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Sufficient Optimality Conditions for Hybrid Systems of Variable Dimension

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Abstract—We consider an optimal control problem for a hybrid system whose continuous motion alternates with discrete variations (switchings) under which the dimension of the state space changes. The moments and the number of switchings are not specified in advance. They are determined as a result of minimizing a functional that incorporates the cost of each switching. The state space may change, for example, when the number of control objects varies, which is typical, in particular, of control problems for groups of a variable number of aircraft. We obtain sufficient optimality conditions for such systems and derive equations for the synthesis of optimal trajectories. The application of optimality conditions is demonstrated in academic examples.

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1. INTRODUCTION

Continuous motion of hybrid systems of variable dimension (HSVDs) is described by differential equations, while instantaneous changes of state (switchings) are described by recurrence equations or inclusions. At the moment of switching, the state space of the system (in particular, its dimension) varies. Control systems with variable state space have been investigated under different names: composite systems [21, 2], systems of variable dimension [14], systems with branching structures [13], stepped systems [16], complex (multistage) processes [12], systems with change of the phase space [4], and hybrid systems with intermediate conditions [18, 9]. Most papers pertain to linear systems and address the issues of stability, controllability, and observability [2, 13]. As a rule, in optimal control problems [21, 16, 4, 18, 9] the moments of change of the state space are either fixed or determined by intermediate conditions, and the switchings of states are uncontrolled. The number of switchings is predetermined; and in the first papers [21, 16, 4] on this subject there was only one switching. For hybrid systems with intermediate conditions, necessary conditions generalizing the maximum principle [17] were obtained in [18, 9]. In these papers, the number of switchings is predetermined and the switchings themselves are uncontrolled.

In the present study, we consider problems in which the number and the moments of switchings are not predetermined, and the switchings themselves are controlled. Here we do not rule out processes with multiple instantaneous switchings [6, 7], which have not been analyzed in problems with a change of the state space. Therefore, the problem of synthesis of optimal HSVDs generalizes similar optimal control problems for continuous–discrete, logical–dynamical, composite, and stepped systems as well as systems with intermediate conditions [2, 18, 9, 20] and systems with a variable or branching structure [14, 13, 11, 5].

As a rule, sufficient optimality conditions for dynamical control systems are related to determining the value function (Hamilton–Jacobi–Bellman (HJB) function). To find the synthesis of optimal HSVDs, we propose to seek auxiliary functions, namely, generators of the value function

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and a two-position value function [8]. We derive differential and recurrence equations for these auxiliary functions using the dynamic programming method [3]; in this case, the ordinary value function turns out to be generally unnecessary. The application of optimality conditions is demonstrated in academic examples of time-optimal problems of group performance.

2. STATEMENT OF THE PROBLEM

On a time interval $T = [t_0, t_{\mathrm{F}}]$, consider a dynamical system with N switchings at times $t_1, \dots, t_N$ that form a nondecreasing finite sequence $\mathcal{T} = \{t_1, \dots, t_N\}$:

$$t_0 \le t_1 \le \ldots \le t_N \le t_{\mathrm{F}}. \tag{2.1}$$

Between unequal consecutive moments of switching, the state of the system changes continuously according to the differential equation

$$\dot{x}_i = f_i(t, x_i(t), u_i(t)), \qquad t \in T_i, \quad i \in \mathcal{N}. \tag{2.2}$$

At the switching moments, it varies discretely according to the recurrence equation

$$x_i(t_i) = g_i(t_i, x_{i-1}(t_i), v_i), \qquad i = 1, \dots, N. \tag{2.3}$$

In (2.2) we use the following notation: $\mathcal{N} \triangleq \{i = 0, 1, \dots, N \mid t_i < t_{i+1}\}$ is the set of the numbers of nonzero (positive-length) partial intervals $T_i = [t_i, t_{i+1}]$ of continuous evolution of the system; $x_i(t)$ is the state of the system at time $t \in T_i$, $x_i(t) \in X_i = \mathbb{R}^{n_i}$; $u_i(t)$ is a control of continuous evolution of the system at time $t \in T_i$, with $u_i(t) \in U_i \subset \mathbb{R}^{p_i}$, where U_i is a given set of admissible control values, $i \in \mathcal{N}$. If $t_i = t_{i+1}$, then the differential equation (2.2) is omitted ($i \notin \mathcal{N}$), the function $x(\cdot)$ is defined at a single point as $x_i(t_i)$, and the value $u(t_i)$ of the control at this point is inessential. In equation (2.3), the notation has the following meaning: v_i is the control of the switching of the system at time $t_i \in \mathcal{T}$, $v_i \in V_i \subset \mathbb{R}^{q_i}$, and V_i is a given set of admissible control values for switchings, $i = 1, \dots, N$. The function $f_i \colon T \times X_i \times U_i \to \mathbb{R}^{n_i}$, $i = 1, \dots, N$, is continuous on the whole domain of definition together with its derivative $\partial f_i / \partial x_i$; the function $g_i \colon T \times X_i \times V_i \to \mathbb{R}^{n_i}$, $i = 1, \dots, N$, is bounded. We assume that equation (2.3) does not contain so-called fictitious switchings, at which the state of the system is preserved, i.e., $x_i(t_i) = x_{i-1}(t_i)$, and there is no actual switching. The possible equality of consecutive moments in (2.1) means that the system undergoes multiple instantaneous switchings [6, 7].

The initial state $x_0(t_0) = x_0$ of the system is fixed, while the terminal state is determined by the first reach of the terminal surface,

$$(t_{\mathrm{F}}, x_N(t_{\mathrm{F}})) \in \mathbf{\Gamma}_N, \tag{2.4}$$

defined by the equation $\Gamma_N(t, x_N) = 0$, where $\Gamma_N \colon [t_0, +\infty) \times X_N \to \mathbb{R}^{l_N}$ is a continuous vector function. Similar terminal conditions can be imposed on the left end of the trajectory [15, 1], or on both ends of the trajectory simultaneously (for example, the periodicity condition).

