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# Analysis in Noncommutative Algebras and Modules

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**Abstract**—In a previous paper, we developed an analysis in associative commutative algebras and in modules over them, which may be useful in problems of contemporary mathematical and theoretical physics. Here we work out similar methods in the noncommutative case.

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## 1. INTRODUCTION

In [15] we presented an analysis in the associative commutative algebras and in modules over them, which is adapted to the problems of contemporary mathematical and theoretical physics. Here we develop a similar analysis in the noncommutative case. We keep the exposition as close as possible to that in [15] in order to demonstrate the differences and similarities between the commutative and noncommutative cases and to make the paper self-contained.

All linear operations are considered over the number field $\mathbb{F} = \mathbb{R}, \mathbb{C}$. As a rule, we assume summation over repeated upper and lower indices. If objects under study have natural topologies, then we assume that the corresponding mappings are continuous (for example, if $\mathcal{S}$ and $\mathcal{S}'$ are topological spaces, then $\mathrm{Hom}(\mathcal{S};\mathcal{S}')$ is the set of all continuous mappings from $\mathcal{S}$ to $\mathcal{S}'$).

We freely use the general notation and definitions introduced in [15].

## 2. PRELIMINARIES

In this paper by an algebra we will mean an *associative noncommutative algebra over the number field* $\mathbb{F}$, and by a module, a *left module over an algebra*, unless explicitly stated otherwise.

Let $\mathcal{A}$ be an algebra. We will use the following notation:

- $\widehat{\mathcal{A}} = \mathcal{A} \oplus_{\mathbb{F}} \mathbb{F}$, the *unital extension* of $\mathcal{A}$; in particular, $\widehat{\mathcal{A}} = \mathcal{A}$ if the algebra $\mathcal{A}$ is *unital*, i.e., if it contains a unit element;
- $\operatorname{cen}\mathcal{A} = \{A \in \mathcal{A} \mid [A,B] = 0 \text{ for all } B \in \mathcal{A}\}$, the *center* of $\mathcal{A}$;
- $\operatorname{ann}\mathcal{A} = \{A \in \mathcal{A} \mid A\cdot B = B\cdot A = 0 \text{ for all } B \in \mathcal{A}\}$, the *annihilator* of $\mathcal{A}$ (here and below, the dot is the multiplication sign).

Clearly,

- $\operatorname{cen}\mathcal{A}$ is a commutative subalgebra of $\mathcal{A}$; in particular, $\operatorname{cen}\mathcal{A} = \mathcal{A}$ if the algebra $\mathcal{A}$ is commutative;
- $\operatorname{ann}\mathcal{A}$ is an ideal (i.e., a two-sided ideal) of $\mathcal{A}$; in particular, $\operatorname{ann}\mathcal{A} = 0$ if the algebra $\mathcal{A}$ is unital.

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Let $\mathcal{M}$ be an $\widehat{\mathcal{A}}$-module (i.e., a left $\widehat{\mathcal{A}}$-module).[1] Then

- $\operatorname{ann}_{\mathcal{A}} \mathcal{S} = \{A \in \mathcal{A} \mid A \cdot \mathcal{S} = 0\} = \{A \in \mathcal{A} \mid A \cdot M = 0 \text{ for all } M \in \mathcal{S}\}$, the *annihilator of a subset* $\mathcal{S} \subset \mathcal{M}$;
- $\operatorname{ann}_{\mathcal{M}} \mathcal{E} = \{M \in \mathcal{M} \mid \mathcal{E} \cdot M = 0\} = \{M \in \mathcal{M} \mid A \cdot M = 0 \text{ for all } A \in \mathcal{E}\}$, the *annihilator of a subset* $\mathcal{E} \subset \mathcal{A}$.

Clearly,

- $\operatorname{ann}_{\mathcal{A}} \mathcal{S}$ is a left ideal of $\mathcal{A}$;
- $\operatorname{ann}_{\mathcal{M}} \mathcal{E}$ is a submodule of the $\mathcal{A}$-module $\mathcal{M}$ if $\mathcal{E} \cdot \mathcal{A} \subset \mathcal{E}$ (i.e., if $A \cdot B \in \mathcal{E}$ for all $A \in \mathcal{E}$ and $B \in \mathcal{A}$);
- $\operatorname{ann}_{\mathcal{M}} \mathcal{E} \subset \operatorname{ann}_{\mathcal{M}}(\mathcal{A} \cdot \mathcal{E})$; in particular, $\operatorname{ann}_{\mathcal{M}} \mathcal{E} = \operatorname{ann}_{\mathcal{M}}(\mathcal{A} \cdot \mathcal{E})$ if the algebra $\mathcal{A}$ is unital.

