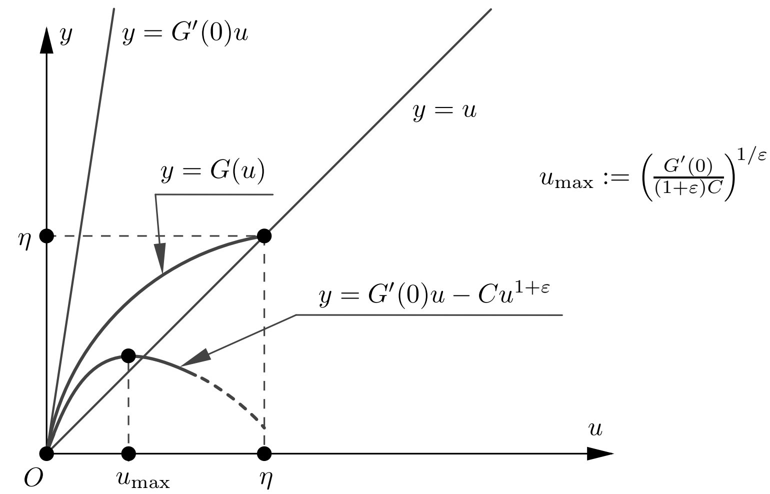

Fig. 1.

Equation (1.1) arises in various applied problems of natural science. In particular, equations of this type are used in the mathematical theory of geographical spread of epidemic and in the theory of Markov processes (see [1, 2, 4, 5]).

It should be noted that the corresponding linear and nonlinear equations on the positive half-line were studied in considerable detail in [2] and [5], respectively.

One can directly verify that equation (1.1) has two vacuum solutions $f(x) \equiv 0$ and $f(x) \equiv \eta$.

In the present paper, we use the method of constructing invariant conic segments for the nonlinear monotone Hammerstein–Stieltjes operator to find a nontrivial nonnegative monotonically nondecreasing solution between the vacua 0 and η for equation (1.1). This solution satisfies the following limit relations:

$$\lim_{x\to-\infty} f(x) = 0, \qquad \lim_{x\to+\infty} f(x) = \eta.$$

In Section 3 we also establish the following inclusions:

$$f \in L_1(-\infty, 0), \qquad \eta - f \in L_1(0, +\infty).$$

A uniqueness theorem in a certain conic segment is proved in Section 4.

In the final Section 5, we give particular examples of functions G that satisfy all the hypotheses of the results proved in the paper.

2. AUXILIARY FACTS

2.1. Properties of Diekmann's function. Introduce Diekmann's function (see [1]) on the interval $[0, +\infty)$ by setting

$$L(\lambda) := G'(0) \int_{-\infty}^{\infty} e^{-\lambda t}\, dF(t), \tag{2.1}$$

where the integral is assumed to converge.

Note that

$$L(0) = G'(0)\big(F(+\infty) - F(-\infty)\big) = G'(0) > 1.$$

Condition (P3) also implies that $L'(0) = -G'(0)\nu(F) < 0$. Since the function $L(\lambda)$ is continuous, there exists a number $\lambda_0 > 0$ such that the inequality

$$L'(\lambda) < 0$$

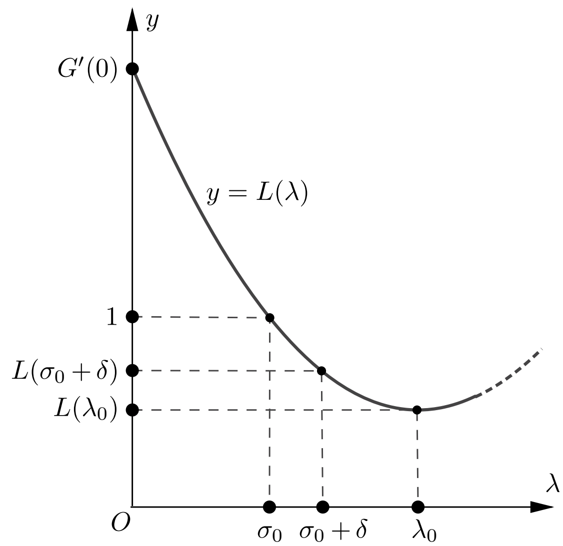

Fig. 2.

holds for all $\lambda \in [0, \lambda_0]$. Therefore,

$$L(\lambda) \text{ decreases in } \lambda \text{ on } [0, \lambda_0]. \tag{2.2}$$

On the other hand,

$$L''(\lambda) = G'(0) \int_{-\infty}^{\infty} t^2 e^{-\lambda t}\, dF(t) > 0$$

(the integral may be equal to $+\infty$), which implies that $L(\lambda)$ is convex on $[0, +\infty)$ (Fig. 2).