The set of admissible processes $\mathcal{D}_0(t_0, x_0)$ is formed by quadruples $d = (\mathcal{T}, x(\cdot), u(\cdot), \{v\})$ consisting of a nondecreasing sequence $\mathcal{T}$ of switching moments (2.1), a sequence $x(\cdot) \triangleq \{x_i(\cdot)\}_{i=0}^N$ of absolutely continuous functions $x_i \colon T_i \to X_i$, a sequence $u(\cdot) \triangleq \{u_i(\cdot)\}_{i=0}^N$ of bounded measurable functions $u_i \colon T_i \to U_i$, and a sequence $v(\cdot) \triangleq \{v_i\}_{i=1}^N$ of vectors $v_i \in V_i$ such that the pairs $(x_i(\cdot), u_i(\cdot))$, $i = 0, 1, \dots, N$, satisfy equation (2.2) almost everywhere on the interval T_i, the triples $(x_{i-1}(t_i), x_i(t_i), v_i)$, $i = 1, \dots, N$, on $\mathcal{T}$ satisfy the recurrence equation (2.3), the condition $x_0(t_0) = x_0$ holds at the initial moment of time, and the terminal condition (2.4) holds at the terminal moment. We stress that the number $N = |\mathcal{T}|$ of switchings and the switching moments $\mathcal{T} = \{t_1, \dots, t_N\}$ are not fixed and may not coincide in different admissible processes.

On the set $\mathcal{D}_0(t_0, x_0)$ of admissible processes, we introduce an objective functional

$$I_0(t_0, x_0, d) = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} f_i^0(t, x_i(t), u_i(t))\, dt + \sum_{i=1}^{N} g_i^+(t_i, x_{i-1}(t_i), v_i) + F_N(t_\mathrm{F}, x_N(t_\mathrm{F})), \tag{2.5}$$

where the functions $f_i^0\colon T \times X_i \times U_i \to \mathbb{R}$ and $F_N\colon [t_0, +\infty) \times X_N \to \mathbb{R}$ are continuous and bounded from below and the functions $g_i^+\colon T \times X_i \times V_i \to \mathbb{R}_+$ are nonnegative, $g_i^+ \geq 0$, $i = 1, \dots, N$. The latter condition allows one to consider each term $g_i^+(t_i, x_{i-1}(t_i), v_i)$ in (2.5) as the cost of (or "penalty" for) switching $x_{i-1}(t_i) \to x_i(t_i)$. In (2.5), the terminal moment t_F is also denoted by t_{N+1}.

It is required to find the minimum value of the functional (2.5) and an optimal process $d^* = (\mathcal{T}^*, x^*(\cdot), u^*(\cdot), \{v^*\})$ for which this value is attained:

$$I_0(t_0, x_0, d^*) = \min_{d \in \mathcal{D}_0(t_0, x_0)} I_0(t_0, x_0, d). \tag{2.6}$$

If the minimum value (2.6) does not exist, then we can address the problem of finding a minimizing sequence of admissible processes [15]. The number of switchings in the processes of a minimizing sequence may either remain finite or indefinitely increase. An infinite number of switchings in an optimal process becomes impossible if we strengthen the nonnegativity condition imposed on the functions g_i^+ in (2.5) by requiring that $g_i^+(t_i, x_i, v_i) \geq \mathrm{const} > 0$. The application of such "penalties" in the objective functional excludes fictitious switchings as well as sequences of processes with an indefinitely increasing number of switchings, because they are certainly nonminimizing.

3. GENERATORS OF THE VALUE FUNCTION

The application of the dynamic programming method [3] is based on the concept of value function (HJB function), which is defined as the minimum value of the functional of remaining losses. Denote by $\mathcal{D}_i(t, x_i)$ the set of admissible processes after the ith switching that satisfy the condition $x_i(t) = x_i$. The remaining switchings occur at times $t_{i+1}, \dots, t_{i+k}$, which form a nondecreasing sequence on the interval $[t, t_\mathrm{F}]$:

$$t \triangleq t_i \leq t_{i+1} \leq \ldots \leq t_{i+k} \leq t_{i+k+1} \triangleq t_\mathrm{F}.$$

The number k of remaining switchings and the switching moments $t_{i+1}, \dots, t_{i+k}$ themselves are not fixed and may not coincide in different admissible processes.

On the set $\mathcal{D}_i(t, x_i)$ of admissible processes after the ith switching, we define the functional of remaining losses as

$$I_i(t, x_i, d) = \sum_{j=i}^{i+k} \int_{t_j}^{t_{j+1}} f_j^0(t, x_j(t), u_j(t))\, dt + \sum_{j=i+1}^{i+k} g_j^+(t_j, x_{j-1}(t_j), v_j) + F_{i+k}(t_\mathrm{F}, x_{i+k}(t_\mathrm{F})). \tag{3.1}$$

The value function $\varphi_i(t, x_i)$ after the ith switching is by definition equal to the functional of remaining losses (3.1) calculated on an optimal process satisfying the initial condition $x_i(t) = x_i$. In other words, the value function is equal to the minimum value of the functional of remaining losses (3.1) on the set of admissible processes $\mathcal{D}_i(t, x_i)$:

$$\varphi_i(t, x_i) = \min_{d \in \mathcal{D}_i(t, x_i)} I_i(t, x_i, d). \tag{3.2}$$

For problem (2.6), we define a *generator* of the value function. The value $\varphi_i^k(t, x_i)$ of a generator is equal to the value of the functional of remaining losses (3.1) calculated on an optimal process among all admissible processes that start from the initial position (t, x_i) after the ith switching and have k (possibly, fictitious) switchings after the ith switching. If we denote by $\mathcal{D}_i^k(t, x_i)$ the set of admissible processes among $\mathcal{D}_i(t, x_i)$ that have k (possibly, fictitious) switchings and by $I_i^k(t, x_i, d)$ the functional (3.1) with a fixed number k of remaining switchings, then

$$\varphi_i^k(t, x_i) = \min_{d \in \mathcal{D}_i^k(t, x_i)} I_i^k(t, x_i, d). \tag{3.3}$$