Let $\mathcal{M}$ and $\mathcal{M}'$ be $\widehat{\mathcal{A}}$-modules. Then

- $\operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \mathcal{M}')$ is the $\operatorname{cen} \widehat{\mathcal{A}}$-module of all $\widehat{\mathcal{A}}$-linear mappings from $\mathcal{M}$ to $\mathcal{M}'$;
- $\operatorname{End}_{\widehat{\mathcal{A}}}(\mathcal{M}) = \operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \mathcal{M})$ is the $\operatorname{cen} \widehat{\mathcal{A}}$-algebra of all $\widehat{\mathcal{A}}$-linear mappings over $\mathcal{M}$.

## 3. MULTIPLIERS IN ALGEBRAS AND MODULES

**3.1. Multipliers in algebras.** Let $\mathcal{A}$ be an algebra.

**Definition 1.** A linear mapping $R \in \operatorname{End}_{\mathbb{F}}(\mathcal{A})$ is called a *multiplier of the algebra* $\mathcal{A}$ if the following *multiplier rule* holds:

$$R(A \cdot B) = RA \cdot B = A \cdot RB \qquad \text{for all} \quad A, B \in \mathcal{A}.$$

Let $\mathfrak{M}(\mathcal{A})$ be the set of all multipliers of an algebra $\mathcal{A}$.

**Proposition 1.** *The following statements hold*:

- *the set* $\mathfrak{M}(\mathcal{A})$ *is a unital subalgebra of the algebra* $\operatorname{End}_{\mathbb{F}}(\mathcal{A})$;
- *the exact sequence of algebras*

$$0 \to \operatorname{ann} \mathcal{A} \to \operatorname{cen} \widehat{\mathcal{A}} \xrightarrow{\operatorname{ad}} \mathfrak{M}(\mathcal{A})$$

  *is defined, where the adjoint action* $\operatorname{ad} \in \operatorname{Hom}_{\mathrm{alg}}(\operatorname{cen} \widehat{\mathcal{A}}; \mathfrak{M}(\mathcal{A}))$ *is specified by the rule* $C \mapsto \operatorname{ad}_C \colon \mathcal{A} \to \mathcal{A}$, $\operatorname{ad}_C A = C \cdot A$;
- *the algebra* $\mathfrak{M}(\mathcal{A})$ *is a* $\operatorname{cen} \widehat{\mathcal{A}}$*-algebra*;
- *for any* $R \in \mathfrak{M}(\mathcal{A})$ *the kernel* $\operatorname{Ker} R = \{A \in \mathcal{A} \mid RA = 0\}$ *and the image* $\operatorname{Im} R = \{A = RB \mid B \in \mathcal{A}\}$ *are ideals* (*i.e., two-sided ideals*) *of the algebra* $\mathcal{A}$;
- *for any* $R \in \mathfrak{M}(\mathcal{A})$ *the image* $R(\operatorname{ann} \mathcal{A})$ *is contained in* $\operatorname{ann} \mathcal{A}$ *and the image* $R(\operatorname{cen} \mathcal{A})$ *is contained in* $\operatorname{cen} \mathcal{A}$;
- *the commutator* $[R', R'']$ *belongs to* $\operatorname{Hom}_{\mathbb{F}}(\mathcal{A}; \operatorname{ann} \mathcal{A})$ *for all* $R', R'' \in \mathfrak{M}(\mathcal{A})$; *in particular, the algebra* $\mathfrak{M}(\mathcal{A})$ *is commutative if* $\operatorname{ann} \mathcal{A} = 0$ (*for example, if the algebra* $\mathcal{A}$ *is unital*).

**3.2. Multipliers in modules.** Let $\mathcal{A}$ be an algebra, and let $\mathcal{M}$ be an $\widehat{\mathcal{A}}$-module.

