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A Generalization of the Yang–Mills Equations

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Abstract—A generalization of the Yang–Mills equations is proposed. It is shown that any solution of the Yang–Mills equations (in the Lorentz gauge) is also a solution of the new generalized equation. It is also shown that the generalized equation has solutions that do not satisfy the Yang–Mills equations.

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The Yang–Mills equations (introduced in 1954) are a class of equations that depend on the gauge Lie group. In theoretical physics, as gauge groups one mainly uses unitary gauge groups or more general semisimple gauge groups. It is well known that the Maxwell equations (1862) can be viewed as a particular case of the Yang–Mills equations with abelian gauge group U(1). The Standard Model (1970s) uses the Yang–Mills equations with gauge groups U(1), SU(2), and SU(3) to describe the interactions of elementary particles.

In the present study, we propose a generalization of the Yang–Mills equations (4.6) (in the pseudo-Euclidean space $\mathbb{R}^{r,s}$). We show that any solution of the Yang–Mills equations (in the Lorentz gauge) is also a solution of the new generalized equation. We also show that the generalized equation has solutions that do not satisfy the Yang–Mills equations.

Section 1 shows how the Maxwell equations for the potential and strength of an electromagnetic field can be expressed using the technique of differential forms (using the operators of exterior differentiation d and codifferentiation δ). We propose a method for generalizing the Maxwell equations that consists in replacing each of the operators d and δ by $d+\delta$ and discuss the equations obtained.

In Section 2, we consider the Yang–Mills equations and discuss the possibility of expressing these equations using the technique of differential forms with values in a Lie algebra.

In Section 3, we introduce the technique of genforms, which can be viewed as a generalization of the technique of differential forms. For genforms, not only the operation of exterior multiplication but also the operation of Clifford multiplication is defined.

In Section 4, we write the Yang–Mills equations using the technique of genforms and, on this basis, propose a generalization of the Yang–Mills equations.

In Section 5, we consider the question of internal consistency of the new equations in some particular cases. Namely, we consider the generalized Maxwell equations in the pseudo-Euclidean spaces $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$, and $\mathbb{R}^{1,3}$. The well-posedness of the Cauchy problem is proved by the symmetrization method, i.e., by reducing the Cauchy problem for the generalized system of Maxwell equations to the Cauchy problem for a Friedrichs symmetric hyperbolic system of first-order equations.

1. MAXWELL EQUATIONS

Pseudo-Euclidean space $\mathbb{R}^{r,s}$. Let r and s be nonnegative integers and $n=r+s\geq 1$. Denote by $\mathbb{R}^{r,s}$ the n-dimensional (pseudo-)Euclidean space [16] with Cartesian coordinates x^μ, $\mu=1,\dots,n$ (the tensor indices of the space $\mathbb{R}^{r,s}$ will be denoted by lowercase Greek letters $\mu,\nu,\alpha,\dots$

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running through the values $1,\dots,n$), and with the metric tensor defined by a diagonal matrix

$$\eta = \|\eta^{\mu\nu}\| = \|\eta_{\mu\nu}\|$$

with r ones and s minus ones on the diagonal.

In $\mathbb{R}^{r,s}$, we consider the changes of coordinates from the (pseudo)orthogonal group $\mathrm{O}(r,s)$,

$$x^\mu \to \acute{x}^\mu = p^\mu_\nu x^\nu, \tag{1.1}$$

where

$$P = \|p^\mu_\nu\| \in \mathrm{O}(r,s) \qquad \text{and} \qquad \mathrm{O}(r,s) = \bigl\{P = \|p^\mu_\nu\| \in \mathrm{GL}(n,\mathbb{R})\colon\ P^{\mathrm{T}}\eta P = \eta\bigr\},$$

with P^{T} the transposed matrix.

In the space $\mathbb{R}^{r,s}$, we will consider tensors (and tensor fields) defined in the coordinates x^μ by their real or complex components $u^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l}$; under the change of coordinates (1.1), the components of the tensor (tensor field) are transformed according to the standard rule of tensor analysis

$$u^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l} \to \acute{u}^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l} = p^{\mu_1}_{\alpha_1}\dots p^{\mu_k}_{\alpha_k} q^{\beta_1}_{\nu_1}\dots q^{\beta_l}_{\nu_l} u^{\alpha_1\dots\alpha_k}_{\beta_1\dots\beta_l}, \tag{1.2}$$

where

$$Q = \|q^\beta_\nu\| = P^{-1}.$$

Denote the set of tensors with k contravariant and l covariant indices by T^k_l. We will write $u \in \mathrm{T}^k_l$ or $u^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l} \in \mathrm{T}^k_l$.

Maxwell equations. In modern theoretical and mathematical physics, the Maxwell equations, which underlie electromagnetism, are often expressed in terms of the *potential* a_μ and *strength* $f_{\mu\nu}$ of the electromagnetic field as the following equations [14] in the Minkowski space[1] $\mathbb{R}^{1,3}$:

$$\partial_\mu a_\nu - \partial_\nu a_\mu - f_{\mu\nu} = 0, \qquad \partial_\mu f^{\mu\nu} = j^\nu, \tag{1.3}$$

where the real tensor fields $a_\mu \in \mathrm{T}_1$, $f_{\mu\nu} \in \mathrm{T}_2$, and $j^\nu \in \mathrm{T}^1$ depend smoothly on $x \in \mathbb{R}^{1,3}$ and $\partial_\mu = \partial/\partial x^\mu$ are partial derivatives. To raise or lower tensor indices, we use the metric tensor. The vector j^ν is called the *current vector*. Equations (1.3) imply the following equality (vanishing of the 4-divergence of the current vector):

$$\partial_\nu j^\nu = 0. \tag{1.4}$$

Equations (1.3) are invariant under the following *gauge transformation*:

$$a_\mu \to a_\mu + \partial_\mu\lambda, \qquad f_{\mu\nu} \to f_{\mu\nu}, \qquad j^\nu \to j^\nu,$$

where $\lambda = \lambda(x)$ is a scalar smooth real function.

Equations (1.3) are invariant under those changes of coordinates of the Minkowski space that belong to the Lorentz group $\mathrm{O}(1,3)$.

Note also that equations (1.3) can be naturally viewed as equations in the pseudo-Euclidean space $\mathbb{R}^{r,s}$.

Substituting $f_{\mu\nu}$ from the first equation in (1.3) into the second, we obtain an equation for the potential a_μ:

$$(\partial^\mu\partial_\mu)a_\nu - \partial_\nu(\partial^\mu a_\mu) = j_\nu. \tag{1.5}$$

[1] For simplicity, we use a system of units in which the speed of light is equal to 1. In addition, we use Einstein's convention on summation over repeated indices.

Differential forms [3]. The Maxwell equations (1.3) can be expressed in terms of differential forms (see [14]). Every covariant skew-symmetric tensor field with components $a_{\mu_1\dots\mu_k}$ in $\mathbb{R}^{r,s}$ with Cartesian coordinates x^μ can be assigned a *differential form*

$$A = \frac{1}{k!} a_{\mu_1\dots\mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$$

of degree k, where dx^μ are the differentials of the coordinates x^μ. Denote the set of differential forms of degree k by $\Lambda_k(\mathbb{R}^{r,s})$, $0 \le k \le n$.

Given a family of $n+1$ differential forms $\overset{k}{A} \in \Lambda_k(\mathbb{R}^{r,s})$ of degrees from 0 to n, we can consider an inhomogeneous differential form

$$A = \sum_{k=0}^{n} \overset{k}{A} \in \Lambda(\mathbb{R}^{r,s}) = \bigoplus_{k=0}^{n} \Lambda_k(\mathbb{R}^{r,s}).$$

The set of differential forms $\Lambda(\mathbb{R}^{r,s})$ is equipped with the operations of addition and multiplication by a scalar field (these operations are defined pointwise, at every point $x \in \mathbb{R}^{r,s}$). For differential forms, there is also an operation of exterior multiplication. If $A \in \Lambda_k(\mathbb{R}^{r,s})$ and $B \in \Lambda_l(\mathbb{R}^{r,s})$, then

$$A \wedge B = (-1)^{kl} B \wedge A \in \Lambda_{k+l}(\mathbb{R}^{r,s}).$$

For differential forms in $\Lambda_k(\mathbb{R}^{r,s})$, one introduces the operator of exterior differentiation (generalized gradient)

$$d\colon \Lambda_k(\mathbb{R}^{r,s}) \to \Lambda_{k+1}(\mathbb{R}^{r,s})$$

and the operator of codifferentiation (generalized divergence)

$$\delta\colon \Lambda_k(\mathbb{R}^{r,s}) \to \Lambda_{k-1}(\mathbb{R}^{r,s}).$$

These operators can be related to each other by the Hodge operator (see [14])

$$*\colon \Lambda_k(\mathbb{R}^{r,s}) \to \Lambda_{n-k}(\mathbb{R}^{r,s}),$$

namely,

$$\delta = \epsilon * d *,$$

where $\epsilon = \pm 1$ depending on k, r, and s. An important property of the operators d and δ is that

$$d^2 = 0 \qquad \text{and} \qquad \delta^2 = 0. \tag{1.6}$$

Note that

$$(d\delta + \delta d) = \partial^\mu \partial_\mu = \Box$$

is the d'Alembertian. The operators d and δ are discussed in greater detail in Section 3.

Maxwell equations in terms of differential forms. It is well known [14] that the Maxwell equations in the pseudo-Euclidean space $\mathbb{R}^{r,s}$ (in particular, in the Minkowski space $\mathbb{R}^{1,3}$) can be expressed in terms of the differential forms

$$A = a_\mu dx^\mu \in \Lambda_1(\mathbb{R}^{r,s}), \qquad J = j_\mu dx^\mu \in \Lambda_1(\mathbb{R}^{r,s}), \qquad F = \frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu \in \Lambda_2(\mathbb{R}^{r,s}) \tag{1.7}$$

as the equalities

$$dA - F = 0, \qquad \delta F = J, \tag{1.8}$$

and condition (1.4) turns into

$$\delta J = 0.$$

Equation (1.5) is expressed as

$$\delta dA = J, \tag{1.9}$$

or, which is the same,

$$(d\delta + \delta d)A - d(\delta A) = J.$$

Consider two systems of equations

$$(d+\delta)A - F = 0, \qquad \delta F = J \tag{1.10}$$

and

$$dA - F = 0, \qquad (d+\delta)F = J, \tag{1.11}$$

which are obtained from equations (1.8) by replacing the operator d or δ with the operator $d+\delta$ (in the case of the system of equations (1.10), one should assume that $F \in \Lambda_2(\mathbb{R}^{r,s}) \oplus \Lambda_0(\mathbb{R}^{r,s})$). Since both systems (1.10) and (1.11) reduce to the same equation (1.9) in view of (1.6), all three systems of equations (1.8), (1.10), and (1.11) can be assumed to be equivalent to each other and equivalent to the system of Maxwell equations (1.3).