Suppose that

$$L(\lambda_0) < 1. \tag{2.3}$$

Then, by the intermediate value theorem, there exists a (unique) number $\sigma_0 \in (0, \lambda_0)$ such that

$$L(\sigma_0) = 1. \tag{2.4}$$

Note also that for $\delta \in (0, \lambda_0 - \sigma_0)$ we have

$$L(\delta + \sigma_0) < 1. \tag{2.5}$$

2.2. A priori estimate for the left end of Diekmann's conic segment. Consider the following function introduced in [1]:

$$\Phi(x) := \max\bigl\{\eta e^{\sigma_0 x} - M e^{(\delta+\sigma_0)x}, 0\bigr\}, \qquad x \in \mathbb{R}, \tag{2.6}$$

where $M > 0$ and $\delta \in (0, \lambda_0 - \sigma_0)$ are numerical parameters.

Note that for $M > \eta$ the function $\Phi(x)$ attains its maximum at the point

$$x_{\max} = \frac{1}{\delta} \ln \frac{\eta\sigma_0}{M(\delta + \sigma_0)} < 0$$

and $\Phi(x) = 0$ for $x \geq (1/\delta)\ln(\eta/M)$.

These properties immediately imply that if $\delta \in (0, \min\{\varepsilon\sigma_0, \lambda_0 - \sigma_0\})$, then inequality (2.5) holds together with the following upper bound for the function $\Phi(x)$:

$$\Phi^{1+\varepsilon}(x) \leq \eta^{1+\varepsilon} e^{(\delta+\sigma_0)x}, \qquad x \in \mathbb{R}. \tag{2.7}$$

Estimates (2.5) and (2.7) play an important role in further considerations.

3. SOLVABILITY OF EQUATION (1.1)

3.1. Successive approximations. Consider the following successive approximations proposed by Diekmann for equation (1.1):

$$f_0(x)=\begin{cases}\eta e^{\sigma_0 x}, & x\le 0,\\ \eta, & x>0,\end{cases} \tag{3.1}$$

$$f_{n+1}(x)=\int\limits_{-\infty}^{\infty} G(f_n(x-t))\,dF(t), \qquad x\in\mathbb{R},\quad n=0,1,2,\dots. \tag{3.2}$$

Using properties (P1)–(P3) and (N1)–(N4) as well as inequalities (2.5), (2.7) and formula (2.4), we can easily prove by induction on n that

(A) $f_n(x)$ does not increase in n for $x\in\mathbb{R}$;

(B) $f_n(x)$ does not decrease in x on $\mathbb{R}$, $n=0,1,2,\dots$;

(C) for $M>\max\{\eta, c\eta^{1+\varepsilon}L(\sigma_0+\delta)/(G'(0)(1-L(\sigma_0+\delta)))\}$ and $\delta\in(0,\min\{\varepsilon\sigma_0,\lambda_0-\sigma_0\})$, the following inequality holds:

$$f_n(x)\ge\Phi(x), \qquad n=0,1,2,\dots, \quad x\in\mathbb{R};$$

(D) $f_n\in C(\mathbb{R})$, $n=0,1,2,\dots$.

Thus, properties (A)–(D) imply that the sequence of continuous functions $\{f_n(x)\}_{n=0}^{\infty}$ has a pointwise limit as $n\to\infty$: $\lim_{n\to\infty} f_n(x)=f(x)$; moreover, by Beppo Levi's limit theorem (see [6]), the function $f(x)$ satisfies equation (1.1). On the other hand, since the pre-kernel F is continuous and the function $G(f(x))$ is bounded, we can state that $f\in C(\mathbb{R})$ (due to the properties of the convolution).

Properties (A) and (C) also imply that

$$\Phi(x)\le f(x)\le f_0(x), \qquad x\in\mathbb{R}. \tag{3.3}$$

3.2. The limit of the solution at $\pm\infty$. First note that the limit function $f(x)$ is monotonically nondecreasing on $\mathbb{R}$ according to property (B). Therefore, in view of (3.3), we can state that there exist limits

$$0<l:=\lim_{x\to+\infty} f(x)\le\eta \qquad \text{and} \qquad \lim_{x\to-\infty} f(x)=0. \tag{3.4}$$

Passing to the limit on both sides of equation (1.1) as $x\to+\infty$ and using the well-known limit relation for the convolution operation (see [3]), we obtain

$$l=G(l), \qquad l\in(0,\eta]. \tag{3.5}$$

Since the function G is concave and satisfies property (N2), we conclude from (3.5) that $l=\eta$. Thus, $\lim_{x\to+\infty} f(x)=\eta$.