**Definition 2.** A pair $\mathbf{R} = (\Delta_R, R) \in \operatorname{End}_{\mathbb{F}}(\mathcal{M}) \times \mathfrak{M}(\mathcal{A})$ is called a *multiplier of the* $\widehat{\mathcal{A}}$*-module* $\mathcal{M}$ if the following *multiplier rule* holds:

$$\Delta_R(A \cdot M) = RA \cdot M = A \cdot \Delta_R M \qquad \text{for all} \quad A \in \mathcal{A}, \quad M \in \mathcal{M}.$$

Let $\mathfrak{M}(\mathcal{M})$ be the set of all multipliers of the $\widehat{\mathcal{A}}$-module $\mathcal{M}$.

[1] Note that by definition the set of all $\widehat{\mathcal{A}}$-modules coincides with the set of all $\mathcal{A}$-modules, but the first notation emphasizes the fact that the algebra $\mathcal{A}$ may be non-unital.

The direct product $\operatorname{End}_{\mathbb{F}}(\mathcal{M}) \times \mathfrak{M}(\mathcal{A})$ has the structure of a unital algebra with the componentwise operations:

- $\lambda'\mathbf{R}' + \lambda''\mathbf{R}'' = (\lambda'\Delta_{R'} + \lambda''\Delta_{R''}, \lambda' R' + \lambda'' R'')$ for $\lambda', \lambda'' \in \mathbb{F}$, and
- $\mathbf{R}' \circ \mathbf{R}'' = (\Delta_{R'} \circ \Delta_{R''}, R' \circ R'')$

for all $\mathbf{R}' = (\Delta_{R'}, R'), \mathbf{R}'' = (\Delta_{R''}, R'') \in \operatorname{End}_{\mathbb{F}}(\mathcal{M}) \times \mathfrak{M}(\mathcal{A})$ (here and below, the small circle is the composition sign). The unit element of this algebra is $\mathbf{id}_{\mathfrak{M}(\mathcal{M})} = (\mathrm{id}_{\mathcal{M}}, \mathrm{id}_{\mathcal{A}})$.

**Proposition 2.** *The following statements hold*:

- *the set $\mathfrak{M}(\mathcal{M})$ is a unital subalgebra of the algebra $\operatorname{End}_{\mathbb{F}}(\mathcal{M}) \times \mathfrak{M}(\mathcal{A})$;*
- *for any $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$ the kernel $\operatorname{Ker}\Delta_R = \{M \in \mathcal{M} \mid \Delta_R M = 0\}$ and the image $\operatorname{Im}\Delta_R = \{M = \Delta_R N \mid N \in \mathcal{M}\}$ are submodules of the $\widehat{\mathcal{A}}$-module $\mathcal{M}$;*
- *for any $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$ the image $\operatorname{Im}\Delta_R(\operatorname{ann}_{\mathcal{M}}\mathcal{A})$ is contained in $\operatorname{ann}_{\mathcal{M}}\mathcal{A}$ and the image $\operatorname{Im}R(\operatorname{ann}_{\mathcal{A}}\mathcal{M})$ is contained in $\operatorname{ann}_{\mathcal{A}}\mathcal{M}$;*
- *for any $\mathbf{R}' = (\Delta_{R'}, R'), \mathbf{R}''(\Delta_{R''}, R'') \in \mathfrak{M}(\mathcal{M})$ the image $[\Delta_{R'}, \Delta_{R''}](\mathcal{A}\cdot\mathcal{M})$ is zero, so the commutator $[\Delta_{R'}, \Delta_{R''}]$ belongs to $\operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$; in particular, the algebra $\mathfrak{M}(\mathcal{M})$ is commutative if the algebra $\mathcal{A}$ is unital (here $\mathcal{A}\cdot\mathcal{M} = \{A\cdot M \mid A \in \mathcal{A},\ M \in \mathcal{M}\} \subset \mathcal{M}$).*

**Proposition 3.** *The adjoint action defines the algebra morphism*

$$\mathbf{ad}\colon\ \operatorname{cen}\widehat{\mathcal{A}} \to \mathfrak{M}(\mathcal{M}), \qquad C \mapsto \mathbf{ad}_C = (\mathrm{ad}_C, \mathrm{ad}_C),$$