Now, consider the system of equations

$$(d+\delta)A - F = 0, \qquad (d+\delta)F = J \tag{1.12}$$

for $A, J \in \Lambda_1(\mathbb{R}^{r,s})$ and $F \in \Lambda_2(\mathbb{R}^{r,s}) \oplus \Lambda_0(\mathbb{R}^{r,s})$. This system is obtained from system (1.8) by replacing both operators d and δ with the operator $d+\delta$. Clearly, this system of equations reduces to the equation

$$(d\delta + \delta d)A = J. \tag{1.13}$$

Such systems of equations (1.12) have been considered by many authors (see, for example, [9, 5]).

Now we consider a few questions related to the above variants of the Maxwell equations written in terms of differential forms. All these questions are simple, but they are important for us because they provide a basis for considering more complicated questions related to the Yang–Mills equations.

If we supplement the system of equations (1.12) with the equation $\delta A = 0$, then any solution of the resulting system of equations

$$(d+\delta)A - F = 0, \qquad (d+\delta)F = J, \qquad \delta A = 0 \tag{1.14}$$

is obviously also a solution of system (1.11), which is equivalent to the system of Maxwell equations (1.3).

Let us show that in the case of signature $(r,s) = (1, n-1)$ of the pseudo-Euclidean space, the condition $\delta A = 0$ for all $x \in \mathbb{R}^{r,s}$ in the system of equations (1.14) can be replaced by the two conditions

$$\begin{aligned} \delta J = 0 \qquad &\text{for} \quad x^1 > 0, \\ \delta A = 0, \qquad \partial_1(\delta A) = 0 \qquad &\text{for} \quad x^1 = 0. \end{aligned} \tag{1.15}$$

One of these conditions is a condition on the right-hand side J, and the other is defined on the hyperplane $x^1 = 0$. To prove this fact, we rewrite system (1.12) in the form (1.13) and act by the operator δ on both sides of the first equation. We obtain

$$\delta d\delta A = (\delta d)(\delta A) = \Box(\delta A) = 0.$$

Setting $b := \delta A \in \Lambda_0(\mathbb{R}^{1,n-1})$ and taking into account (1.15), we obtain the Cauchy problem in the domain $x^1 > 0$ for the function (0-form) $b = b(x)$ with the initial data specified for $x^1 = 0$:

$$\Box b = 0 \qquad \text{for} \quad x^1 > 0, \tag{1.16}$$
$$\partial_1 b = 0, \qquad b = 0 \qquad \text{for} \quad x^1 = 0.$$

It is well known from the theory of partial differential equations that the Cauchy problem (1.16) for the hyperbolic equation $\Box b = 0$ has a unique solution, and hence $b \equiv 0$ for all $x^1 > 0$ (changing x^1 to $-x^1$, we obtain the same result for $x^1 < 0$). The statement is proved.

Let us show that system (1.12) can be viewed as a generalization of the system of Maxwell equations. It is obvious that if $A, J \in \Lambda_1(\mathbb{R}^{r.s})$ is a solution of the Maxwell equations (1.8) in the Lorentz gauge $\delta A = 0$, then these 1-forms are also a solution of equations (1.12).

Let us show that under the conditions $J \neq 0$ and $\delta J = 0$, among the solutions of equations (1.12), there are solutions that do not satisfy the Maxwell equations (1.8) (but satisfy the additional condition $\delta d\delta A = 0$, $\delta A \neq 0$). Indeed, from the condition $\delta J = 0$ we obtain

$$\delta J = \delta(d\delta + \delta d)A = \delta d\delta A = 0. \tag{1.17}$$

Now, take two functions (0-forms) $j = j(x)$ and $a = a(x)$ that satisfy the system of equations

$$\Box a = j, \qquad \Box j = 0, \tag{1.18}$$

which can be rewritten in terms of differential forms as

$$\delta da = j, \qquad \delta dj = 0.$$

The following statement is valid: if the functions a and j satisfy equations (1.18), then the 1-forms

$$A = da, \qquad J = dj$$

satisfy the system of equations

$$d\delta A = J, \qquad dA = 0 \tag{1.19}$$

subject to the condition $\delta J = 0$. This solution (A, J) gives a particular subclass of solutions to the system of equations

$$(d\delta + \delta d)A = J, \qquad \delta J = 0.$$

It remains to show that such a solution to system (1.18) for $J = dj \neq 0$ does not satisfy the Maxwell equations

$$\delta dA = J.$$

This is obvious, since for $dA = 0$ we have $\delta dA = 0$, and if the right-hand side J is nonzero, this yields a contradiction:

$$0 = \delta dA = J \neq 0.$$

The statement is proved.

We call the system of equations (1.12) a *generalized system of Maxwell equations* (expressed in the technique of differential forms).

Thus, we have proved the following propositions.

Proposition 1. *If we supplement the system of equations* (1.12) *with the equation* $\delta A = 0$, *then any solution of the resulting system of equations* (1.14) *is also a solution to the system of equations* (1.11), *which is equivalent to the system of Maxwell equations* (1.3).

Proposition 2. *In the case of signature* $(r,s)=(1,n-1)$ *of the pseudo-Euclidean space, the condition* $\delta A=0$ *from Proposition 1 can be relaxed to condition* (1.15). *In this case, the condition* $\delta J=0$ *on the right-hand side is additionally required.*

Proposition 3. *Any solution of the Maxwell equations* (1.8) *in the Lorentz gauge* $\delta A=0$ *is also a solution to the system of equations* (1.12) *with the right-hand side satisfying the condition* $\delta J=0$.

Proposition 4. *The system of equations* (1.12) *subject to the additional condition* $\delta J=0$ *has certain solutions that do not satisfy the Maxwell equations* (1.8).

2. YANG–MILLS EQUATIONS

Let K be a semisimple Lie group and L be the real Lie algebra of the Lie group K. The Lie algebra L is a real vector space of dimension N with basis $t^1,\dots,t^N$. Multiplication of elements in L is defined by a Lie bracket $[a,b]=-[b,a]$ satisfying the Jacobi identity. Multiplication of the basis elements is defined by real structure constants $c_l^{rs}=-c_l^{sr}$, $r,s,l=1,\dots,N$, of the Lie algebra L:

$$[t^r,t^s]=c_l^{rs}t^l. \tag{2.1}$$

In this study, we represent the elements of the Lie algebra L and the Lie group K by square matrices of appropriate size or by elements of the Clifford algebra $\mathcal{C}\ell(p,q)$. In both cases, the Lie bracket is given by the commutator $[a,b]=ab-ba$, with either the matrix multiplication or multiplication of elements of the Clifford algebra on the right-hand side.

Denote by $L\mathrm{T}^a_b$ the set of tensor fields of the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of type (a,b) and rank $a+b$ with values in the Lie algebra L.

In the pseudo-Euclidean space $\mathbb{R}^{p,q}$, consider the equations

$$\partial_\mu a_\nu-\partial_\nu a_\mu-\rho[a_\mu,a_\nu]-f_{\mu\nu}=0,\qquad \partial_\mu f^{\mu\nu}-\rho[a_\mu,f^{\mu\nu}]=j^\nu, \tag{2.2}$$

in which $a_\mu\in L\mathrm{T}_1$, $j^\nu\in L\mathrm{T}^1$, $f_{\mu\nu}=-f_{\nu\mu}\in L\mathrm{T}_2$, and ρ is a real constant. These equations are called *Yang–Mills equations* (system of Yang–Mills equations). Usually, it is assumed that a_μ and $f_{\mu\nu}$ are unknowns and j^ν is a known vector with values in the Lie algebra L.

Equations (2.2) are said to define the *Yang–Mills field* $(a_\mu,f_{\mu\nu})$. In this case, a_μ is the *potential* and $f_{\mu\nu}$ is the *strength* of the Yang–Mills field. The vector j^ν is called a *nonabelian current* (in the case of abelian group K, the vector j^ν is called a *current*).

We can substitute the components of the skew-symmetric tensor $f_{\mu\nu}$ defined by the first equation in (2.2) into the second equation to obtain a single second-order equation for the potential of the Yang–Mills field:

$$\partial_\mu\bigl(\partial^\mu a^\nu-\partial^\nu a^\mu-\rho[a^\mu,a^\nu]\bigr)-\rho\bigl[a_\mu,\partial^\mu a^\nu-\partial^\nu a^\mu-\rho[a^\mu,a^\nu]\bigr]=j^\nu. \tag{2.3}$$

Let us look at equations (2.2) from another angle. Let $a_\mu\in L\mathrm{T}_1$ be an arbitrary given L-valued covector that depends smoothly on $x\in\mathbb{R}^{p,q}$. Introduce the notation

$$f_{\mu\nu}:=\partial_\mu a_\nu-\partial_\nu a_\mu-\rho[a_\mu,a_\nu],\qquad j^\nu:=\partial_\mu f^{\mu\nu}-\rho[a_\mu,f^{\mu\nu}]. \tag{2.4}$$

Now we can consider the expression $\partial_\nu j^\nu-\rho[a_\nu,j^\nu]$ and verify by straightforward calculations that

$$\partial_\nu j^\nu-\rho[a_\nu,j^\nu]=0. \tag{2.5}$$

This equality is called a *nonabelian conservation law* (in the case of abelian group K, we have $\partial_\nu j^\nu=0$, i.e., the divergence of the vector j^ν is zero).

Thus, the nonabelian conservation law (2.5) is a consequence of the Yang–Mills equations (2.2).