It also immediately follows from (3.3) that

$$f\in L_1(-\infty,0) \qquad \text{and} \qquad \int\limits_{-\infty}^{0} f(x)\,dx\le\frac{\eta}{\sigma_0}. \tag{3.6}$$

3.3. Integral asymptotics of the solution at $+\infty$. Since $\lim_{x\to+\infty} f(x)=\eta$ and $f(x)$ does not decrease on $\mathbb{R}$, there exists an $r>0$ such that the inequality

$$f(x)\geq\frac{\eta}{2} \tag{3.7}$$

holds for $x\geq r$. We fix such an r. Since $f\in C(\mathbb{R})$, we obviously have $\eta-f\in L_1(0,r)$. Let us estimate the difference $\eta-f(x)$ for all $x\geq r$. To this end, we introduce the additional notation

$$\widetilde{F}(x)=1-F(x)\qquad\text{and}\qquad\widetilde{G}(u)=\eta-G(u).$$

In view of condition (P1) and the monotonicity of f, from (1.1) we obtain

$$\begin{aligned}
\eta-f(x)&=\eta\int_{-\infty}^{\infty}dF(t)-\int_{-\infty}^{\infty}G(f(x-t))\,dF(t)\\
&=\eta\int_{x}^{\infty}dF(t)+\eta\int_{-\infty}^{x}dF(t)-\int_{-\infty}^{x}G(f(x-t))\,dF(t)-\int_{x}^{\infty}G(f(x-t))\,dF(t)\\
&=\eta\widetilde{F}(x)+\int_{-\infty}^{x}\widetilde{G}(f(x-t))\,dF(t)-\int_{x}^{\infty}G(f(x-t))\,dF(t)\\
&\leq\eta\widetilde{F}(x)+\int_{-\infty}^{0}\widetilde{G}(f(x-t))\,dF(t)+\int_{0}^{x}\widetilde{G}(f(x-t))\,dF(t)\\
&\leq\eta\widetilde{F}(x)+\widetilde{G}(f(x))F(0)-\int_{0}^{x}\widetilde{G}(f(t))\,d_tF(x-t)\\
&=\eta\widetilde{F}(x)+F(0)\widetilde{G}(f(x))-\int_{0}^{r}\widetilde{G}(f(t))\,d_tF(x-t)-\int_{r}^{x}\widetilde{G}(f(t))\,d_tF(x-t)\\
&\leq\eta\widetilde{F}(x)+F(0)\widetilde{G}(f(x))+\eta(F(x)-F(x-r))+\int_{0}^{x-r}\widetilde{G}(f(x-t))\,dF(t)\\
&\leq\eta\widetilde{F}(x)+F(0)\widetilde{G}(f(x))+\eta\widetilde{F}(x-r)+\int_{0}^{x-r}\widetilde{G}(f(x-t))\,dF(t).
\end{aligned}$$

Thus, the monotonicity of f and inequality (3.3) imply the two-sided estimate

$$0\leq\eta-f(x)\leq\eta\widetilde{F}(x)+F(0)\widetilde{G}(f(x))+\eta\widetilde{F}(x-r)+\int_{0}^{x-r}\widetilde{G}(f(x-t))\,dF(t),\qquad x\in\mathbb{R}. \tag{3.8}$$

Let $R>r$ be an arbitrary number. Integrating both sides of inequality (3.8) with respect to x from r to R and applying the Fubini theorem (see [6]) and conditions (P1) and (P2), we find

$$\begin{aligned}
&\int_{r}^{R}(\eta-f(x))\,dx\\
&\qquad\leq\eta\int_{r}^{\infty}\widetilde{F}(x)\,dx+F(0)\int_{r}^{R}\widetilde{G}(f(x))\,dx+\eta\int_{r}^{\infty}\widetilde{F}(x-r)\,dx+\int_{r}^{R}\int_{0}^{x-r}\widetilde{G}(f(x-t))\,dF(t)\,dx
\end{aligned}$$