*where $\mathrm{ad}_C M = C\cdot M$ and $\mathrm{ad}_C A = C\cdot A$ for all $C, A \in \mathcal{A}$ and $M \in \mathcal{M}$. The kernel $\operatorname{Ker}\mathbf{ad}$ coincides with $\operatorname{ann}_{\operatorname{cen}\mathcal{A}}\mathcal{M}$. In other words, the exact sequence of algebras*

$$0 \to \operatorname{ann}_{\operatorname{cen}\mathcal{A}}\mathcal{M} \to \operatorname{cen}\widehat{\mathcal{A}} \xrightarrow{\mathbf{ad}} \mathfrak{M}(\mathcal{M})$$

*is defined. In particular, $\mathfrak{M}(\mathcal{M})$ is a $\operatorname{cen}\widehat{\mathcal{A}}$-algebra.*

**Proposition 4.** *Let $\mathcal{M}$ be a free $\widehat{\mathcal{A}}$-module with an $\widehat{\mathcal{A}}$-basis $\mathbf{b} = \{b_i \in \mathcal{M} \mid i \in I\}$ indexed by a set $I$, so that $\mathcal{M} = \{M = A^i\cdot b_i \mid A^i \in \widehat{\mathcal{A}}\}$. Then an algebra injection $\mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{M})$ is defined by the componentwise action,*

$$\mathfrak{M}(\mathcal{A}) \ni R \mapsto \mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M}), \qquad \Delta_R(A^i\cdot b_i) = RA^i\cdot b_i.$$

There is another way to treat multipliers of an $\widehat{\mathcal{A}}$-module $\mathcal{M}$. Namely, consider the mapping

$$\pi \in \operatorname{Hom}_{\operatorname{cen}\widehat{\mathcal{A}}}(\mathfrak{M}(\mathcal{M}); \mathfrak{M}(\mathcal{A}))\colon \qquad \mathbf{R} = (\Delta_R, R) \mapsto R.$$

**Proposition 5.** *The set $\mathfrak{M}_\pi(\mathcal{A}) = \operatorname{Im}\pi = \{R \in \mathfrak{M}(\mathcal{A}) \mid \pi^{-1}(R) \neq \varnothing\}$ is a subalgebra of the $\operatorname{cen}\widehat{\mathcal{A}}$-algebra $\mathfrak{M}(\mathcal{A})$.*

**Theorem 1.** *For every $R \in \mathfrak{M}_\pi(\mathcal{A})$ the inverse image $\pi^{-1}(R)$ is a $\operatorname{cen}\widehat{\mathcal{A}}$-affine space over the $\operatorname{cen}\widehat{\mathcal{A}}$-module $\operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$. In particular, the triple $\pi\colon \mathfrak{M}(\mathcal{M}) \to \mathfrak{M}(\mathcal{A})$ is a $\operatorname{cen}\widehat{\mathcal{A}}$-affine fiber bundle over the $\operatorname{cen}\widehat{\mathcal{A}}$-module $\operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$.*

**Proof.** Indeed, let $\mathbf{R}' = (\Delta'_R, R), \mathbf{R}'' = (\Delta''_R, R) \in \pi^{-1}(R)$. Then for the difference $\Delta'_R - \Delta''_R$ we have

$$(\Delta'_R - \Delta''_R)(A\cdot M) = (R - R)A\cdot M = 0 = A\cdot(\Delta'_R - \Delta''_R)M$$

for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$, i.e., $\Delta'_R - \Delta''_R \in \operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$. On the other hand, let $\mathbf{R} = (\Delta_R, R) \in \pi^{-1}(R)$ and $\rho \in \operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$. Then

$$(\Delta_R + \rho)(A\cdot M) = \Delta_R(A\cdot M) + \rho(A\cdot M) = RA\cdot M + A\cdot\rho M = RA\cdot M + 0 = RA\cdot M$$

for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$; i.e., $\mathbf{R}' = (\Delta_R + \rho, R) \in \pi^{-1}(R)$. $\square$

Consider the set of all $*$-*sections* of the bundle $\pi$,

$$\mathcal{S}_*(\pi) = \bigl\{\boldsymbol{\Delta} \in \mathrm{Hom}_*(\mathfrak{M}_\pi(\mathcal{A}); \mathfrak{M}(\mathcal{M})) \bigm| \pi \circ \boldsymbol{\Delta} = \mathrm{id}_{\mathfrak{M}(\mathcal{A})}\bigr\}, \qquad * = \mathbb{F}, \mathrm{cen}\,\widehat{\mathcal{A}}.$$