Suppose that the tensor fields a_μ, $f_{\mu\nu}$, and j^ν satisfy the Yang–Mills equations (2.2). Take a smooth scalar field with values in a Lie group $S = S(x) \in K$ and consider the transformed tensor fields

$$\acute{a}_\mu = S^{-1} a_\mu S - S^{-1}\partial_\mu S, \qquad \acute{f}_{\mu\nu} = S^{-1} f_{\mu\nu} S, \qquad \acute{j}^\nu = S^{-1} j^\nu S. \tag{2.6}$$

Then these tensor fields satisfy the same Yang–Mills equations

$$\partial_\mu \acute{a}_\nu - \partial_\nu \acute{a}_\mu - \rho[\acute{a}_\mu, \acute{a}_\nu] - \acute{f}_{\mu\nu} = 0, \qquad \partial_\mu \acute{f}^{\mu\nu} - \rho[\acute{a}_\mu, \acute{f}^{\mu\nu}] = \acute{j}^\nu;$$

i.e., equations (2.2) are invariant under the transformations (2.6). The transformation (2.6) is called a *gauge transformation* (or *gauge symmetry*), and the group K is called the *gauge group* of the Yang–Mills equations (2.2).

Differential forms with values in the Lie algebra L. Considering the tensor product $L \otimes \Lambda(\mathbb{R}^{r,s})$ of the Lie algebra L and the set of differential forms $\Lambda(\mathbb{R}^{r,s})$, we arrive at differential forms with values in the Lie algebra L. Differential forms of degree k in $L \otimes \Lambda_k(\mathbb{R}^{r,s})$ can be written as

$$A = \frac{1}{k!}\, a_{\mu_1\dots\mu_k} \otimes dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k},$$

where $a_{\mu_1\dots\mu_k}$ are the components of a covariant skew-symmetric tensor field with values in the Lie algebra L.

Consider the Yang–Mills equations (2.2) and introduce the notation

$$A = a_\mu \otimes dx^\mu \in L \otimes \Lambda_1(\mathbb{R}^{r,s}), \qquad J = j_\mu \otimes dx^\mu \in L \otimes \Lambda_1(\mathbb{R}^{r,s}),$$
$$F = \frac{1}{2} f_{\mu\nu} \otimes dx^\mu \wedge dx^\nu \in L \otimes \Lambda_2(\mathbb{R}^{r,s}),$$

where $f_{\mu\nu} = -f_{\nu\mu}$.

It is known that the Yang–Mills equations can be expressed (represented) in terms of differential forms in $L \otimes \Lambda(\mathbb{R}^{r,s})$ as

$$dA - \rho A \wedge A - F = 0, \qquad d * F - \rho(A \wedge *F - *F \wedge A) = *J.$$

In the next section, we describe the technique of genforms, which allows one to express the Yang–Mills equations and see the possibility of their generalization.

3. SET OF GENFORMS $\Lambda^{[h]}(\mathbb{R}^{r,s})$

Some elements of the technique of genforms have been used by the author since the early 2000s in a number of publications devoted to the model Dirac equation and other field equations (see [11, Ch. 6] as well as [10, 12, 17][2]).

Clifford algebra $\mathcal{C}\ell(r,s)$. The construction of the Clifford algebra $\mathcal{C}\ell(r,s)$ is discussed in detail in [11, Ch. 3]. It has generators e^a, $a = 1,\dots,n$, $n = r+s$, identity e, and two multiplications: the Clifford multiplication $U, V \to UV$ and exterior multiplication $U, V \to U \wedge V$. The generators satisfy the conditions that define the Clifford multiplication,

$$e^a e^b + e^b e^a = 2\eta^{ab} e, \qquad a, b = 1, \dots, n, \tag{3.1}$$

and the exterior multiplication is defined by the formula ($k \geq 2$)

$$e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_k} = e^{[a_1} e^{a_2} \ldots e^{a_k]},$$

where the square brackets denote the operation of alternation of indices.

[2] In these papers, the term "h-forms" was used instead of the present term "genforms."

The dimension of the Clifford algebra $\mathcal{C}\ell(r,s)$ as a vector space is equal to 2^n, and as a basis of $\mathcal{C}\ell(r,s)$ one can take the elements

$$e, e^a, e^{a_1a_2}, \dots, e^{12\dots n}, \qquad a_1 < a_2 < \dots,$$

numbered by ordered multi-indices of length from 0 to n, where the indices $a, a_1, \dots$ run through the values from 1 to n and

$$e^{a_1\dots a_k} = e^{a_1}\dots e^{a_k} = e^{a_1}\wedge\dots\wedge e^{a_k} \qquad \text{for} \quad a_1 < \dots < a_k. \tag{3.2}$$

The vector subspace of $\mathcal{C}\ell(r,s)$ spanned by the basis elements $e^{a_1\dots a_k}$ numbered by ordered multi-indices of length k is denoted by $\mathcal{C}\ell_k(r,s)$. The dimension of the vector space $\mathcal{C}\ell_k(r,s)$ is equal to (the binomial coefficient) C_n^k. We have

$$\mathcal{C}\ell(r,s) = \bigoplus_{k=0}^{n} \mathcal{C}\ell_k(r,s).$$

In the Clifford algebra $\mathcal{C}\ell(r,s)$, there are projection operators

$$\pi_k\colon\ \mathcal{C}\ell(r,s) \to \mathcal{C}\ell_k(r,s), \qquad k = 0,1,\dots,n,$$

and a reversal operator $\sim$:

$$(UV)^\sim = V^\sim U^\sim \quad \forall U,V \in \mathcal{C}\ell(r,s), \qquad (e^a)^\sim = e^a, \quad a = 1,\dots,n.$$

Any element $U \in \mathcal{C}\ell_k(r,s)$ can be expanded in the basis elements with real coordinates $u_{a_1\dots a_k}$ numbered by ordered multi-indices:

$$U = \sum_{a_1<\dots<a_k} u_{a_1\dots a_k} e^{a_1\dots a_k}. \tag{3.3}$$

Let us extend the system of C_n^k numbers $u_{a_1\dots a_k}$ numbered by ordered multi-indices to a system of n^k numbers $u_{a_1\dots a_k}$ numbered by arbitrary multi-indices $a_1,\dots,a_k = 1,\dots n$ by means of the antisymmetry condition

$$u_{a_1\dots a_k} = u_{[a_1\dots a_k]}.$$

In this case, the element (3.3) can be expressed in the following forms (cf. (3.2)):

$$U = \sum_{a_1<\dots<a_k} u_{a_1\dots a_k} e^{a_1\dots a_k} = \frac{1}{k!} u_{a_1\dots a_k} e^{a_1}\dots e^{a_k} = \frac{1}{k!} u_{a_1\dots a_k} e^{a_1}\wedge\dots\wedge e^{a_k}. \tag{3.4}$$

In field theory, as a rule, complexified Clifford algebras $\mathbb{C}\otimes\mathcal{C}\ell(r,s)$ are used.

Tetrads in $\mathbb{R}^{r,s}$. Let y_a^μ, $a = 1,\dots,n$, be n orthonormal vectors in the (pseudo-)Euclidean space $\mathbb{R}^{r,s}$, $r+s=n$, with coordinates x^μ. This set of vectors is called a *tetrad*. Tetrads satisfy the orthonormality condition

$$y_a^\mu y_b^\nu \eta^{ab} = \eta^{\mu\nu}. \tag{3.5}$$

One can pass from one tetrad y_a^μ to another $\acute{y}_a^\mu$ by a (pseudo)orthogonal transformation with respect to the Latin index,

$$y_a^\mu \to \acute{y}_a^\mu = q_a^b y_b^\mu, \tag{3.6}$$

where $Q = \|q_a^b\| \in \mathrm{O}(r,s)$. In this case, the tetrad $\acute{y}_a^\mu$ also satisfies condition (3.5). Therefore, we can speak of a class of tetrads corresponding to the (pseudo)orthogonal Lie group $\mathrm{O}(r,s)$.

Genforms. Let y^μ_a be a tetrad in $\mathbb{R}^{r,s}$, and let $\mathcal{C}\ell(r,s)$ be the Clifford algebra with generators e^a satisfying conditions (3.1). In $\mathbb{R}^{r,s}$, define a $\mathcal{C}\ell_1(r,s)$-valued vector

$$h^\mu := y^\mu_a e^a \in \mathcal{C}\ell_1(r,s)\mathrm{T}^1,$$

which we will call a *genvector*. Note that the components of the genvector h^μ, the components of the tetrad y^μ_a, and the generators e^a of the Clifford algebra do not depend on the point $x\in\mathbb{R}^{r,s}$. Using equalities (3.5) and (3.1), one can easily verify that the components of the genvector h^μ satisfy the same conditions

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu}e, \qquad \mu,\nu = 1,\dots,n,$$

as the generators e^a (one should replace e by h and the Latin indices a and b by the Greek (tensor) indices μ and ν). Thus, we arrive at a variant of the Clifford algebra $\mathcal{C}\ell(r,s)$ with generators h^μ, which is associated with the (pseudo-)Euclidean space $\mathbb{R}^{r,s}$ with Cartesian coordinates x^μ, $\mu=1,\dots,n$. By analogy with formulas (3.4), the elements of such a geometrized algebra are written as *genforms*

$$U = \sum_{k=0}^{n} \frac{1}{k!} u_{\mu_1\dots\mu_k} h^{\mu_1}\wedge\dots\wedge h^{\mu_k}, \tag{3.7}$$

where $u_{\mu_1\dots\mu_k} = u_{[\mu_1\dots\mu_k]}$ are the components of a covariant skew-symmetric tensor field of rank k in the space $\mathbb{R}^{r,s}$.