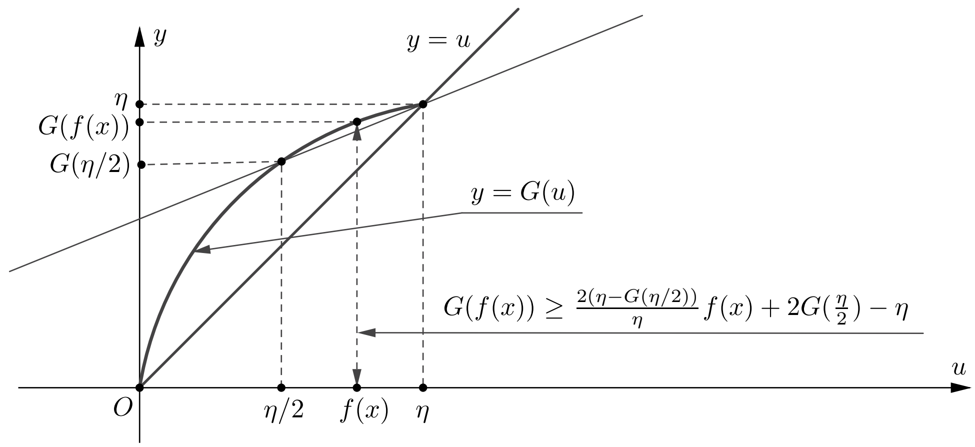

Fig. 3.

$$
\begin{aligned}
&= \eta \int\limits_r^\infty \widetilde{F}(x)\,dx + \eta \int\limits_0^\infty \widetilde{F}(x)\,dx + F(0) \int\limits_r^R \widetilde{G}(f(x))\,dx + \int\limits_0^{R-r} \int\limits_{r+t}^R \widetilde{G}(f(x-t))\,dx\,dF(t) \\
&= \eta \int\limits_r^\infty \widetilde{F}(x)\,dx + \eta \int\limits_0^\infty \widetilde{F}(x)\,dx + F(0) \int\limits_r^R \widetilde{G}(f(x))\,dx + \int\limits_0^{R-r} \int\limits_{r}^{R-t} \widetilde{G}(f(y))\,dy\,dF(t) \\
&\le \eta \int\limits_r^\infty \widetilde{F}(x)\,dx + \eta \int\limits_0^\infty \widetilde{F}(x)\,dx + F(0) \int\limits_r^R \widetilde{G}(f(x))\,dx + \int\limits_0^{\infty} \int\limits_{r}^{R} \widetilde{G}(f(y))\,dy\,dF(t) \\
&= \eta \int\limits_r^\infty \widetilde{F}(x)\,dx + \eta \int\limits_0^\infty \widetilde{F}(x)\,dx + F(0) \int\limits_r^R \widetilde{G}(f(x))\,dx + (1-F(0)) \int\limits_r^R \widetilde{G}(f(x))\,dx,
\end{aligned}
$$

which implies

$$
\int\limits_r^R \big(G(f(x)) - f(x)\big)\,dx \le \eta \int\limits_r^\infty \widetilde{F}(x)\,dx + \eta \int\limits_0^\infty \widetilde{F}(x)\,dx. \tag{3.9}
$$

Let us compose an equation of the straight line (Fig. 3) passing through the points $(\eta/2, G(\eta/2))$ and (η,η):

$$
y = \frac{2(\eta - G(\eta/2))}{\eta} u + 2G\Big(\frac{\eta}{2}\Big) - \eta.
$$

In view of inequality (3.7) and the concavity of G, we can state that

$$
G(f(x)) \ge \frac{2(\eta - G(\eta/2))}{\eta} f(x) + 2G\Big(\frac{\eta}{2}\Big) - \eta, \qquad x \ge r. \tag{3.10}
$$

Inequality (3.10) implies

$$
G(f(x)) - f(x) \ge (\eta - f(x)) \frac{2G(\eta/2) - \eta}{\eta}, \qquad x \ge r. \tag{3.11}
$$

Note that $\alpha := (2G(\eta/2) - \eta)/\eta > 0$, because $G(u) > u$ on $(0,\eta)$.

Using (3.9) and (3.11), we arrive at the following chain of inequalities:

$$\alpha\int\limits_r^R(\eta-f(x))\,dx\le\int\limits_r^R\big(G(f(x))-f(x)\big)\,dx\le\eta\int\limits_r^\infty\widetilde F(x)\,dx+\eta\int\limits_0^\infty\widetilde F(x)\,dx.$$

Letting here $R\to+\infty$, we obtain the inclusion $\eta-f\in L_1(r,+\infty)$ and the inequality

$$\int\limits_r^\infty(\eta-f(x))\,dx\le\frac{1}{\alpha}\int\limits_r^\infty\big(G(f(x))-f(x)\big)\,dx\le\frac{\eta}{\alpha}\left(\int\limits_r^\infty\widetilde F(x)\,dx+\int\limits_0^\infty\widetilde F(x)\,dx\right). \tag{3.12}$$

Thus, since f is continuous, we finally conclude that $\eta-f\in L_1(0,+\infty)$.