The set $\mathcal{S}_{\mathbb{F}}(\pi)$ has the structure of a linear space, and the set $\mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\pi)$ has the structure of a $\mathrm{cen}\,\widehat{\mathcal{A}}$-module. We will call the elements of the $\mathrm{cen}\,\widehat{\mathcal{A}}$-module $\mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\pi)$ *covariant multipliers.*

**Definition 3.** The *curvature* $F(\boldsymbol{\Delta}) \in \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}\bigl(\bigotimes^2_{\mathrm{cen}\,\widehat{\mathcal{A}}} \mathfrak{M}_\pi(\mathcal{A}); \mathrm{End}_{\widehat{\mathcal{A}}}(\mathcal{M})\bigr)$ of a covariant multiplier $\boldsymbol{\Delta} \in \mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\pi)$ is defined by the *residual rule*

$$\boldsymbol{\Delta}_R \circ \boldsymbol{\Delta}_S - \boldsymbol{\Delta}_{R\circ S} = (F(\boldsymbol{\Delta})_{R,S}, 0) \qquad \text{for all} \quad R, S \in \mathfrak{M}_\pi(\mathcal{A}),$$

where

$$F(\boldsymbol{\Delta})_{R,S} = \Delta_R \circ \Delta_S - \Delta_{R\circ S}.$$

Every algebra $\mathcal{A}$ has the adjoint structure of an $\widehat{\mathcal{A}}$-module defined as $\widehat{\mathcal{A}} \times \mathcal{A} \ni (A, M) \mapsto A \cdot M$. We denote the resulting $\widehat{\mathcal{A}}$-module by $\boldsymbol{\mathcal{A}}$.

**Proposition 6.** *Let $\boldsymbol{\mathcal{A}}$ be the adjoint $\widehat{\mathcal{A}}$-module of an algebra $\mathcal{A}$. Then*

- *there is a natural algebra injection* $\iota_{\mathfrak{M}}\colon \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\boldsymbol{\mathcal{A}})$, $R \mapsto \mathbf{R} = (R, R)$;
- *the curvature* $F(\iota_{\mathfrak{M}})$ *is zero*;
- $\mathfrak{M}(\boldsymbol{\mathcal{A}}) = \mathrm{Im}\,\iota_{\mathfrak{M}} \oplus \mathrm{Hom}_{\widehat{\mathcal{A}}}(\boldsymbol{\mathcal{A}}; \mathrm{ann}\,\mathcal{A})$, $\mathbf{R} = (\Delta_R, R) = (R, R) + (\Delta_R - R, 0)$.

## 4. GAUGE TRANSFORM OF MULTIPLIERS

Let $\mathcal{A}$ be an algebra, and let $\mathcal{M}$ be an $\widehat{\mathcal{A}}$-module. Let $\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ be the group of all automorphisms of the $\widehat{\mathcal{A}}$-module $\mathcal{M}$.

**Definition 4.** Every automorphism $G \in \mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ defines the following *gauge transform* on the $\mathrm{cen}\,\widehat{\mathcal{A}}$-algebra $\mathfrak{M}(\mathcal{M})$:

$$\mathbf{R} = (\Delta_R, R) \mapsto \mathrm{ad}_G \mathbf{R} = (\mathrm{ad}_G \Delta_R, R), \qquad \mathrm{ad}_G \Delta_R = G \circ \Delta_R \circ G^{-1}.$$

**Theorem 2.** *The gauge transform on the* $\mathrm{cen}\,\widehat{\mathcal{A}}$*-algebra* $\mathfrak{M}(\mathcal{M})$ *defines the action*

$$\mathrm{ad}\colon\ \mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M}) \to \mathrm{End}_{\text{alg-}\mathrm{cen}\,\widehat{\mathcal{A}}}(\mathfrak{M}(\mathcal{M})), \qquad G \mapsto \mathrm{ad}_G.$$