Thus, in the technique of genforms, a covariant skew-symmetric tensor field of rank k defined in $\mathbb{R}^{r,s}$ (with coordinates x^μ) by components $u_{\mu_1\dots\mu_k}$ is assigned a genform of degree k,

$$u_{\mu_1\dots\mu_k} \to \frac{1}{k!} u_{\mu_1\dots\mu_k} h^{\mu_1}\wedge\dots\wedge h^{\mu_k},$$

and a family of covariant skew-symmetric tensor fields of ranks from 0 to n is assigned an (inhomogeneous) genform (3.7). Denote the set of genforms of degree k by $\Lambda^{[h]}_k(\mathbb{R}^{r,s})$ and the set of inhomogeneous genforms by

$$\Lambda^{[h]}(\mathbb{R}^{r,s}) = \bigoplus_{k=0}^{n} \Lambda^{[h]}_k(\mathbb{R}^{r,s}).$$

For genforms, all the same operations are defined as for the Clifford algebra, namely, the exterior and Clifford multiplications of genforms $U,V\to U\wedge V$ and $U,V\to UV$, the reversal operation $U\to U^\sim$, and the projections onto the subspaces of genforms of degrees $k=0,1,\dots,n$:

$$\pi_k\colon\ \Lambda^{[h]}(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_k(\mathbb{R}^{r,s}).$$

In this notation, the expression $[h]$ indicates the root symbol of the genvector h^μ. If we consider complexified genforms in $\mathbb{C}\otimes\Lambda^{[h]}(\mathbb{R}^{r,s})$, which are expressed in the form (3.7) with complex-valued functions $u_{\mu_1\dots\mu_k} = u_{\mu_1\dots\mu_k}(x)$, then for such genforms the operation of (complex) conjugation is defined,

$$U \to \overline{U} = \sum_{k=0}^{n} \frac{1}{k!} \overline{u}_{\mu_1\dots\mu_k} h^{\mu_1}\wedge\dots\wedge h^{\mu_k}, \tag{3.8}$$

where $\overline{u}_{\mu_1\dots\mu_k}$ are complex conjugate functions. Note that from the viewpoint of conjugation operation $U\to\overline{U}$, the generators h^μ are assumed to be real ($\overline{h}^\mu = h^\mu$).

The main difference of the set of genforms from the set of differential forms is in the presence of the operation of Clifford multiplication of genforms, which defines the structure of the Clifford algebra on the set of genforms. An attempt to introduce the Clifford multiplication of differential forms was made by Kähler [9] in 1962. To incorporate the electromagnetic field potential into the version of the Dirac equation proposed by him,[3] Kähler introduced a special multiplication of 1-forms by forms of arbitrary degree and established that, as applied to the differentials of coordinates dx^μ, the multiplication introduced by him satisfies the formula

$$dx^\mu dx^\nu + dx^\nu dx^\mu = 2g^{\mu\nu}. \tag{3.9}$$

Here the right-hand side contains the components of the metric tensor of a (pseudo-)Riemannian manifold. In the case of a diagonal metric $g^{\mu\nu}$, formula (3.9) appears in the definition of the Clifford algebra $\mathcal{C}\ell(r,s)$ under the assumption that dx^μ are generators of the Clifford algebra. Therefore, Kähler called the introduced multiplication the *Clifford multiplication of differential forms*. Later, in 1974, the construction of the Clifford multiplication of differential forms was rediscovered by Atiyah [1], who considered the construction in less detail. Up to now, the construction of the Clifford multiplication of differential forms has not been part of the mainstream in mathematics and is familiar only to a narrow circle of specialists. I can guess that such a situation occurs because formula (3.9) is difficult to accept, since this formula is not consistent with the existing view of the differentials of coordinates as a basis of the cotangent space to a manifold. The construction of genforms described here (see also [11]) seems to be more consistent and logical. Note that it does not coincide with the construction of the Clifford multiplication of differential forms. Rather, the apparatus of genforms can be viewed as a development of Kähler's apparatus on a somewhat different basis: Kähler's apparatus is based on differential forms, while the apparatus of genforms is based on Clifford algebras. Note that the Clifford multiplication of differential forms and the related operators $d+\delta$ were used in studies related to the Dirac equation and its modifications [8, 9, 2, 15, 11]. In the present study, we focus on the Maxwell and Yang–Mills equations and their generalizations.

On the relationship between the Clifford algebra $\mathcal{C}\ell(r,s)$ and the set of genforms $\Lambda^{[h]}(\mathbb{R}^{r,s})$. Let $\mathbb{R}^{r,s}$ be a (pseudo-)Euclidean space with Cartesian coordinates x^μ. Denote the set of smooth functions[4] of the variable $x\in\mathbb{R}^{r,s}$ with values in the Clifford algebra $\mathcal{C}\ell(r,s)$ by $\mathcal{C}\ell(\mathbb{R}^{r,s})$:

$$\mathcal{C}\ell(\mathbb{R}^{r,s}) = \bigl\{U\colon\ \mathbb{R}^{r,s}\to\mathcal{C}\ell(r,s)\bigr\}.$$

One can add functions in $\mathcal{C}\ell(\mathbb{R}^{r,s})$ together and multiply them by each other in the Clifford and exterior ways pointwise (at every point $x\in\mathbb{R}^{r,s}$).

Let us show that an arbitrary tetrad y^μ_a defines a one-to-one correspondence between the elements of the sets $\mathcal{C}\ell(\mathbb{R}^{r,s})$ and $\Lambda^{[h]}(\mathbb{R}^{r,s})$, where $h^\mu = y^\mu_a e^a$. Indeed, if real antisymmetric coefficients

$$u_{a_1\dots a_k} = u_{a_1\dots a_k}(x) = u_{[a_1\dots a_k]}$$

define an element of the algebra

$$U = U(x) = \frac{1}{k!}\,u_{a_1\dots a_k}\,e^{a_1}\wedge\dots\wedge e^{a_k}\in\mathcal{C}\ell(\mathbb{R}^{r,s}),$$

then we can use the formulas $h^\mu = y^\mu_a e^a$ and $e^a = y^a_\mu h^\mu$ (where e^a are generators of the Clifford algebra $\mathcal{C}\ell(r,s)$ and h^μ are generators of the algebra $\Lambda^{[h]}(\mathbb{R}^{r,s})$) to rewrite this element U as

$$U = U(x) = \frac{1}{k!}\bigl(u_{a_1\dots a_k}y^{a_1}_{\mu_1}\dots y^{a_k}_{\mu_k}\bigr)h^{\mu_1}\wedge\dots\wedge h^{\mu_k} = \frac{1}{k!}\,u_{\mu_1\dots\mu_k}\,h^{\mu_1}\wedge\dots\wedge h^{\mu_k}\in\Lambda^{[h]}_k(\mathbb{R}^{r,s}),$$

[3] This version of the Dirac equation describes a 16-component wave function. It was proposed (without details) in 1928 by Ivanenko and Landau [8] and is sometimes called the ILK-equation in the literature.

[4] We assume that the smoothness of the functions under consideration is sufficient for the validity of the arguments used. For example, for the results of the present study, it suffices to assume that the functions are three times continuously differentiable with respect to $x\in\mathbb{R}^{r,s}$. This remark also applies to the tensor fields in question.

where

$$u_{\mu_1\dots\mu_k} = u_{\mu_1\dots\mu_k}(x) = u_{[\mu_1\dots\mu_k]} = u_{a_1\dots a_k} y^{a_1}_{\mu_1}\dots y^{a_k}_{\mu_k}.$$

This correspondence between the elements of the sets $\mathcal{C}\ell(\mathbb{R}^{r,s})$ and $\Lambda^{[h]}(\mathbb{R}^{r,s})$ is one-to-one.[5]

Operator ð. Introduce the differential operator $ð\colon \Lambda^{[h]}(\mathbb{R}^{r,s}) \to \Lambda^{[h]}(\mathbb{R}^{r,s})$,

$$ð = h^\mu \partial_\mu, \tag{3.10}$$

which acts on genforms $U = U(x) \in \Lambda^{[h]}(\mathbb{R}^{r,s})$ according to the rule $ðU = h^\mu \partial_\mu U$, where $\partial_\mu = \partial/\partial x^\mu$ are partial derivatives. We assume that the components of the vector h^μ are independent of $x \in \mathbb{R}^{r,s}$, i.e., $\partial_\mu h^\nu = 0$ for all $\mu, \nu = 1, \dots, n$.

The differential operator ð plays a key role in our theory. Using the operator ð, below we define some other differential operators in $\Lambda^{[h]}(\mathbb{R}^{r,s})$.

Operators d and δ. Note another obvious property of the multiplication of elements of the Clifford algebra $\mathcal{C}\ell(r,s)$:

$$\overset{1}{U}\overset{k}{V} = \overset{k+1}{W} + \overset{k-1}{W} \in \mathcal{C}\ell_{k+1}(r,s) \oplus \mathcal{C}\ell_{k-1}(r,s) \qquad \forall \overset{1}{U} \in \mathcal{C}\ell_1(r,s), \quad \overset{k}{V} \in \mathcal{C}\ell_k(r,s).$$

Using this property, we introduce operators[6]

$$d, \delta\colon\ \Lambda^{[h]}(\mathbb{R}^{r,s}) \to \Lambda^{[h]}(\mathbb{R}^{r,s})$$

such that $d\colon \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_{k+1}(\mathbb{R}^{r,s})$ and $\delta\colon \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_{k-1}(\mathbb{R}^{r,s})$. By definition, set

$$d\overset{k}{A} := \pi_{k+1}(ð\overset{k}{A}), \qquad \delta\overset{k}{A} := \pi_{k-1}(ð\overset{k}{A}),$$

where $\overset{k}{A} \in \Lambda^{[h]}_k(\mathbb{R}^{r,s})$, $k = 0, 1, \dots, n$.

Here are the basic properties of these operators:

(1) $d + \delta = ð$;

(2) $d^2 = 0$ and $\delta^2 = 0$;

(3) $d\overset{n}{A} = 0$ and $\delta\overset{0}{A} = 0$.