Note that all possible shifts of the constructed solution $f(x)$ also satisfy equation (1.1).

Thus, the above considerations lead to the following theorem.

Theorem 1. *Under conditions* (P1)–(P3), (N1)–(N4), *and* (2.3), *equation* (1.1) *has a one-parameter family of nontrivial nonnegative monotonically nondecreasing continuous bounded solutions on* $\mathbb{R}$ *of the form* $f_c(x)=f(x+c)$, $c\in\mathbb{R}$, *where the function* $f(x)$ *satisfies equation* (1.1) *and has the following additional properties*:

(I) $\lim_{x\to-\infty}f(x)=0$ *and* $\lim_{x\to+\infty}f(x)=\eta$;

(II) $f\in L_1(-\infty,0)$ *and* $\int_{-\infty}^0 f(x)\,dx\le\eta/\sigma_0$;

(III) $\eta-f\in L_1(0,+\infty)$.

Remark 1. It should be noted that, on the one hand, the result proved above generalizes Theorem 6 from [1] (in [1] the number ε is 1 and $F(x)$ is an absolutely continuous function on $\mathbb{R}$); on the other hand, our result complements the theorem from [1], since we have also proved the additional property (III) of the solution f.

4. THE UNIQUENESS OF A SOLUTION IN A SPECIFIC CONIC SEGMENT

As shown above, equation (1.1) has a one-parameter family of nonnegative nontrivial solutions of the form $f_c(x)=f(x+c)$, $c\in\mathbb{R}$, with certain properties (see Theorem 1). These solutions are generated by the main solution $f(x)$. A natural question arises as to whether the constructed solution $f(x)$ is unique in the conic segment $[\Phi(x),f_0(x)]$.

Introduce the notation

$$\mathcal{P}:=\big\{f\in C(\mathbb{R})\colon\ \Phi(x)\le f(x)\le f_0(x),\ x\in\mathbb{R}\big\}. \tag{4.1}$$

The following theorem holds.

Theorem 2. *Suppose that all conditions of Theorem* 1 *are satisfied. Then equation* (1.1) *has a unique solution in the function class* $\mathcal{P}$.

Proof. Assume the contrary: equation (1.1) has two solutions $f,\widetilde f\in\mathcal{P}$. Let us show that in this case the function

$$e^{-(\delta+\sigma_0)x}|f(x)-\widetilde f(x)|$$

is bounded on $\mathbb{R}$. Indeed, for $x\ge(1/\delta)\ln(\eta/M)$, by the definition of the functions $f_0(x)$ and $\Phi(x)$, we have

$$e^{-(\delta+\sigma_0)x}|f(x)-\widetilde f(x)|\le 2\eta e^{-(\delta+\sigma_0)x}\le 2\eta\left(\frac{M}{\eta}\right)^{(1+\sigma_0/\delta)}<+\infty.$$

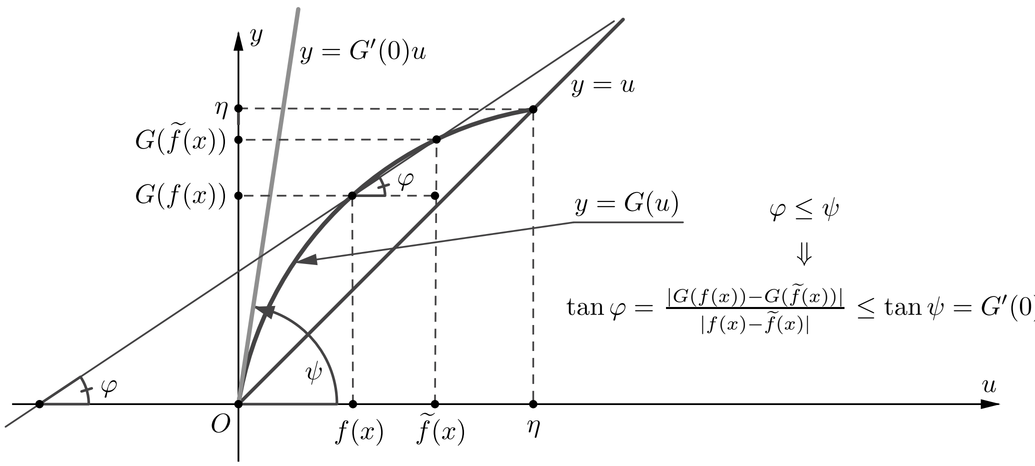

Fig. 4.