The value function is determined by its generators as

$$\varphi_i(t, x_i) = \min_{k \in \mathbb{Z}_+} \varphi_i^k(t, x_i). \tag{3.4}$$

An important role in finding the generators is played by the so-called [8] *two-position* value function $\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau})$, $i = 0, 1, \ldots, N$. It is defined as a solution to the Lagrange problem for system (2.2) with fixed ends of the trajectory:

$$\begin{gathered} \dot{x}_i = f_i(t, x_i(t), u_i(t)), \quad \theta \le t \le \tau, \qquad x_i(\theta) = x_{i\theta}, \quad x_i(\tau) = x_{i\tau}, \\ \int\limits_{\theta}^{\tau} f_i^0(t, x_i(t), u_i(t))\, dt \to \min. \end{gathered} \tag{3.5}$$

This function as a function of the initial position, $(t, x_i) \to \psi_i(t, x_i | \tau, x_{i\tau})$, satisfies the HJB equation with zero terminal conditions

$$\begin{gathered} \min_{u \in U_i} \left[\frac{\partial \psi_i(t, x_i | \tau, x_{i\tau})}{\partial t} + \frac{\partial \psi_i(t, x_i | \tau, x_{i\tau})}{\partial x_i} f_i(t, x_i, u_i) + f_i^0(t, x_i, u_i) \right] = 0, \\ \psi_i(\tau, x_{i\tau} | \tau, x_{i\tau}) = 0. \end{gathered} \tag{3.6}$$

Taking account of the “symmetry” $\psi_i(\tau, x_{i\tau} | \theta, x_{i\theta}) = -\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau})$, we can write the “opposite” equation for the function $(t, x_i) \to \psi_i(\theta, x_{i\theta} | t, x_i)$ of the terminal position. Minimizing the left-hand side of the HJB equation (3.6), we obtain a so-called optimal *two-position* control

$$\mathbf{u}_i(t, x_i | \tau, x_{i\tau}) = \arg\min_{u \in U_i} \left[\frac{\partial \psi_i(t, x_i | \tau, x_{i\tau})}{\partial t} + \frac{\partial \psi_i(t, x_i | \tau, x_{i\tau})}{\partial x_i} f_i(t, x_i, u_i) + f_i^0(t, x_i, u_i) \right]. \tag{3.7}$$

We assume that the function $\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau})$ is defined for all pairs of positions $(\theta, x_{i\theta}) \in T \times X_i$, $(\tau, x_{i\tau}) \in T \times X_i$, $\theta < \tau$. If a solution to problem (3.5) does not exist, then we set the value of the two-position value function to be $+\infty$: $\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau}) = +\infty$.

4. EQUATIONS FOR THE GENERATORS OF THE VALUE FUNCTION

To derive equations for the generators of the value function, we apply the Bellman optimality principle modified for problems with switchings. According to this principle, an optimal process with k remaining switchings becomes an optimal process with $k - 1$ switchings after the first switching. We assume that there exist two-position value functions $\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau})$ and controls $\mathbf{u}_i(t, x_i | \tau, x_{i\tau})$ that satisfy the initial problem (3.6).

Each zero generator $\varphi_i^0(t, x_i)$, $i = 0, 1, \ldots$, is determined by the value of the functional (3.1) on an optimal process without jumps. These functions satisfy the HJB equation

$$\min_{u \in U_i} \left[\frac{\partial \varphi_i^0(t, x_i)}{\partial t} + \frac{\partial \varphi_i^0(t, x_i)}{\partial x_i} f_i(t, x_i, u_i) + f_i^0(t, x_i, u_i) \right] = 0 \tag{4.1}$$

with the terminal condition

$$\varphi_i^0(t_{\mathrm{F}}, x_i) = F_i(t_{\mathrm{F}}, x_{iF}), \qquad (t_{\mathrm{F}}, x_{iF}) \in \mathbf{\Gamma}_i. \tag{4.2}$$

To find the other generators, we use a recurrent procedure. Suppose that for a positive integer k the generators φ_i^{k-1}, $i \in \mathbb{Z}_+$, are known. Then, according to the optimality principle, the generators φ_i^k, $i \in \mathbb{Z}_+$, satisfy the equation

$$\varphi_i^k(t, x_i) = \min_{t \le \tau \le t_{\mathrm{F}}} \min_{\widehat{x}_i \in X_i} \left\{ \psi_i(t, x_i | \tau, \widehat{x}_i) + \min_{v \in V_{i+1}} \left[\varphi_{i+1}^{k-1}(\tau, g_{i+1}(\tau, \widehat{x}_i, v)) + g_{i+1}^+(\tau, \widehat{x}_i, v) \right] \right\}. \tag{4.3}$$

Indeed, continuous motion after the ith switching on the interval $[t, \tau]$ preceding the first of the remaining k switchings is governed, in view of (3.5) and (3.6), by the optimal control (3.7), which transfers the system from the position (t, x_i) to a position $(\tau, \widehat{x}_i)$ in which a jump occurs. The optimality of the $(i+1)$th switching is guaranteed by the minimization with respect to the control v. Therefore, the expression in braces is equal to the minimum value of the functional of remaining losses for the given position $(\tau, \widehat{x}_i)$ of switching. The first two minimization operations in (4.3) establish the best position for this switching. Thus, the right-hand side of (4.3) yields the minimum value of the functional (3.1) with k switchings remaining after the ith switching, which determines the generator (3.3). Then, according to (3.2) and (3.4), one calculates the minimum value of the functional (3.1) using the generators of the value function:

$$\min_{d \in \mathcal{D}_i(t, x_i)} I_i(t, x_i, d) = \varphi_i(t, x_i) = \min_{k \in \mathbb{Z}_+} \varphi_i^k(t, x_i). \tag{4.4}$$

The initial conditions for equation (4.3) are given by the zero generators $\varphi_i^0(t, x_i)$, $i \in \mathbb{Z}_+$, i.e., by the value functions for processes without switchings. Each of these functions is sought as a solution to the HJB equation (4.1) with terminal condition (4.2). However, if the two-position value function $\psi_i(\theta, x_{i\theta} | \tau, x_{i\tau})$ is known, then the generator $\varphi_i^0(t, x_i)$ can be obtained by solving the finite-dimensional minimization problem:

$$\varphi_i^0(t, x_i) = \min_{(t_{\mathrm{F}}, x_F) \in \mathbf{\Gamma}_i} \left[\psi_i(t, x_i | t_{\mathrm{F}}, x_F) + F_i(t_{\mathrm{F}}, x_F) \right]. \tag{4.5}$$

This equality expresses the relationship between the solutions of the problems with Lagrange and Bolza functionals.