The operators d and δ are first-order operators, since they contain the first derivatives ∂_μ. Now we define operators of order $n \ge 1$ ($d_{(1)} = d$, $\delta_{(1)} = \delta$):

$$d_{(n)} = \begin{cases} (d\delta)^{n/2} & \text{if } n \text{ is even}, \\ (d\delta)^{(n-1)/2} d & \text{if } n \text{ is odd}, \end{cases} \qquad \delta_{(n)} = \begin{cases} (\delta d)^{n/2} & \text{if } n \text{ is even}, \\ (\delta d)^{(n-1)/2}\delta & \text{if } n \text{ is odd}. \end{cases}$$

Here we use shorthand notation such as $(d\delta)^2 = d\delta d\delta$. Note that

$$d_{(n)}\colon\ \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_k(\mathbb{R}^{r,s}), \qquad \delta_{(n)}\colon\ \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \qquad \text{if } n \text{ is even},$$

$$d_{(n)}\colon\ \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_{k+1}(\mathbb{R}^{r,s}), \qquad \delta_{(n)}\colon\ \Lambda^{[h]}_k(\mathbb{R}^{r,s}) \to \Lambda^{[h]}_{k-1}(\mathbb{R}^{r,s}) \qquad \text{if } n \text{ is odd}.$$

[5] One can say that genforms give a *representation* of the set $\mathcal{C}\ell(\mathbb{R}^{r,s})$ (the set of functions with values in the Clifford algebra). This point of view is quite relevant when considering objects of the (pseudo-)Euclidean space $\mathbb{R}^{r,s}$. However, when considering manifolds, it turns out that an appropriate generalization of genforms leads to more natural structures on a manifold compared with Clifford algebras. Therefore, on a manifold, the theory should be constructed starting from the set of genforms rather than from the Clifford algebra.

[6] It is easy to see that these operators d and δ are analogous to the operators of exterior differentiation d and codifferentiation δ in the theory of differential forms (therefore, we keep the same notation d and δ for them).

Note also that

$$ð^2 = (d+\delta)^2 = d\delta + \delta d = \partial^\mu \partial_\mu = \square,$$

where $\square$ is the n-dimensional d'Alembert operator of signature (r,s). For $n \geq 1$, we have

$$ð^n = (d+\delta)^n = d_{(n)} + \delta_{(n)} = \begin{cases} \square^{n/2} & \text{if } n \text{ is even,} \\ \square^{(n-1)/2}(d+\delta) & \text{if } n \text{ is odd.} \end{cases}$$

A remarkable property of the technique of genforms is the fact that in the situations where Clifford multiplication is not needed and only exterior multiplication is used, the technique of genforms is actually indistinguishable from the technique of differential forms (one should formally replace dx^μ by h^μ). Therefore, the technique of genforms can be viewed as a generalization of the technique of differential forms.

For example, we expressed the Maxwell equations in terms of the differential forms (1.7) as equations (1.8). Using the genforms

$$A = a_\mu h^\mu \in \Lambda_1^{[h]}(\mathbb{R}^{r,s}), \qquad J = j_\mu h^\mu \in \Lambda_1^{[h]}(\mathbb{R}^{r,s}), \qquad F = \frac{1}{2} f_{\mu\nu} h^\mu \wedge h^\nu \in \Lambda_2^{[h]}(\mathbb{R}^{r,s}), \tag{3.11}$$

we can express the Maxwell equations in the same form $dA = F$, $\delta F = J$.

If $A, J \in \Lambda_1^{[h]}(\mathbb{R}^{r,s})$ and $F \in \Lambda_2^{[h]}(\mathbb{R}^{r,s}) \oplus \Lambda_0^{[h]}(\mathbb{R}^{r,s})$, then the generalized Maxwell equations (1.12) are expressed in terms of genforms in the same way: $(d+\delta)A = F$, $(d+\delta)F = J$, and $\delta J = 0$.

In addition, note that all what has been said above about genforms in $\Lambda^{[h]}(\mathbb{R}^{r,s})$ is also valid for complexified genforms in $\mathbb{C} \otimes \Lambda^{[h]}(\mathbb{R}^{r,s})$.

4. YANG–MILLS EQUATIONS AND GENFORMS WITH VALUES IN THE LIE ALGEBRA L

For the representation of the Yang–Mills equations, we need genforms with values in the Lie algebra L. Considering the tensor product $L \otimes \Lambda^{[h]}(\mathbb{R}^{r,s})$ of the Lie algebra L and the set of genforms $\Lambda^{[h]}(\mathbb{R}^{r,s})$, we arrive at genforms with values in the Lie algebra L. Genforms of degree k in $L \otimes \Lambda_k^{[h]}(\mathbb{R}^{r,s})$ can be written as

$$A = \frac{1}{k!} a_{\mu_1 \ldots \mu_k} \otimes h^{\mu_1} \wedge \ldots \wedge h^{\mu_k},$$

where $a_{\mu_1 \ldots \mu_k}$ are the components of a covariant skew-symmetric tensor field with values in the Lie algebra L, i.e., $a_{\mu_1 \ldots \mu_k} \colon \mathbb{R}^{r,s} \to L$.

We will use the following linear differential operators $ð, d, \delta \colon L \otimes \Lambda^{[h]}(\mathbb{R}^{r,s}) \to L \otimes \Lambda^{[h]}(\mathbb{R}^{r,s})$:

$$ðA := h^\mu \partial_\mu A, \qquad dA := \pi_{k+1}(ðA), \qquad \delta A := \pi_{k-1}(ðA),$$

where $A \in L \otimes \Lambda_k^{[h]}(\mathbb{R}^{r,s})$, $k = 0, 1, \ldots, n$. These operators have the properties $d + \delta = ð$, $d^2 = 0$, and $\delta^2 = 0$. We will also use the reversal operator $A \to \widetilde{A} = (-1)^{k(k-1)/2} A$ and the conjugation operator

$$A \to \overline{A} = \frac{1}{k!} a^\dagger_{\mu_1 \ldots \mu_k} \otimes h^{\mu_1} \wedge \ldots \wedge h^{\mu_k}, \tag{4.1}$$

where $a^\dagger_{\mu_1 \ldots \mu_k}$ is the Hermitian conjugate matrix (for every fixed value of the multi-index $\mu_1 \ldots \mu_k$) or the Hermitian conjugate element of the Clifford algebra (for the Hermitian conjugation of elements of the Clifford algebra, see [11]). It is obvious that the conjugation operation (4.1) is a generalization of the operation of complex conjugation (3.8) of complexified genforms in $\mathbb{C}\otimes\Lambda^{[h]}(\mathbb{R}^{r,s})$. Recall that for the Hermitian conjugation of matrices (or elements of the Clifford algebra) we have the property $(ab)^\dagger = b^\dagger a^\dagger$.

Let $\rho\in\mathbb{R}$ be a constant, a_μ be the components of a covector field with values in the Lie algebra L, and $f_{\mu\nu} = -f_{\nu\mu}$ be the components of a covariant skew-symmetric tensor field with values in the Lie algebra L. Set

$$A := a_\mu\otimes h^\mu,\qquad F := \frac{1}{2}f_{\mu\nu}\otimes(h^\mu h^\nu).$$

Then the following formulas are valid (where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]$):

$$A^2 = \frac{1}{2}(a_\mu a_\nu - a_\nu a_\mu)\otimes(h^\mu h^\nu) + a^\mu a_\mu\otimes e,$$

$$(\widetilde{A^2}) = -\frac{1}{2}(a_\mu a_\nu - a_\nu a_\mu)\otimes(h^\mu h^\nu) + a^\mu a_\mu\otimes e,\qquad \frac{1}{2}(A^2-(\widetilde{A^2})) = \frac{1}{2}(a_\mu a_\nu - a_\nu a_\mu)\otimes(h^\mu h^\nu),$$

$$\eth A = h^\lambda\partial_\lambda A = (d+\delta)A = \frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu)\otimes(h^\mu h^\nu) + \partial^\mu a_\mu\otimes e,$$

$$dA = \frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu)\otimes(h^\mu h^\nu),\qquad \delta A = \partial^\mu a_\mu\otimes e,$$

$$F = dA - \frac{\rho}{2}(A^2-(\widetilde{A^2})) = \frac{1}{2}\big(\partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu,a_\nu]\big)\otimes(h^\mu h^\nu),$$

$$\eth F = (\partial^\mu f_{\mu\nu})\otimes h^\nu + \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}\partial_\lambda f_{\mu\nu}\otimes(h^\lambda h^\mu h^\nu),$$

$$AF = (a^\mu f_{\mu\nu})\otimes h^\nu + \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}a_\lambda f_{\mu\nu}\otimes(h^\lambda h^\mu h^\nu),$$

$$FA = -(f_{\mu\nu}a^\mu)\otimes h^\nu + \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}f_{\mu\nu}a_\lambda\otimes(h^\mu h^\nu h^\lambda),$$

$$\widetilde{F}A = -(f_{\mu\nu}a^\mu)\otimes h^\nu - \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}f_{\mu\nu}a_\lambda\otimes(h^\mu h^\nu h^\lambda),$$

$$AF+\widetilde{F}A = [a^\mu, f_{\mu\nu}]\otimes h^\nu + \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}[a_\lambda, f_{\mu\nu}]\otimes(h^\lambda h^\mu h^\nu),$$

$$\eth F - \rho(AF+\widetilde{F}A) = \big(\partial^\mu f_{\mu\nu} - \rho[a^\mu, f_{\mu\nu}]\big)\otimes h^\nu + \frac{1}{2}\sum_{\mu\neq\lambda\neq\nu}\big(\partial_\lambda f_{\mu\nu} - \rho[a_\lambda, f_{\mu\nu}]\big)\otimes(h^\lambda h^\mu h^\nu). \tag{4.2}$$

Lemma (on Bianchi's differential identity). *If* $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]$, *then*

$$\sum_{\mu\neq\lambda\neq\nu}\big(\partial_\lambda f_{\mu\nu} - \rho[a_\lambda, f_{\mu\nu}]\big)\otimes(h^\lambda h^\mu h^\nu)\equiv 0. \tag{4.3}$$

Proof. The assertion of the lemma is a consequence of Bianchi's differential identity [18, p. 269]

$$D_\mu f_{\alpha\beta} + D_\alpha f_{\beta\mu} + D_\beta f_{\mu\alpha} = 0,$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]$ and $D_\mu f_{\alpha\beta} = \partial_\mu f_{\alpha\beta} - \rho[a_\mu, f_{\alpha\beta}]$.