If $x \in (-\infty, (1/\delta)\ln(\eta/M))$, then

$$-Me^{(\delta+\sigma_0)x} \le f(x) - \widetilde{f}(x) \le Me^{(\delta+\sigma_0)x}, \qquad \text{or} \qquad e^{-(\delta+\sigma_0)x}|f(x) - \widetilde{f}(x)| \le M.$$

Thus,

$$\alpha := \sup_{x\in\mathbb{R}} e^{-(\delta+\sigma_0)x}|f(x) - \widetilde{f}(x)| < +\infty. \tag{4.2}$$

Since G is concave on the interval $[0,\eta]$, it follows from conditions (N1) and (N3) that (Fig. 4)

$$\big|G(f(x)) - G(\widetilde{f}(x))\big| \le G'(0)|f(x) - \widetilde{f}(x)|, \qquad x \in \mathbb{R}. \tag{4.3}$$

In view of (4.2) and (4.3), from equation (1.1) we obtain

$$\begin{aligned}|f(x) - \widetilde{f}(x)| &\le \int\limits_{-\infty}^{\infty} \big|G(f(x-t)) - G(\widetilde{f}(x-t))\big|\,dF(t) \le G'(0)\int\limits_{-\infty}^{\infty}\big|f(x-t) - \widetilde{f}(x-t)\big|\,dF(t)\\ &= G'(0)\int\limits_{-\infty}^{\infty}\big|f(x-t) - \widetilde{f}(x-t)\big|\,e^{-(\delta+\sigma_0)(x-t)}\,e^{(\delta+\sigma_0)x}\,e^{-(\delta+\sigma_0)t}\,dF(t)\\ &\le \alpha G'(0)e^{(\delta+\sigma_0)x}\int\limits_{-\infty}^{\infty} e^{-(\delta+\sigma_0)t}\,dF(t) = \alpha e^{(\delta+\sigma_0)x}L(\delta+\sigma_0),\end{aligned}$$

which implies

$$e^{-(\delta+\sigma_0)x}|f(x) - \widetilde{f}(x)| \le \alpha L(\delta+\sigma_0), \qquad x \in \mathbb{R}. \tag{4.4}$$

It follows from (4.4) that

$$\alpha \le \alpha L(\delta+\sigma_0). \tag{4.5}$$

Since

$$L(\delta+\sigma_0) < 1$$

for $\delta \in (0, \min\{\varepsilon, \lambda_0 - \sigma_0\})$, we obtain $\alpha = 0$ in view of (4.5). Therefore, $f(x) = \widetilde{f}(x)$ on $\mathbb{R}$. $\square$

Remark 2. Applying arguments similar to those used in the proof of Theorem 2, one can verify that

$$\alpha_n := \sup_{x\in\mathbb{R}} e^{-(\delta+\sigma_0)x}|f_{n+1}(x) - f_n(x)| < +\infty.$$

From (3.1), (3.2), and (4.3), by induction on n, one can easily derive the estimate

$$\alpha_n \le \alpha_0 L^n(\delta+\sigma_0), \qquad n = 0, 1, 2, \dots .$$

Thus, we get the following uniform estimate for the sequence of functions $\chi_n(x) := e^{-(\delta+\sigma_0)x} f_n(x)$:

$$|\chi_{n+1}(x) - \chi_n(x)| \le \alpha_0 L^n(\delta+\sigma_0),$$

where $0 < L(\delta+\sigma_0) < 1$ for $\delta \in (0, \min\{\varepsilon, \lambda_0 - \sigma_0\})$.

Remark 3. Applying the method of V. S. Vladimirov and Ya. I. Volovich (see [7, proof of Theorem 3]), we now show that if $1 > F(0) > 0$, then the boundary value problem

$$\lim_{x\to+\infty} f(x) = \eta \tag{4.6}$$

for equation (1.1) has no positive bounded solutions $f(x) \le \eta$, $x \in \mathbb{R}$, such that

$$\tan\alpha_0 := \frac{G(f(x_0))}{f(x_0)} \ge \frac{1}{F(0)} \tag{4.7}$$

for some $x_0 \in \mathbb{R}$.