5. OPTIMAL POSITIONAL CONTROL

When solving equations (4.1) and (4.3), one performs four minimization operations. Minimizing the left-hand side of (4.1), one determines an optimal positional control of continuous motion without switchings:

$$\mathbf{u}_i(t, x_i) = \arg\min_{u \in U_i} \left[\frac{\partial \varphi_i^0(t, x_i)}{\partial t} + \frac{\partial \varphi_i^0(t, x_i)}{\partial x_i} f_i(t, x_i, u_i) + f_i^0(t, x_i, u_i) \right]. \tag{5.1}$$

Minimizing the right-hand side of (4.3) yields an optimal positional control of the switching of the system,

$$\mathbf{v}_{i+1}^k(\tau, x_i) = \arg\min_{v \in V_{i+1}} \left[\varphi_{i+1}^{k-1}(\tau, g_{i+1}(\tau, x_i, v)) + g_{i+1}^+(\tau, x_i, v) \right], \tag{5.2}$$

and an optimal position $(\boldsymbol{\tau}_i^k, \mathbf{x}_i^k)$ of the first of the remaining k switchings, i.e., an optimal moment

$$\boldsymbol{\tau}_i^k(t, x_i) = \arg\min_{t \le \tau \le t_{\mathrm{F}}} \min_{\widehat{x}_i \in X_i} \left\{ \psi_i(t, x_i | \tau, \widehat{x}_i) + \min_{v \in V_{i+1}} \left[\varphi_{i+1}^{k-1}(\tau, g_{i+1}(\tau, \widehat{x}_i, v)) + g_{i+1}^+(\tau, \widehat{x}_i, v) \right] \right\} \tag{5.3}$$

and an optimal state before the switching,

$$\mathbf{x}_i^k(t,x_i) = \underset{\widehat{x}_i\in X_i}{\arg\min}\ \min_{t\le\tau\le t_{\mathrm{F}}}\Big\{\psi_i(t,x_i|\tau,\widehat{x}_i) + \min_{v\in V_{i+1}}\big[\varphi_{i+1}^{k-1}(\tau,g_{i+1}(\tau,\widehat{x}_i,v)) + g_{i+1}^{+}(\tau,\widehat{x}_i,v)\big]\Big\}. \quad (5.4)$$

Notice that the positional constructions (5.2)–(5.4) depend not only on the position $(t,x_i)\in T\times X_i$ of the system after the ith switching but also on the number k of remaining switchings. An optimal number of switchings is determined in (4.4):

$$\mathbf{k}_i(t,x_i) = \underset{k\in\mathbb{Z}_+}{\arg\min}\,\varphi_i^k(t,x_i). \quad (5.5)$$

Another minimization operation is performed when one finds the two-position optimal control (3.6). This control of the continuous motion of the system does not depend on the number of remaining switchings. Taking into account the relation (4.5) between the two-position value function and the zero generator, we can express the control (5.1) in the Bolza problem without switchings in terms of the two-position control (3.7):

$$\mathbf{u}_i(t,x_i) = \mathbf{u}_i(t,x_i|\boldsymbol{\tau}_i^0,\mathbf{x}_i^0), \quad (5.6)$$

where $(\boldsymbol{\tau}_i^0,\mathbf{x}_i^0)$ is an optimal terminal position of the process without switchings that starts from the position (t,x_i):

$$(\boldsymbol{\tau}_i^0,\mathbf{x}_i^0) = \underset{(t_{\mathrm{F}},x_F)\in\boldsymbol{\Gamma}_i}{\arg\min}\big[\psi_i(t,x_i|t_{\mathrm{F}},x_F) + F_i(t_{\mathrm{F}},x_F)\big]. \quad (5.7)$$

Thus, the optimal positional control for the systems under consideration forms a whole "*control complex*" of positional constructions, which consists of the following functions: $\mathbf{u}_i(t,x_i)$ and $\mathbf{u}_i(t,x_i|\tau,\widehat{x}_i)$, the optimal controls (5.1) and (3.7) of the continuous motion of the system; $\mathbf{v}_{i+1}^k(\tau,x_i)$, the optimal control (5.2) of the $(i+1)$th switching; $\boldsymbol{\tau}_i^k(t,x_i)$, the optimal moment (5.3) of the first of the k remaining switchings; $\mathbf{x}_i^k(t,x_i)$, the optimal state (5.4) before this switching; and $\mathbf{k}_i(t,x_i)$, the optimal number (5.5) of switchings of the process starting from position (t,x_i).

The positional constructions (3.7) and (5.1)–(5.5) allow one to find an optimal process. Indeed, suppose that the system is in the position (t_0,x_0), i.e., it satisfies the initial condition $x(t_0)=x_0$. Then, for this position, we determine the optimal number $N=\mathbf{k}_0(t_0,x_0)$ of remaining switchings and the position $(t_1,x_0(t_1))$ of the first switching: $t_1=\boldsymbol{\tau}_0^N(t_0,x_0)$ and $x_0(t_1)=\mathbf{x}_0^N(t_0,x_0)$. If $t_1=t_0$, then the system immediately switches to another state, $x_0(t_1)\to x_1(t_1)=g_1(t_1,x_0(t_1),v_1)$, under the control $v_1=\mathbf{v}_1^N(t_1,x_0(t_1))$. If $t_1>t_0$, then the state of the system varies continuously on the interval $[t_0,t_1]$ according to equation (2.2) with program control $u_0(t)=\mathbf{u}_0(t,x_0(t)|t_1,x_0(t_1))$, and at the end of this interval a jump $x_0(t_1)\to x_1(t_1)=g_1(t_1,x_0(t_1),v_1)$ occurs. Thus, the system arrives at a position $(t_1,x_1(t_1))$, in which one performs the same operations except for finding the optimal number of switchings, because this number is equal to $N-1$. If the optimal number of switchings in the initial position (t_0,x_0) is zero ($\mathbf{k}_0(t_0,x_0)=0$), then there are no switchings and the state of the system varies continuously all the time according to equation (2.2) under the control $u_0(t)=\mathbf{u}_0(t,x_0(t))$.