**Proof.** Indeed, let $G \in \mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$. Then

$$\mathrm{ad}_G \Delta_R(A \cdot M) = (G \circ \Delta_R \circ G^{-1})(A \cdot M) = A \cdot ((G \circ \Delta_R \circ G^{-1})M) = A \cdot \mathrm{ad}_G \Delta_R M$$

for all $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$, $A \in \mathcal{A}$, and $M \in \mathcal{M}$. Further,

$$\begin{aligned}\mathrm{ad}_G \Delta_R(A \cdot M) &= (G \circ \Delta_R \circ G^{-1})(A \cdot M) = (G \circ \Delta_R)(A \cdot G^{-1}M)\\ &= G(RA \cdot G^{-1}M) = RA \cdot (G \circ G^{-1}M) = RA \cdot M\end{aligned}$$

for all $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$, $A \in \mathcal{A}$, and $M \in \mathcal{M}$. Next,

$$\begin{aligned}\mathrm{ad}_G(\mathbf{R}' \circ \mathbf{R}'') &= \bigl(G \circ (\Delta_{R'} \circ \Delta_{R''}) \circ G^{-1}, R' \circ R''\bigr)\\ &= \bigl((G \circ \Delta_{R'} \circ G^{-1}) \circ (G \circ \Delta_{R''} \circ G^{-1}), R' \circ R''\bigr) = \mathrm{ad}_G \mathbf{R}' \circ \mathrm{ad}_G \mathbf{R}''\end{aligned}$$

for all $\mathbf{R}' = (\Delta_{R'}, R'), \mathbf{R}'' = (\Delta_{R''}, R'') \in \mathfrak{M}(\mathcal{M})$. Finally,

$$\mathrm{ad}_G(C \circ \Delta_R) = G \circ C \circ \Delta_R \circ G^{-1} = C \circ (G \circ \Delta_R \circ G^{-1}) = C \circ \mathrm{ad}_G \Delta_R$$

for all $C \in \operatorname{cen}\widehat{\mathcal{A}}$ and $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$. Now, let $G, H \in \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$. Then

$$\begin{aligned}\operatorname{ad}_{G\circ H}\Delta_R &= (G\circ H)\circ\Delta_R\circ(G\circ H)^{-1} = G\circ(H\circ\Delta_R\circ H^{-1})\circ G^{-1}\\ &= G\circ\operatorname{ad}_H\Delta_R\circ G^{-1} = (\operatorname{ad}_G\circ\operatorname{ad}_H)\Delta_R\end{aligned}$$

for all $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$. $\square$

**Remark 1.** The difference $\operatorname{ad}_G\Delta_R - \Delta_R$ belongs to $\operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{M}; \operatorname{ann}_{\mathcal{M}}\mathcal{A})$ for all $G \in \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ and $\mathbf{R} = (\Delta_R, R) \in \mathfrak{M}(\mathcal{M})$. Indeed, in this case, $\pi(\operatorname{ad}_G\mathbf{R}) = \pi(\mathbf{R})$ (see Theorem 1).

**Definition 5.** Multipliers $\mathbf{R}' = (\Delta_{R'}, R'), \mathbf{R}'' = (\Delta_{R''}, R'') \in \mathfrak{M}(\mathcal{M})$ are called *gauge equivalent*, $\mathbf{R}' \sim_{\mathfrak{M}} \mathbf{R}''$, if $\mathbf{R}' = \operatorname{ad}_G\mathbf{R}''$ for some $G \in \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$.

**Proposition 7.** *The gauge transform on the* $\operatorname{cen}\widehat{\mathcal{A}}$*-algebra* $\mathfrak{M}(\mathcal{M})$ *defines a gauge equivalence relation* $\sim_{\mathfrak{M}}$ *in* $\mathfrak{M}(\mathcal{M})$.

We call the corresponding quotient space $\mathfrak{M}(\mathcal{M})/{\sim_{\mathfrak{M}}}$ the *moduli space* of $\mathfrak{M}(\mathcal{M})$.

**Definition 6.** Every automorphism $G \in \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ defines the following pointwise *gauge transform* on the $\operatorname{cen}\widehat{\mathcal{A}}$-module $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)$:

$$\boldsymbol{\Delta} \mapsto \operatorname{ad}_G\boldsymbol{\Delta}\colon\ \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{M}), \qquad R \mapsto \operatorname{ad}_G\boldsymbol{\Delta}_R = (\operatorname{ad}_G\Delta_R, R).$$