By the lemma and formula (4.2), we have

$$\eth F - \rho(AF+\widetilde{F}A) = \big(\partial^\mu f_{\mu\nu} - \rho[a^\mu, f_{\mu\nu}]\big)\otimes h^\nu,$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]$. Therefore, the Yang–Mills equations (2.2) in the technique of genforms with values in the Lie algebra L are represented by the equations

$$dA - \frac{\rho}{2}(A^2 - (\widetilde{A^2})) - F = 0, \qquad \eth F - \rho(AF + \widetilde{F}A) = J, \tag{4.4}$$

where

$$A = a_\mu \otimes h^\mu \in L \otimes \Lambda_1^{[h]}(\mathbb{R}^{r,s}), \qquad J = j_\nu \otimes h^\nu \in L \otimes \Lambda_1^{[h]}(\mathbb{R}^{r,s}),$$
$$F = \frac{1}{2} f_{\mu\nu} \otimes (h^\mu h^\nu) \in L \otimes \Lambda_2^{[h]}(\mathbb{R}^{r,s}).$$

Recall that the Yang–Mills equations (2.2) imply the equality (2.5). If the Lie algebra L is the real Lie algebra of a unitary Lie group (i.e., if the elements of the Lie algebra L are anti-Hermitian matrices $U^\dagger = -U$), then one can verify that a consequence of equations (4.4) is expressed in the technique of genforms as

$$\delta J - \rho\big(AJ - \widetilde{\widetilde{JA}}\big) = 0. \tag{4.5}$$

Note that for the Maxwell equations there are three equivalent expression in the technique of differential forms (genforms), namely, (1.8), (1.10), and (1.11), while for the Yang–Mills equations there is only one expression (4.4) in the technique of genforms with values in Lie algebra.

Generalized Yang–Mills equations. Writing the Maxwell equations in the technique of differential forms (1.8), (1.10), and (1.11) made it possible to see the generalized Maxwell equations (1.12). We have shown that all solutions of the Maxwell equations (in the Lorentz gauge) are contained among the solutions of system (1.12). Moreover, we have shown that system (1.12) subject to the condition $\delta J = 0$ on its right-hand side also has solutions that do not satisfy the Maxwell equations.

Similar arguments lead to the system of equations

$$\eth A - \frac{\rho}{2}(A^2 - (\widetilde{A^2})) - F = 0, \qquad \eth F - \rho(AF + \widetilde{F}A) = J, \tag{4.6}$$

where $A \in L \otimes \Lambda_1^{[h]}(\mathbb{R}^{r,s})$, $F \in L \otimes \big(\Lambda_0^{[h]}(\mathbb{R}^{r,s}) \oplus \Lambda_2^{[h]}(\mathbb{R}^{r,s})\big)$, and $J \in L \otimes \Lambda_1^{[h]}(\mathbb{R}^{r,s})$, which provides a basis for further study.

Formula (4.2) and the previous formulas from the same block, as well as the lemma on Bianchi's differential identity, allow us to write equations (4.6) in more detail as

$$F = \eth A - \frac{\rho}{2}(A^2 - (\widetilde{A^2})) = \frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]) \otimes (h^\mu h^\nu) + b \otimes e, \tag{4.7}$$

$$J = \eth F - \rho(AF + \widetilde{F}A) = \big(\partial^\mu f_{\mu\nu} - \rho[a^\mu, f_{\mu\nu}]\big) \otimes h^\nu + \big(\partial_\nu b - \rho(a_\nu b + b a_\nu)\big) \otimes h^\nu, \tag{4.8}$$

where $A = a_\mu \otimes h^\mu$, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - \rho[a_\mu, a_\nu]$, and $b = \partial^\mu a_\mu$.

If we supplement the system of equations (4.6) with the equation $\delta A = 0$, then, obviously, any solution of the resulting system of equations

$$\eth A - \frac{\rho}{2}(A^2 - (\widetilde{A^2})) - F = 0, \qquad \eth F - \rho(AF + \widetilde{F}A) = J, \qquad \delta A = 0 \tag{4.9}$$

is also a solution of system (4.4), which is equivalent to the system of Yang–Mills equations (2.2).

When considering the system of equations (1.14), which arises in the theory of Maxwell equations, we have proved that in the case of signature $(r, s) = (1, n-1)$ of the pseudo-Euclidean space, the condition $\delta A = 0$ can be relaxed by replacing it with condition (1.15) and the condition $\delta J = 0$.

We conjecture that a similar statement also holds for the condition $\delta A = 0$ from the system of equations (4.9). However, the analysis of this problem has not yet been brought to a rigorous proof.

The system of equations (4.6) with the additional condition $\delta J - \rho(AJ - \widetilde{JA}) = 0$ on the right-hand side can be viewed as a generalization of the Yang–Mills equations. Indeed, if $A, J \in L \otimes \Lambda_1^{[h]}(\mathbb{R}^{r,s})$ is a solution to the Yang–Mills equations (4.4) in the Lorentz gauge $\delta A = 0$, then these A and J are also a solution to equations (4.6). On the other hand, among the solutions of equations (4.6), there are solutions that do not satisfy the Yang–Mills equations (4.4) (this fact has been proved above for the Maxwell equations, which are a particular case of the Yang–Mills equations). We will call equations (4.6) a *generalized system of Yang–Mills equations* (expressed in the technique of genforms).

In future publications, we are going to consider the possibility of using the system of equations (4.6) *instead of* the Yang–Mills equations. We hope that the solutions of the new equations will be given a physical interpretation.

5. RELATION TO THE THEORY OF FRIEDRICHS SYMMETRIC HYPERBOLIC SYSTEMS OF FIRST-ORDER EQUATIONS

In the previous section, we introduced a new class of systems of equations (4.6). Equations in this class (we view them as generalized Yang–Mills equations) depend on the pseudo-Euclidean space $\mathbb{R}^{r,s}$ and the real Lie algebra L. In this connection, a question arises as to whether the new equations are internally consistent. Can one ensure that these equations are neither overdetermined nor underdetermined? How the Cauchy problem is posed for these equations, and is the Cauchy problem well-posed in the sense of Hadamard (a solution exists, is unique, and depends continuously on the initial data, the right-hand sides, and the coefficients of the equations)? In this section, we answer these questions in some particular cases of systems (4.6). Namely, as the real algebra L, we will consider only the real Lie algebra $\mathfrak{u}(1)$ of the unitary abelian Lie group $\mathrm{U}(1)$ of complex numbers with unit absolute value; i.e., we restrict the analysis to the generalized Maxwell equations (1.12). As the pseudo-Euclidean space $\mathbb{R}^{r,s}$, we consider three spaces $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$, and $\mathbb{R}^{1,3}$.

Let us outline the further analysis. We express the Cauchy problem for equations (1.12) as a Cauchy problem for the so-called (*Friedrichs*) *symmetric hyperbolic systems of equations* (SHSEs). The theory of SHSEs (including the questions of well-posedness of the Cauchy problem for SHSEs) was developed by Friedrichs [6], Dezin [4], Godunov [7], Mizohata [13], and others. The application of the theory of SHSEs will allow one to make a conclusion about the consistency of system (1.12) and about the well-posedness of the Cauchy problem for this system of equations.

Let $n \geq 2$ be an integer, $\mathbb{R}^n$ be the Euclidean space with Cartesian coordinates $x^1, \dots, x^n$, and $\Omega \subset \mathbb{R}^n$ be a bounded open domain such that $\Omega \subset \{x \in \mathbb{R}^n \colon x^1 > 0\}$. Consider the Cauchy problem for the system of linear first-order partial differential equations

$$\sum_{i=1}^{n} H_i \partial_i u + Qu = j, \qquad x \in \Omega, \tag{5.1}$$

$$u = \psi, \qquad x \in S. \tag{5.2}$$

Here $H_1, \dots, H_n$ and Q are square $N \times N$ matrices that depend smoothly on $x = (x^1, \dots, x^n)$; $u = u(x)$ is the N-dimensional column vector of unknown functions; $j = j(x)$ is the N-dimensional column vector of functions on the right-hand side; S is an open domain in the plane $x^1 = 0$ that is obtained from the $(n-1)$-dimensional set $\overline{\Omega} \cap \{x^1 = 0\}$ by removing the boundary, which is assumed to be smooth; and $\psi = \psi(x^2, \dots, x^n)$ is the N-dimensional column vector of the initial functions (defined for $x^1 = 0$).

If, for any $x \in \Omega$, the matrices $H_1, \dots, H_n$ are symmetric ($H_i^{\mathrm{T}} = H_i$) and the matrix H_1 is positive definite, then the system of first-order equations (5.1) is called a Friedrichs symmetric x^1-hyperbolic system of equations in the domain Ω.

Suppose that the following conditions hold for the Cauchy problem (5.1), (5.2):

(a) the matrices H_i are symmetric, the matrix H_1 is positive definite, and there exists a constant $\gamma > 0$ such that $(H_1\xi, \xi) > \gamma(\xi, \xi)$ for all nonzero real N-dimensional vectors ξ and all $x \in \Omega$;

(b) for $x^1 > 0$, the domain $\Omega \subset \mathbb{R}^n$ is bounded by a surface $\partial\Omega$ such that the matrix $\sum_{i=1}^n H_i\tau_i$, where $\tau = (\tau_1, \dots, \tau_n)$ is the outward normal vector, is positive definite for all points $x \in \partial\Omega$;[7]

(c) the matrix-valued functions

$$H_i = H_i(x), \qquad Q = Q(x), \qquad j = j(x), \qquad x \in \Omega,$$

are smooth (infinitely differentiable)[8] functions of $x \in \Omega$; the boundary $\partial\Omega$ of Ω for $x^1 > 0$ is smooth; and the vector function $\psi = \psi(\acute{x})$ is a smooth function of $\acute{x} \in S$. The boundary of S is also smooth.

According to the theory of SHSEs, under conditions (a)–(c), there exists a classical (continuously differentiable) solution to the Cauchy problem (5.1), (5.2). Also, there is an a priori estimate according to which the solution of problem (5.1), (5.2) is unique and stable with respect to small variations of the functions $H_i(x)$, $Q(x)$, $j(x)$, and $\psi(\acute{x})$ in the appropriate norm. This precisely means that the Cauchy problem (5.1), (5.2) for SHSEs subject to conditions (a)–(c) is well-posed in the sense of Hadamard.[9]

We will prove the well-posedness of the Cauchy problem for the system of equations (1.12) by reducing it to an equivalent Cauchy problem for an SHSE (thus we can use the above-formulated general result on the well-posedness of the Cauchy problem for SHSEs). In order not to overload the exposition, we assume that the initial data of the Cauchy problem for $x^1 = 0$ are defined by compactly supported functions and that the solution of the Cauchy problem is considered for all $x^1 > 0$.