6. SUFFICIENT OPTIMALITY CONDITIONS

Sufficient optimality conditions in classical control problems for dynamical systems [3, 15] are related to the value function (HJB or Krotov functions). The conditions proposed below do not involve the value function. Instead, the sequence of generators of the value function and the two-position value function are used (see Section 3). In this case, the conventional optimal positional control is replaced by the "control complex" of positional constructions (see Section 5).

Theorem. *If for problem* (2.1)–(2.6) *there exist sequences of functions* ψ_i *and* φ_i^k, $i \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, *that satisfy equations* (3.6) *and* (4.1)–(4.3) *on the domain of definition, then, for a process* $d = (\mathcal{T}, x(\cdot), u(\cdot), \{v\}) \in \mathcal{D}_0(t_0, x_0)$ *with switching moments* $t_1, \dots, t_N$ *that form a nondecreasing sequence* (2.1) *to be optimal, it is sufficient that the following conditions hold*:

$$N = \mathbf{k}_0(t_0, x_0), \tag{6.1}$$

$$u_i(t) = \mathbf{u}_i\big(t, x_i(t)|t_{i+1}, x_i(t_{i+1})\big), \qquad t \in T_i, \quad i \in \mathcal{N}, \tag{6.2}$$

$$v_i = \mathbf{v}_i^{N-i+1}(t_i, x_{i-1}(t_i)), \qquad i = 1, \dots, N, \tag{6.3}$$

$$t_i = \boldsymbol{\tau}_{i-1}^{N-i+1}(t_{i-1}, x_{i-1}(t_{i-1})), \qquad i = 1, \dots, N, \tag{6.4}$$

$$x_{i-1}(t_i) = \mathbf{x}_{i-1}^{N-i+1}(t_{i-1}, x_{i-1}(t_{i-1})), \qquad i = 1, \dots, N, \tag{6.5}$$

where $T_i = [t_i, t_{i+1}]$ *and* $\mathcal{N} = \{i = 0, 1, \dots, N \mid t_i < t_{i+1}\}$. *For* $N = 0$ *formulas* (6.3)–(6.5) *should be omitted, and the control* (6.2) *for* $i = N \in \mathcal{N}$ *is replaced by* $u_N(t) = \mathbf{u}_N(t, x_N(t))$, $t \in T_N$.

Proof. Let us compare the value of the functional $I_0(t_0, x_0, d)$ calculated on an admissible process d with the value $\varphi_0(t_0, x_0)$ of the value function. Suppose that the process d has switchings (i.e., $N > 0$). On the control v_{i+1} obtained by formula (6.3), the minimum value of the expression in square brackets in (4.3) is attained for $\tau = t_{i+1}$ and $\widehat{x} = x_i(t_{i+1})$:

$$\begin{aligned}\min_{v \in V_{i+1}} &\big[\varphi_{i+1}^{k-1}\big(t_{i+1}, g_{i+1}(t_{i+1}, x_i(t_{i+1}), v)\big) + g_{i+1}^{+}(t_{i+1}, x_i(t_{i+1}), v)\big] \\ &= \varphi_{i+1}^{k-1}\big(t_{i+1}, g_{i+1}(t_{i+1}, x_i(t_{i+1}), v_{i+1})\big) + g_{i+1}^{+}(t_{i+1}, x_i(t_{i+1}), v_{i+1}) \\ &= \varphi_{i+1}^{k-1}[t_{i+1}] + g^{+}[t_{i+1}].\end{aligned} \tag{6.6}$$

Henceforth, the time point in square brackets as an argument of the function means that the value of this function is calculated on the admissible process d at this point of time. In (6.6), we used the notation $g^{+}[t_{i+1}] \triangleq g^{+}(t_{i+1}, x(t_{i+1}), v_{i+1})$ and $\varphi_{i+1}^{k-1}[t_{i+1}] \triangleq \varphi_{i+1}^{k-1}(t_{i+1}, x_{i+1}(t_{i+1}))$ and the fact that $x_{i+1}(t_{i+1}) = g_{i+1}(t_{i+1}, x_i(t_{i+1}), v_{i+1})$ according to (2.3).

The position $(t_{i+1}, x_i(t_{i+1}))$ satisfying conditions (6.4) and (6.5) guarantees that the expression in braces in (4.3) attains its minimum value for $\tau = t_{i+1}$ and $\widehat{x} = x_i(t_{i+1})$. Therefore,

$$\varphi_i^k[t_i] = \psi_i\big(t_i, x_i(t_i)|t_{i+1}, x_i(t_{i+1})\big) + \varphi_{i+1}^{k-1}[t_{i+1}] + g^{+}[t_{i+1}]. \tag{6.7}$$

Under the control (6.2), the minimum in equation (3.6) is attained for $\tau = t_{i+1}$ and $\widehat{x} = x_i(t_{i+1})$. Hence, taking into account (2.2), we obtain the following equality for the derivative of the function $\psi_i^k[t] \triangleq \psi_i^k(t, x_i(t)|t_{i+1}, x_i(t_{i+1}))$:

$$\frac{d}{dt}\psi_i^k[t] + f_i^0(t, x_i(t), u_i(t)) = 0. \tag{6.8}$$

Integrating equation (6.8) over the interval $T_i = [t_i, t_{i+1}]$, for the zero terminal condition $\psi_i^k[t_{i+1}] = 0$ we obtain

$$\psi_i^k[t_{i+1}] - \psi_i^k[t_i] + \int\limits_{t_i}^{t_{i+1}} f_i^0(t, x_i(t), u_i(t))\,dt = 0 \qquad \Rightarrow \qquad \psi_i^k[t_i] = \int\limits_{t_i}^{t_{i+1}} f_i^0[t]\,dt, \tag{6.9}$$

where $f_i^0[t] \triangleq f_i^0(t, x_i(t), u_i(t))$. Substituting (6.9) into (6.7), we arrive at the equality

$$\varphi_{i+1}^{k-1}[t_{i+1}] - \varphi_i^k[t_i] + g^{+}[t_{i+1}] + \int\limits_{t_i}^{t_{i+1}} f_i^0[t]\,dt = 0. \tag{6.10}$$