Indeed, assume the contrary: equation (1.1) (with the boundary condition (4.6)) has a positive bounded solution $f(x) \le \eta$. Let us show that in this case there exists an $x_1 > x_0$ such that $f(x_1) < f(x_0)$. If such an x_1 did not exist, then the inequality $f(x) \ge f(x_0)$ would hold for all $x \ge x_0$. In this case, since the function G is monotone, it follows from (1.1) that

$$f(x_0) > -\int_{x_0}^{\infty} G(f(t))\, d_t F(x_0 - t) \ge -G(f(x_0)) \int_{x_0}^{\infty} d_t F(x_0 - t) = G(f(x_0))F(0),$$

or $G(f(x_0))/f(x_0) < 1/F(0)$. The last inequality contradicts condition (4.7).

Introduce the notation

$$\mathcal{D} := \{x > x_0 \colon\ f(x) < f(x_0)\}.$$

It follows from the above that $\mathcal{D} \neq \varnothing$. Let

$$T := \sup \mathcal{D}.$$

By the definition of supremum, there exists a sequence of points $\{x_k\}_{k=1}^{\infty} \subset \mathcal{D}$ such that $x_k \uparrow T$ as $k \to \infty$.

Suppose that $T < \infty$. Then the continuity of the solution (which follows from the continuity of the convolution of an integrable function and a bounded function) implies the convergence

$$f(x_k) \to f(T) \qquad \text{as} \quad k \to \infty.$$

Therefore, $f(T) \le f(x_0)$. Since G is monotone, we find that

$$G(f(T)) \le G(f(x_0)).$$

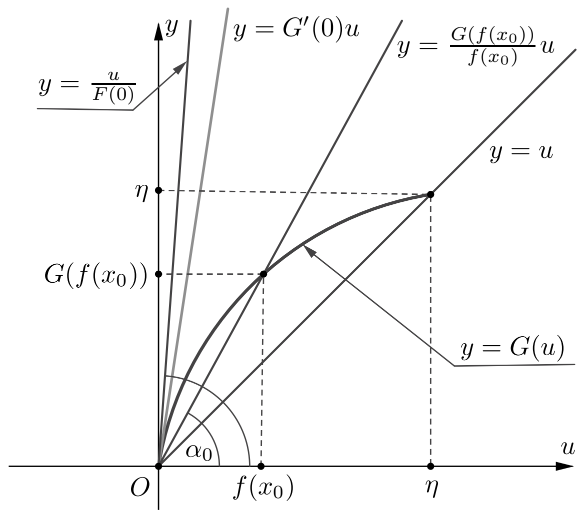

Fig. 5.

The concavity of G and the above inequalities imply that

$$\frac{G(f(T))}{f(T)} \geq \frac{G(f(x_0))}{f(x_0)} \geq \frac{1}{F(0)}. \tag{4.8}$$

It follows from (4.8) (as shown above) that there exists a $T_1 > T$ such that

$$f(T_1) < f(T) \leq f(x_0).$$

This contradicts the definition of supremum. Therefore, $T = \infty$.

Since $f(x_k) < f(x_0)$ ($x_k \in \mathcal{D}$, $k = 1, 2, \dots$), we can again apply the concavity and monotonicity of G to obtain

$$\frac{G(f(x_k))}{f(x_k)} \geq \frac{G(f(x_0))}{f(x_0)} \geq \frac{1}{F(0)}. \tag{4.9}$$

Letting $k \to \infty$ in (4.9), we arrive at the inequality

$$1 = \frac{G(\eta)}{\eta} \geq \frac{1}{F(0)},$$

which contradicts the condition $1 > F(0) > 0$.

Thus, for the existence of a positive bounded solution $f(x)$, the inequalities $G'(0) < 1/F(0)$ and $0 < F(0) < 1$ must hold. The geometric meaning of this fact is illustrated in Fig. 5.

5. EXAMPLES

In conclusion, we give examples of functions F and G for which all hypotheses of the above Theorems 1 and 2 are satisfied.

In applications (see [1]), one can find functions G and F of the following form:

$$G(u) = \gamma(1 - e^{-u}), \qquad u \in \mathbb{R}^+ := [0, +\infty), \quad \gamma > 1, \tag{5.1}$$

$$F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-(t-c)^2}\, dt, \qquad x \in \mathbb{R}, \quad c > 0. \tag{5.2}$$

Let us show that the functions (5.1) and (5.2) satisfy conditions (P1)–(P3) and (N1)–(N4), respectively, and condition (2.3).