Consider the case of $n = 2$ and the pseudo-Euclidean space $\mathbb{R}^{1,1}$ with Cartesian coordinates x^1, x^2. The metric tensor is defined by the diagonal matrix $\eta = \mathrm{diag}(1, -1)$. The set of genforms $\Lambda^{[h]}(\mathbb{R}^{1,1})$ is regarded (at every point $x \in \mathbb{R}^{1,1}$) as a four-dimensional vector space with basis e, h^1, h^2, h^{12}, where $h^{12} = h^1h^2$ and

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu}e, \qquad \mu, \nu = 1, 2.$$

Let

$$A = a_1h^1 + a_2h^2 \in \Lambda_1^{[h]}(\mathbb{R}^{1,1}), \qquad J = j_1h^1 + j_2h^2 \in \Lambda_1^{[h]}(\mathbb{R}^{1,1}),$$

$$F = fe + f_{12}h^{12} \in \Lambda_0^{[h]}(\mathbb{R}^{1,1}) \oplus \Lambda_2^{[h]}(\mathbb{R}^{1,1}), \qquad \eth = h^1\partial_1 + h^2\partial_2.$$

We have

$$\eth A = (\partial_1 a_1 - \partial_2 a_2)e + (\partial_1 a_2 - \partial_2 a_1)h^{12}, \tag{5.3}$$

$$\eth F = (\partial_1 f + \partial_2 f_{12})h^1 + (\partial_2 f + \partial_1 f_{12})h^2. \tag{5.4}$$

[7] This means that the domain Ω is a domain of dependence of the solution to system (5.1) on the initial data defined for $x^1 = 0$ in the $(n-1)$-dimensional domain S. The part of the boundary of Ω on which $x^1 > 0$ is called a *Hamilton–Jacobi hat* (see [7]).

[8] The requirement of infinite differentiability of the data appearing in the Cauchy problem (5.1), (5.2) can be relaxed, and this important question is the subject of study in the theory of SHSEs. In the present paper, we do not consider this question.

[9] The existence of a solution to the Cauchy problem was proved in [6, 7] by the finite-difference approximation method and in [13] by the methods of semigroup theory.

Consider the following Cauchy problem for system (1.12):

$$\eth A - F = 0, \qquad \eth F = J \qquad \text{for} \quad x^1 > 0, \tag{5.5}$$

$$A = \dot{A} = \dot{a}_1 h^1 + \dot{a}_2 h^2, \qquad F = \dot{F} = \dot{f} e + \dot{f}_{12} h^{12} \qquad \text{for} \quad x^1 = 0, \tag{5.6}$$

where $\dot{u}_1$, $\dot{u}_2$, $\dot{f}$, and $\dot{f}_{12}$ are given smooth functions of x^2. In view of equalities (5.3) and (5.4), the Cauchy problem (5.5), (5.6) takes the form

$$\partial_1 a_1 - \partial_2 a_2 - f = 0, \qquad \partial_1 f + \partial_2 f_{12} = j_1, \tag{5.7}$$

$$\partial_1 a_2 - \partial_2 a_1 - f_{12} = 0, \qquad \partial_2 f + \partial_1 f_{12} = j_2. \tag{5.8}$$

Introducing a column of unknown functions $u = u(x) = (a_1, a_2, f, f_{12})^{\mathrm{T}}$ and a column of the right-hand side $j = j(x) = (0, 0, j_1, j_2)^{\mathrm{T}}$, we obtain the following Cauchy problem for an SHSE:

$$\begin{aligned} H_1 \partial_1 u + H_2 \partial_2 u + Q u = j \qquad &\text{for} \quad x^1 > 0, \\ u = \dot{u}(x^2) \qquad &\text{for} \quad x^1 = 0, \end{aligned} \tag{5.9}$$

where H_1 is the fourth-order identity matrix,

$$H_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\dot{u} = \dot{u}(x^2) = (\dot{a}_1, \dot{a}_2, \dot{f}, \dot{f}_{12})^{\mathrm{T}}$ is the vector function of initial data.

The well-posedness of the Cauchy problem (5.9) implies the well-posedness of the original Cauchy problem (5.5), (5.6).

Consider the case of $n = 3$ and the pseudo-Euclidean space $\mathbb{R}^{1,2}$ with Cartesian coordinates x^1, x^2, x^3. The metric tensor is defined by the diagonal matrix $\eta = \mathrm{diag}(1, -1, -1)$. The set of genforms $\Lambda^{[h]}(\mathbb{R}^{1,2})$ is regarded (at every point $x \in \mathbb{R}^{1,2}$) as an eight-dimensional vector space with basis $e, h^1, h^2, h^3, h^{12}, h^{13}, h^{23}, h^{123}$, where

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu} e, \qquad \mu, \nu = 1, 2, 3.$$

Let

$$A = a_1 h^1 + a_2 h^2 + a_3 h^3 \in \Lambda_1^{[h]}(\mathbb{R}^{1,2}), \qquad J = j_1 h^1 + j_2 h^2 + j_3 h^3 \in \Lambda_1^{[h]}(\mathbb{R}^{1,2}),$$

$$F = f e + f_{12} h^{12} + f_{13} h^{13} + f_{23} h^{23} \in \Lambda_0^{[h]}(\mathbb{R}^{1,2}) \oplus \Lambda_2^{[h]}(\mathbb{R}^{1,2}), \qquad \eth = h^1 \partial_1 + h^2 \partial_2 + h^3 \partial_3.$$

Our concern is the Cauchy problem

$$\begin{aligned} \eth A - F = 0, \qquad \eth F = J \qquad &\text{for} \quad x^1 > 0, \\ A = \dot{A}, \qquad F = \dot{F} \qquad &\text{for} \quad x^1 = 0, \end{aligned} \tag{5.10}$$

where

$$\dot{A} = \dot{a}_1 h^1 + \dot{a}_2 h^2 + \dot{a}_3 h^3, \qquad \dot{F} = \dot{f} e + \dot{f}_{12} h^{12} + \dot{f}_{13} h^{13} + \dot{f}_{23} h^{23},$$

and $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, $\dot{f}$, $\dot{f}_{12}$, $\dot{f}_{13}$, and $\dot{f}_{23}$ are given smooth functions of x^2 and x^3 that satisfy the condition

$$\partial_2 \dot{a}_3 - \partial_3 \dot{a}_2 - \dot{f}_{23} = 0 \tag{5.11}$$

(below, we will see that this condition follows from the first equation in (5.15)).

We have

$$\eth A = (\partial_1 a_1 - \partial_2 a_2 - \partial_3 a_3)e + (\partial_1 a_2 - \partial_2 a_1)h^{12} + (\partial_1 a_3 - \partial_3 a_1)h^{13} + (\partial_2 a_3 - \partial_3 a_2)h^{23},$$

$$\eth F = (\partial_1 f + \partial_2 f_{12} + \partial_3 f_{13})h^1 + (\partial_1 f_{12} + \partial_2 f + \partial_3 f_{23})h^2 + (\partial_1 f_{13} - \partial_2 f_{23} + \partial_3 f)h^3 + (\partial_1 f_{23} - \partial_2 f_{13} + \partial_3 f_{12})h^{123}.$$

Let us write the system of equations $\eth A - F = 0$, $\eth F = J$ componentwise:

$$\partial_1 a_1 - \partial_2 a_2 - \partial_3 a_3 - f = 0, \qquad \partial_1 f + \partial_2 f_{12} + \partial_3 f_{13} = j_1, \tag{5.12}$$

$$\partial_1 a_2 - \partial_2 a_1 - f_{12} = 0, \qquad \partial_1 f_{12} + \partial_2 f + \partial_3 f_{23} = j_2, \tag{5.13}$$

$$\partial_1 a_3 - \partial_3 a_1 - f_{13} = 0, \qquad \partial_1 f_{13} - \partial_2 f_{23} + \partial_3 f = j_3, \tag{5.14}$$

$$\partial_2 a_3 - \partial_3 a_2 - f_{23} = 0, \qquad \partial_1 f_{23} - \partial_2 f_{13} + \partial_3 f_{12} = 0. \tag{5.15}$$

As a result, we have obtained a system of eight equations in seven unknowns a_1, a_2, a_3, f, f_{12}, f_{13}, and f_{23}. However, this system is not overdetermined. Indeed, equations (5.13)–(5.15) are dependent, because the last equation of the system is a consequence of the first equations in (5.13)–(5.15). Therefore, we could just discard the second equation in (5.15) to obtain a system of seven equations in seven unknowns. But this variant does not suit us, because the resulting system of seven equations is not an SHSE. This easily follows from the fact that since the matrix H_1 is positive definite, each equation of the SHSE must contain a term with the partial derivative ∂_1, whereas the first equation in (5.15) does not contain a term with such a derivative.

Our proposal is to obtain an SHSE from system (5.12)–(5.15) by discarding the first equation in (5.15). As a result, we obtain the following Cauchy problem for an SHSE:

$$\begin{aligned} H_1\partial_1 u + H_2\partial_2 u + H_3\partial_3 u + Qu &= j \qquad \text{for} \quad x^1 > 0,\\ u &= \dot u \qquad \text{for} \quad x^1 = 0, \end{aligned} \tag{5.16}$$

where H_1 is the seventh-order identity matrix,

$$H_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and $\dot u = \dot u(x^2, x^3) = (\dot a_1, \dot a_2, \dot a_3, \dot f, \dot f_{12}, \dot f_{13}, \dot f_{23})^{\mathrm T}$ is the initial vector function. Although we have dropped the first equation in (5.15), we require that the initial data of the Cauchy problem (5.16) should satisfy condition (5.11).