Let us transform equation (4.1) for the function $\varphi_N^0(t, x_N)$ in a similar way. Under the control $u_N(t) = \mathbf{u}_N(t, x_N(t))$, the minimum in (4.1) is attained, and according to (2.2) the equation can be rewritten as

$$\frac{d}{dt}\varphi_N^0(t, x_N(t)) + f_N^0(t, x_N(t), u_N(t)) = 0. \tag{6.11}$$

Integrating equation (6.11) over the interval $T_N = [t_N, t_{\mathrm{F}}]$ and taking into account the terminal condition (4.2), we obtain

$$F_N[t_{\mathrm{F}}] - \varphi_N^0[t_{\mathrm{F}}] + \int\limits_{t_N}^{t_{\mathrm{F}}} f_N^0[t]\,dt = 0. \tag{6.12}$$

Here $F_N[t_{\mathrm{F}}] \triangleq F_N(t_{\mathrm{F}}, x_N(t_{\mathrm{F}}))$, $f_N^0[t] \triangleq f_N^0(t, x_N(t), u_N(t))$, and $\varphi_N^0[t_{\mathrm{F}}] \triangleq \varphi_N^0[t_{\mathrm{F}}]$. Summing equalities (6.10) over $i = 0, 1, \ldots, N-1$ (with $k = N - i$) and adding equality (6.12) to the sum, we arrive at the relation

$$\sum_{i=0}^{N} \int\limits_{t_i}^{t_{i+1}} f_i^0[t]\,dt + \sum_{i=1}^{N} g_i^+(t_i, x_{i-1}(t_i), v_i) + F_N(t_{\mathrm{F}}, x_N(t_{\mathrm{F}})) - \varphi_0^N[t_0] = 0. \tag{6.13}$$

Here, as before, $t_{N+1} \triangleq t_{\mathrm{F}}$. Comparing relation (6.13) with the functional (2.5), we conclude that $\varphi_0^N[t_0] = I_0(t_0, x_0, d)$. Then conditions (6.1) and (5.5) imply that $\varphi_0(t_0, x_0) = \varphi_0^N[t_0] = I_0(t_0, x_0, d)$, i.e., the functional (2.5) on the admissible process d is equal to the value function. According to (3.2), this means that the process d is optimal. Thus, the theorem is proved for $N > 0$.

In the absence of switchings (in the case of $N = 0$), taking into account the terminal condition (4.2), from equation (4.1) with the control $u_0(t) = \mathbf{u}_0(t, x_0(t))$ we obtain

$$F_0[t_{\mathrm{F}}] - \varphi_0^0[t_{\mathrm{F}}] + \int\limits_{t_0}^{t_{\mathrm{F}}} f_0^0[t]\,dt = 0 \qquad \Leftrightarrow \qquad \varphi_0^0[t_0] = I_0(t_0, x_0, d). \tag{6.14}$$

Since $N = \mathbf{k}_0(t_0, x_0) = 0$, we have $\varphi_0^0[t_0] = \varphi_0(t_0, x_0)$. Therefore, $\varphi_0(t_0, x_0) = I_0(t_0, x_0, d)$. Thus, the value of the functional on the process d is equal to the value function. This implies the optimality of the process d. The theorem is proved. □

7. TIME-OPTIMAL PROBLEM

Consider a particular case of problem (2.1)–(2.6):

$$\begin{gathered}
0 \le t_1 \le \ldots \le t_N \le T, \\
\dot{x}_i = f_i(x_i(t), u_i(t)), \qquad t_i \le t \le i_{i+1}, \\
x_i(t_i) = g_i(x_{i-1}(t_i), v_i), \qquad i = 1, \ldots, N, \\
x_0(0) = x_0, \qquad x_N(T) = x_{NT}, \\
I = T + \sum_{i=1}^{N} g_i^+(x_{i-1}(t_i), v_i) \to \min.
\end{gathered} \tag{7.1}$$

The initial x_0 and terminal x_{NT} states of the system are fixed. It is required to find the minimum value of the functional and an optimal control on which it is attained. The number of switchings N and the switching times $t_1, \ldots, t_N$ are not predetermined and are to be found under minimization (7.1).

Comparing this with the general statement, we conclude that the system is stationary, the terminal state is specified, and the final time of the process is free. It is required to minimize the time T of reaching the terminal state, penalized by the total cost of switchings. For $g_i^+ = 0$, we obtain a time-optimal problem [17, 15, 1]. In such a statement, the generators of the value function depend only on the state $\varphi_i^k(t,x_i) = \varphi_i^k(x_i)$, and the two-position function has the form $\psi_i(\theta, x_{i\theta}|\tau, x_{i\tau}) = \tau - \theta$. Denote the minimum value of this difference by $\psi_i(x_{i\theta}|x_{i\tau})$. This function is a solution of the classical time-optimal problem

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t)), \qquad x_i(\theta) = x_{i\theta}, \quad x_i(\tau) = x_{i\tau}, \qquad \int_\theta^\tau dt \to \min. \tag{7.2}$$

Taking into account this notation, we write equations (4.1)–(4.3) for the generators of the value function in the time-optimal problem (7.1):

$$\min_{u\in U_i}\left[\frac{\partial\varphi_i^0(x_i)}{\partial x_i} f_i(x_i,u) + 1\right] = 0, \qquad \varphi_i^0(x_{iT}) = 0, \qquad i \in \mathcal{N}, \tag{7.3}$$

$$\varphi_i^k(x_i) = \min_{\widehat{x}_i\in X_i}\left\{\psi_i(x_i|\widehat{x}_i) + \min_{v\in V_{i+1}}\left[\varphi_{i+1}^{k-1}(g_{i+1}(\widehat{x}_i,v)) + g_{i+1}^+(\widehat{x}_i,v)\right]\right\}, \qquad k \in \mathbb{Z}_+. \tag{7.4}$$

The positional constructions of the "control complex" (see Section 5) are independent of time and can be expressed in the same way as in (3.7) and (4.1)–(4.5).