First note that

$$F(-\infty) = 0, \qquad F(+\infty) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-c)^2}\, dt = 1, \qquad F(x) \uparrow \quad \text{on } \mathbb{R},$$

and, moreover, $F(x)$ is an absolutely continuous function on the set $\mathbb{R}$. On the other hand, in view of (4.2) we have

$$\int_0^{\infty} (1 - F(x))\, dx = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \int_x^{\infty} e^{-(t-c)^2}\, dt\, dx = \frac{1}{\sqrt{\pi}} \int_0^{\infty} te^{-(t-c)^2}\, dt < +\infty,$$

$$\int_{-\infty}^{\infty} x^2\, dF(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-(x-c)^2}\, dx < +\infty,$$

$$\nu(F) = \int_{-\infty}^{\infty} x\, dF(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-(x-c)^2}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (t+c) e^{-t^2}\, dt = c > 0.$$

Condition (N1) immediately follows from the structure of the function G. Since $G(0) = 0$, the equation $\gamma(1 - e^{-u}) = u$, $\gamma > 1$, has a unique positive solution η, and $G''(u) = -\gamma e^{-u} < 0$, $u \in \mathbb{R}^+$, it follows that condition (N2) is also satisfied. The validity of condition (N3) follows from the inequality $e^{-u} \geq 1 - u$, $u \in \mathbb{R}^+$.

Let us turn to condition (N4). For the function of the form (5.1), we should choose numbers $\varepsilon > 0$ and $c > 0$ such that

$$\gamma(1 - e^{-u}) \geq \gamma u - cu^{1+\varepsilon}, \qquad u \in \mathbb{R}^+. \tag{5.3}$$

Consider the function

$$\psi(u) = \gamma - \gamma e^{-u} - \gamma u + cu^{1+\varepsilon}, \qquad u \in \mathbb{R}^+.$$

It is clear that

$$\psi(0) = 0 \qquad \text{and} \qquad \psi'(u) = \gamma e^{-u} - \gamma + c(1+\varepsilon)u^{\varepsilon}.$$

So we readily find that taking, for example, the numbers $\varepsilon = 1$ and $c = \gamma/2$ guarantees the inequality

$$\psi'(u) \geq 0, \qquad u \in \mathbb{R}^+.$$

Therefore, $\psi(u)$ does not decrease in u on $\mathbb{R}^+$, which implies (5.3).

Finally, we discuss condition (2.3). For the functions (5.1) and (5.2), the function $L(\lambda)$ takes the form

$$L(\lambda) = \frac{\gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-c)^2} e^{-\lambda t}\, dt = \gamma e^{\lambda^2/4 - c\lambda} = \gamma e^{\lambda^2/4 - \gamma\lambda/2},$$

since $c = \gamma/2$. For this function, the minimum point is $\lambda_{\min} = \gamma$. It is clear that

$$L(\lambda_{\min}) = \gamma e^{-\gamma^2/4} < 1,$$

since $e^{\gamma^2/4} > \gamma^2/4 + 1 \geq \gamma$. Therefore, taking γ as λ_0, we arrive at condition (2.3).

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REFERENCES

1. O. Diekmann, “Thresholds and travelling waves for the geographical spread of infection,” J. Math. Biol. **6** (2), 109–130 (1978).
2. N. B. Engibaryan, “Convolution equations containing singular probability distributions,” Izv. Math. **60** (2), 249–280 (1996) [transl. from Izv. Ross. Akad. Nauk, Ser. Mat. **60** (2), 21–48 (1996)].
3. N. B. Engibaryan, “Conservative systems of integral convolution equations on the half-line and the entire line,” Sb. Math. **193** (6), 847–867 (2002) [transl. from Mat. Sb. **193** (6), 61–82 (2002)].
4. W. Feller, *An Introduction to Probability Theory and Its Applications* (J. Wiley and Sons, New York, 1966), Vol. 2.
5. Kh. A. Khachatryan and A. S. Petrosyan, “On the solvability of one class of nonlinear integral equations with a noncompact Hammerstein–Stieltjes-type operator on the semiaxis,” Ukr. Math. J. **67** (1), 120–145 (2015) [transl. from Ukr. Mat. Zh. **67** (1), 106–127 (2015)].
6. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Nauka, Moscow, 1981; Dover Publ., Mineola, NY, 1999).
7. V. S. Vladimirov and Ya. I. Volovich, “Nonlinear dynamics equation in p-adic string theory,” Theor. Math. Phys. **138** (3), 297–309 (2004) [transl. from Teor. Mat. Fiz. **138** (3), 355–368 (2004)].

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