Let us show that the resulting Cauchy problem (5.16) is equivalent to the Cauchy problem for the system of equations (5.12)–(5.15) with the initial data $u = \dot u$ satisfying condition (5.11). Indeed, let us substitute f_{12} and f_{13} from the first equations in (5.13) and (5.14) into the second equation in (5.15). For $k \equiv \partial_2 a_3 - \partial_3 a_2 - f_{23}$, we obtain (taking account of condition (5.11)) the Cauchy problem

$$\begin{aligned} \partial_1 k &= 0 \qquad \text{for} \quad x^1 > 0,\\ k &= 0 \qquad \text{for} \quad x^1 = 0. \end{aligned}$$

Hence, $k = 0$ for $x^1 > 0$.

The equivalence of the Cauchy problems (5.16) and (5.10) is proved; hence, the well-posedness of the Cauchy problem (5.10) follows from the theory of SHSEs.

Consider the case of $n = 4$ and the pseudo-Euclidean space $\mathbb{R}^{1,3}$ (Minkowski space) with Cartesian coordinates x^1, x^2, x^3, x^4. The metric tensor is defined by the diagonal matrix $\eta = \operatorname{diag}(1, -1, -1, -1)$. The set of genforms $\Lambda^{[h]}(\mathbb{R}^{1,3})$ is regarded (at every point $x \in \mathbb{R}^{1,3}$) as a 16-dimensional vector space with basis

$$e, h^1, h^2, h^3, h^4, h^{12}, h^{13}, h^{14}, h^{23}, h^{24}, h^{34}, h^{123}, h^{124}, h^{134}, h^{234}, h^{1234}, \tag{5.17}$$

where

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu} e, \quad \mu, \nu = 1, 2, 3, 4, \qquad h^{\mu_1 \dots \mu_k} = h^{\mu_1} \dots h^{\mu_k}, \quad \mu_1 < \dots < \mu_k.$$

Let

$$A = a_1 h^1 + a_2 h^2 + a_3 h^3 + a_4 h^4 \in \Lambda_1^{[h]}(\mathbb{R}^{1,3}), \qquad J = j_1 h^1 + j_2 h^2 + j_3 h^3 + j_4 h^4 \in \Lambda_1^{[h]}(\mathbb{R}^{1,3}),$$
$$F = f e + f_{12} h^{12} + f_{13} h^{13} + f_{14} h^{14} + f_{23} h^{23} + f_{24} h^{24} + f_{34} h^{34} \in \Lambda_0^{[h]}(\mathbb{R}^{1,3}) \oplus \Lambda_2^{[h]}(\mathbb{R}^{1,3}),$$
$$\eth = h^1 \partial_1 + h^2 \partial_2 + h^3 \partial_3 + h^4 \partial_4.$$

Our concern is the Cauchy problem

$$\begin{aligned} \eth A - F = 0, \qquad \eth F = J \qquad &\text{for} \quad x^1 > 0, \\ A = \dot{A}, \qquad F = \dot{F} \qquad &\text{for} \quad x^1 = 0, \end{aligned} \tag{5.18}$$

where

$$\dot{A} = \dot{a}_1 h^1 + \dot{a}_2 h^2 + \dot{a}_3 h^3 + \dot{a}_4 h^4,$$
$$\dot{F} = \dot{f} e + \dot{f}_{12} h^{12} + \dot{f}_{13} h^{13} + \dot{f}_{14} h^{14} + \dot{f}_{23} h^{23} + \dot{f}_{24} h^{24} + \dot{f}_{34} h^{34},$$

and $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, $\dot{a}_4$, $\dot{f}$, $\dot{f}_{12}$, $\dot{f}_{13}$, $\dot{f}_{14}$, $\dot{f}_{23}$, $\dot{f}_{24}$, and $\dot{f}_{34}$ are given smooth real functions of x^2, x^3, and x^4 satisfying the conditions

$$\partial_2 \dot{a}_3 - \partial_3 \dot{a}_2 - \dot{f}_{23} = 0, \qquad \partial_2 \dot{a}_4 - \partial_4 \dot{a}_2 - \dot{f}_{24} = 0, \qquad \partial_3 \dot{a}_4 - \partial_4 \dot{a}_3 - \dot{f}_{34} = 0, \tag{5.19}$$

$$\partial_2 \dot{f}_{34} - \partial_3 \dot{f}_{24} + \partial_4 \dot{f}_{23} = 0 \tag{5.20}$$

(below we will see that these conditions follow from equations (5.25)–(5.28)).

A direct calculation yields

$$\begin{aligned} \eth A &= (\partial_1 a_1 - \partial_2 a_2 - \partial_3 a_3 - \partial_4 a_4) e + \sum_{1 \le \mu < \nu \le 4} (\partial_\mu a_\nu - \partial_\nu a_\mu) h^{\mu\nu}, \\ \eth F &= (\partial_1 f + \partial_2 f_{12} + \partial_3 f_{13} + \partial_4 f_{14}) h^1 + (\partial_1 f_{12} + \partial_2 f + \partial_3 f_{23} + \partial_4 f_{24}) h^2 \\ &\quad + (\partial_1 f_{13} - \partial_2 f_{23} + \partial_3 f + \partial_4 f_{34}) h^3 + (\partial_1 f_{14} - \partial_2 f_{24} - \partial_3 f_{34} + \partial_4 f) h^4 \\ &\quad + (\partial_1 f_{23} - \partial_2 f_{13} + \partial_3 f_{12}) h^{123} + (\partial_1 f_{24} - \partial_2 f_{14} + \partial_4 f_{12}) h^{124} \\ &\quad + (\partial_1 f_{34} - \partial_3 f_{14} + \partial_4 f_{13}) h^{134} + (\partial_2 f_{34} - \partial_3 f_{24} + \partial_4 f_{23}) h^{234}. \end{aligned}$$

We write the system of equations $\eth A - F = 0$, $\eth F = J$ componentwise with respect to the basis (5.17):

$$\partial_1 a_1 - \partial_2 a_2 - \partial_3 a_3 - \partial_4 a_4 - f = 0, \qquad \partial_1 f + \partial_2 f_{12} + \partial_3 f_{13} + \partial_4 f_{14} = j_1, \tag{5.21}$$

$$\partial_1 a_2 - \partial_2 a_1 - f_{12} = 0, \qquad \partial_1 f_{12} + \partial_2 f + \partial_3 f_{23} + \partial_4 f_{24} = j_2, \tag{5.22}$$

$$\partial_1 a_3 - \partial_3 a_1 - f_{13} = 0, \qquad \partial_1 f_{13} - \partial_2 f_{23} + \partial_3 f + \partial_4 f_{34} = j_3, \tag{5.23}$$

$$\partial_1 a_4 - \partial_4 a_1 - f_{14} = 0, \qquad \partial_1 f_{14} - \partial_2 f_{24} - \partial_3 f_{34} + \partial_4 f = j_4, \tag{5.24}$$

$$\partial_2 a_3 - \partial_3 a_2 - f_{23} = 0, \qquad \partial_1 f_{23} - \partial_2 f_{13} + \partial_3 f_{12} = 0, \tag{5.25}$$

$$\partial_2 a_4 - \partial_4 a_2 - f_{24} = 0, \qquad \partial_1 f_{24} - \partial_2 f_{14} + \partial_4 f_{12} = 0, \tag{5.26}$$

$$\partial_3 a_4 - \partial_4 a_3 - f_{34} = 0, \qquad \partial_1 f_{34} - \partial_3 f_{14} + \partial_4 f_{13} = 0, \tag{5.27}$$

$$\partial_2 f_{34} - \partial_3 f_{24} + \partial_4 f_{23} = 0. \tag{5.28}$$

As a result, we have obtained a system of 15 equations in 11 unknown functions a_1, a_2, a_3, a_4, f, f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, and f_{34}. In this system of equations, only 11 equations are independent. To get them from system (5.21)–(5.28), we suggest dropping the first equations in (5.25)–(5.27) and equation (5.28) (note that equation (5.28) is a consequence of the first equations in (5.25)–(5.27)). As a result, we obtain the following Cauchy problem for an SHSE:

$$\begin{aligned} H_1\partial_1 u + H_2\partial_2 u + H_3\partial_3 u + H_4\partial_4 u + Qu &= j \qquad \text{for} \quad x^1 > 0, \\ u &= \dot{u} \qquad \text{for} \quad x^1 = 0, \end{aligned} \tag{5.29}$$

where H_1 is the 11th-order identity matrix and H_i, $i = 2,3,4$, are block-diagonal matrices with two blocks H_i' and H_i'' of the fourth and seventh orders, respectively, on the diagonal. These blocks have the form

$$H_2' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_3' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_4' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$H_2'' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_3'' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$H_4'' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\dot{u} = \dot{u}(x^2, x^3, x^4) = (\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4, \dot{f}, \dot{f}_{12}, \dot{f}_{13}, \dot{f}_{14}, \dot{f}_{23}, \dot{f}_{24}, \dot{f}_{34})^{\mathrm{T}}$ is the initial vector function satisfying conditions (5.19) and (5.20).

Let us show that the resulting Cauchy problem (5.29) is equivalent to the Cauchy problem for the system of equations (5.21)–(5.28) with the initial data $u = \dot{u}$ satisfying conditions (5.19) and (5.20). Indeed, let us substitute f_{12}, f_{13}, and f_{14} from the first equations in (5.22)–(5.24) into the second equations in (5.25)–(5.27). As a result, we obtain the following Cauchy problem for $k_{23} \equiv \partial_2 a_3 - \partial_3 a_2 - f_{23}$, $k_{24} \equiv \partial_2 a_4 - \partial_4 a_2 - f_{24}$, and $k_{34} \equiv \partial_3 a_4 - \partial_4 a_3 - f_{34}$ (in view of conditions (5.19) and (5.20)):

$$\partial_1 k_{23} = 0, \qquad \partial_1 k_{24} = 0, \qquad \partial_1 k_{34} = 0 \qquad \text{for} \quad x^1 > 0,$$
$$k_{23} = k_{24} = k_{34} = 0 \qquad \text{for} \quad x^1 = 0.$$

Hence, $k_{23} = k_{24} = k_{34} = 0$ for $x^1 > 0$.

The equivalence of the Cauchy problems (5.29) and (5.18) is proved; hence, the well-posedness of the Cauchy problem (5.18) follows from the theory of SHSEs.

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