8. EXAMPLES

We consider plane motions of a group of a variable number of control objects. Each object moves along the plane with constant linear velocity and bounded angular velocity, i.e., can be described by the Dubins model [10]. First, one control object, a carrier, starts a motion. Then, over time, other control objects separate from it and form a group. The goal of the control is to reach the prescribed terminal positions in minimum time; i.e., this is a time-optimal problem of group performance.

Example 8.1 (simultaneous separation of control objects)**.** Suppose that the motion of the carrier is described by the equations

$$\dot{x}_1^0(t) = V\cos x_3^0(t), \qquad \dot{x}_2^0(t) = V\sin x_3^0(t), \qquad \dot{x}_3^0(t) = u^0(t), \qquad |u^0(t)| \le \Omega, \qquad 0 \le t \le t_1, \tag{8.1}$$

where x_1^0 and x_2^0 are the coordinates of the carrier in the plane, x_3^0 is the angle of inclination of the trajectory to the abscissa, V is the constant linear velocity, u^0 is the angular velocity, and Ω is the maximum possible angular velocity. The initial state is fixed: $x^0(0) = x_0^0$. At time t_1, m control objects separate from the carrier and move according to the equations

$$\dot{x}_1^i(t) = v\cos x_3^i(t), \qquad \dot{x}_2^i(t) = v\sin x_3^i(t), \qquad \dot{x}_3^i(t) = u^i(t), \qquad |u^i(t)| \le \omega, \qquad t_1 \le t. \tag{8.2}$$

The linear velocity v and the maximum angular velocity ω of all objects that separated from the carrier are the same. At the moment of separation, the states of the objects coincide with that of the carrier:

$$x^i(t_1) = x^0(t_1), \qquad i = 1,\dots,m. \tag{8.3}$$

The motion of the ith object terminates at time T^i when the object reaches a given target z^i:

$$x_1^i(T^i) = z_1^i, \qquad x_2^i(T^i) = z_2^i, \qquad i = 1,\dots,m. \tag{8.4}$$

At the moment of hitting the target, the angle x_3^i of inclination of the trajectory may be arbitrary.

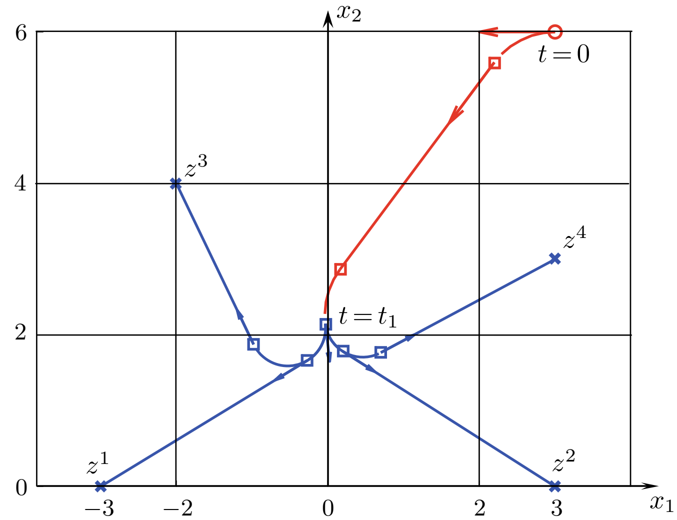

Fig. 1.

The quality of the group control is characterized by the time $T = \max\{T^1, \dots, T^m\}$ of hitting all the targets. It is required to find the minimum time T and a control for which this minimum time is attained, i.e., it is required to solve the time-optimal problem $T \to \min$.

In the formulated problem, there is a single moment of switching t_1, i.e., $N = 1$. There is no control of the switching. Therefore, we have to find an optimal control of the continuous motion and a position of separation. Before the switching, there is a single control object. Writing the auxiliary problem (7.2) for the carrier, we obtain a time-optimal problem of transferring the Dubins machine from a given initial state x to a given terminal state $\widehat{x}$. A solution of this problem is given, for example, in [1, 19]. The optimal trajectory is composed either of three arcs of circles of radius V/Ω or of two arcs of circles connected by a straight line segment. Thus, the optimal two-position control of the carrier is known. Denote by $\Psi(x|\widehat{x})$ the minimum transfer time of the carrier. This function satisfies the HJB equation with zero terminal condition:

$$\min_{|u|\le\Omega}\bigl[\Psi_{x_1}V\cos x_3 + \Psi_{x_2}V\sin x_3 + \Psi_{x_3}u + 1\bigr] = 0, \qquad \Psi(x|\widehat{x}) = 0.$$

After the switching, each object that separated from the carrier should be transferred as fast as possible from the given initial state (8.3) to the given terminal state (8.4). The optimal trajectory of such a motion consists either of two arcs of circles of radius v/ω or of a single arc and a straight line segment. Denote by $\theta(x|z)$ the minimum time of motion of such an object from position x to position z. This function satisfies the HJB equation

$$\min_{|u|\le\omega}\bigl[\theta_{x_1}v\cos x_3 + \theta_{x_2}v\sin x_3 + \theta_{x_3}u + 1\bigr] = 0 \tag{8.5}$$

with zero terminal condition $\theta(x|z) = 0$ for $x_1 = z_1$, $x_2 = z_2$.

Now we find the generators of the value function. The zero generator has the form

$$\varphi_1^0(x^1, \dots, x^m) = \max\bigl\{\theta(x^1|z^1), \dots, \theta(x^m|z^m)\bigr\}.$$

It satisfies equation (7.3), which can easily be verified using (8.5). The first generator satisfies equation (7.4), which for $k = 1$ reads

$$\varphi_0^1(x^0) = \min_{\widehat{x}\in\mathbb{R}^3}\bigl[\Psi(x^0|\widehat{x}) + \varphi_1^0(\widehat{x}, \dots, \widehat{x})\bigr].$$

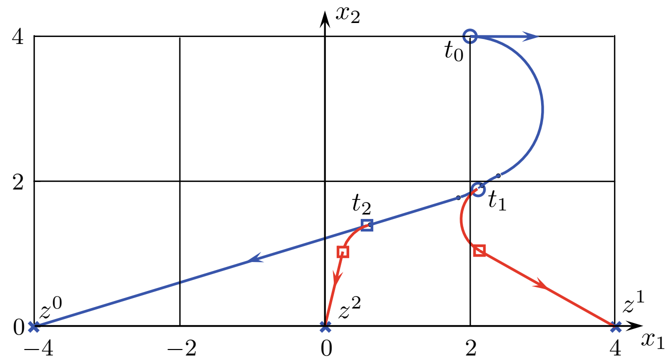

Fig. 2.

We can see that the problem reduces to finite-dimensional minimization, namely, to the search for an optimal switching position $\widehat{x}$. This problem is solved numerically. Figure 1 demonstrates optimal trajectories obtained for the following values of the parameters: $V = 1$, $v = 0.5$, $\Omega = \omega = 1$, $x_0^0 = (3, 6, \pi)$, $z^1 = (-3, 0)$, $z^2 = (3, 0)$, $z^3 = (-2, 4)$, and $z^4 = (3, 3)$. The minimum value of the functional is $\min T = 12.6351$. The separation of trajectories occurs at time $t_1 = 5.098$ at the point $\widehat{x} = (-0.0324, 2.1499, -1.4486)$.

Example 8.2 (sequential separation of control objects)**.** Suppose that the motion of the carrier and the separating objects is described by equations (8.1) and (8.2). The separation of control objects from the carrier occurs sequentially at time points $t_1, \dots, t_N$. The first object separates from the carrier at time t_1 and should hit the target z^1, the second should hit the target z^2, and so on. The motion of the control objects stops when the corresponding targets are hit:

$$x_1^i(T^i) = z_1^i, \qquad x_2^i(T^i) = z_2^i, \qquad i = 0, 1, \dots, N. \tag{8.6}$$

The quality of the group control is characterized by the time $T = \max\{T^0, T^1, \dots, T^N\}$ of hitting all the targets. It is required to find the minimum time T and a control for which this minimum time is attained, i.e., it is required to solve the time-optimal problem $T \to \min$.

In the formulated problem, the number of switchings is N and there is no control of switchings. For the carrier and the separating objects, the time-optimal trajectories from a given initial state to a given terminal state are known, since these are Dubins trajectories [10]. Therefore, we can find the optimal control of continuous motion if we determine the optimal positions of separation of the control objects from the carrier. Hence, the problem reduces to finite-dimensional minimization. Consider its solution for three targets z^0, z^1, and z^2, i.e., for $N = 2$. Let us find the zero generator. After the second switching at time t_2, the group consists of three elements: the carrier and two objects that separated from it. Therefore, the position of the group is determined by the state vectors of the carrier x^0 and the two separated objects x^1 and x^2. The terminal states are defined by (8.6). Hence, the zero generators have the form

$$\varphi_0^0(x^0) = \theta(x^0|z^0), \qquad \varphi_1^0(x^0, x^1) = \max\bigl\{\theta(x^0|z^0), \theta(x^1|z^1)\bigr\},$$
$$\varphi_2^0(x^0, x^1, x^2) = \max\bigl\{\theta(x^0|z^0), \theta(x^1|z^1), \theta(x^2|z^2)\bigr\}.$$

Here, just as in Example 8.1, $\theta(x^i|z^i)$ is the minimum time for the transfer of a control object from the position x^i to the target z^i, $i = 0, 1, 2$. For the first generators, from equation (7.4) we obtain

$$\varphi_0^1(x^0) = \min_{\widehat{x} \in \mathbb{R}^3} \bigl\{\Psi(x^0|\widehat{x}) + \varphi_1^0(\widehat{x}, \widehat{x})\bigr\}, \qquad \varphi_1^1(x^0, x^1) = \min_{\widehat{x} \in \mathbb{R}^3} \bigl\{\Psi(x^0|\widehat{x}) + \varphi_2^0(\widehat{x}, x^1, \widehat{x})\bigr\}.$$

Finally, for the second generator, equation (7.4) has the form

$$\varphi_0^2(x^0) = \min_{\widehat{x}\in\mathbb{R}^3} \big\{ \Psi(x^0|\widehat{x}) + \varphi_1^1(\widehat{x},\widehat{x}) \big\}.$$

These equations were solved numerically for the following values of the parameters: $V = 1$, $v = 0.5$, $\Omega = \omega = 1$, $x_0^0 = (2,4,0)$, $z^0 = (-4,0)$, $z^1 = (0,0)$, and $z^2 = (4,0)$. The minimum value of the functional is $\min T = 9.4759$. The first object separates at time $t_1 = 3.0839$ at the point $x^0(t_1) = (2.1, 1.9, -2.548)$, and the second object separates at time $t_2 = 4.6675$ at the point $x^0(t_2) = (0.6, 1.4, -2.8923)$. The optimal trajectories are shown in Fig. 2.

CONCLUSIONS

We have considered the problem of synthesis of an optimal hybrid control system that operates with switchings at the moments of which the dimension of the state space may change. The number of switchings is not specified in advance and is determined as a result of minimizing an objective functional. Here multiple instantaneous switchings are not excluded. This problem generalizes the problems of synthesis of various classes of hybrid systems. We have developed a new method for deriving sufficient optimality conditions that involves auxiliary functions, the so-called generators of the value function and the two-position value function. Using the dynamic programming method, we have derived equations for finding these auxiliary functions, from which the ordinary value function can then be constructed. The recurrent procedure of the synthesis of a system of variable dimension is more complicated than the synthesis of a continuous–discrete or composite control system. In the general case, it is difficult to obtain its numerical solution. However, if the optimal control of continuous motion is known or can be replaced by a control that is rational from the applied point of view, then the problem becomes finite-dimensional and can be solved relatively easily.

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