**Proposition 8.** *The gauge transform on the* $\operatorname{cen}\widehat{\mathcal{A}}$*-module* $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)$ *defines the action*

$$\begin{aligned}\operatorname{ad}\colon\ &\operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M}) \to \operatorname{End}_{\operatorname{alg-cen}\widehat{\mathcal{A}}}(\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)),\\ &G \mapsto \operatorname{ad}_G\colon\ \mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi) \to \mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi), \qquad \boldsymbol{\Delta} \mapsto \operatorname{ad}_G\boldsymbol{\Delta}.\end{aligned}$$

**Definition 7.** Covariant multipliers $\boldsymbol{\Delta}', \boldsymbol{\Delta}'' \in \mathcal{S}_{\operatorname{cen}\mathcal{A}}(\pi)$ are called *gauge equivalent*, $\boldsymbol{\Delta}' \sim_{\mathfrak{M}} \boldsymbol{\Delta}''$, if $\boldsymbol{\Delta}' = \operatorname{ad}_G\boldsymbol{\Delta}''$ for some $G \in \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$.

**Proposition 9.** *The gauge transform on the* $\operatorname{cen}\widehat{\mathcal{A}}$*-module* $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)$ *defines a gauge equivalence relation* $\sim_{\mathfrak{M}}$ *in* $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)$.

We call the corresponding quotient space $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)/{\sim_{\mathfrak{M}}}$ the *moduli space* of $\mathcal{S}_{\operatorname{cen}\widehat{\mathcal{A}}}(\pi)$.

**Remark 2.** Clearly, instead of the whole group $\operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ one can consider a suitable subgroup $\mathcal{G} \subset \operatorname{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$.

## 5. HOCHSCHILD COHOMOLOGY OF MULTIPLIERS

Let $\mathcal{A}$ be an algebra, let $\mathcal{K}$ and $\mathcal{M}$ be $\widehat{\mathcal{A}}$-modules, and let $\mathcal{U} = \mathcal{A}, \mathcal{K}$ and $\mathcal{V} = \mathcal{A}, \mathcal{M}$.

**Definition 8.** The $\widehat{\mathcal{A}}$-module $C(\mathcal{U}, \mathcal{V}) = \bigoplus_{q\in\mathbb{Z}} C^q(\mathcal{U}, \mathcal{V})$ of *multiplier cochains over* $\mathfrak{M}(\mathcal{U})$ *with coefficients in* $\mathfrak{M}(\mathcal{V})$ is defined by the rule

$$C^q(\mathcal{U}, \mathcal{V}) = \begin{cases} 0, & q < 0,\\ \mathfrak{M}(\mathcal{V}), & q = 0,\\ \operatorname{Hom}_{\operatorname{cen}\widehat{\mathcal{A}}}\bigl(\bigotimes^q_{\operatorname{cen}\widehat{\mathcal{A}}}\mathfrak{M}(\mathcal{U}); \mathfrak{M}(\mathcal{V})\bigr), & q > 0.\end{cases}$$

In particular, the set $C(\mathcal{U}, \mathcal{V})$ has the natural structure of a tensor $\operatorname{cen}\widehat{\mathcal{A}}$-algebra defined as

$$(c^p \otimes c^q)(\eta_1, \dots, \eta_{p+q}) = c^p(\eta_1, \dots, \eta_p) \circ c^q(\eta_{p+1}, \dots, \eta_{p+q})$$

for all $c^p \in C^p(\mathcal{U}, \mathcal{V})$, $c^q \in C^q(\mathcal{U}, \mathcal{V})$, and $\eta_1, \dots, \eta_{p+q} \in \mathfrak{D}(\mathcal{U})$.

**Definition 9.** Let a multiplier $\kappa \in C^1(\mathcal{U},\mathcal{V}) = \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\mathfrak{M}(\mathcal{U});\mathfrak{M}(\mathcal{V}))$ be fixed. The endomorphism $\delta = \delta_\kappa \in \mathrm{End}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(C(\mathcal{U},\mathcal{V}))$ is defined by the *Hochschild rule*

$$\delta c(\eta_1,\dots,\eta_{q+1}) = \kappa\eta_1 \circ c(\eta_2,\dots,\eta_{q+1}) + \sum_{1\le r\le q} (-1)^r c(\eta_1,\dots\eta_r\circ\eta_{r+1}\dots,\eta_{q+1}) + (-1)^{q+1} c(\eta_1,\dots,\eta_q)\circ\kappa\eta_{q+1}$$

for all $q \in \mathbb{Z}_+$, $c \in C^q(\mathcal{U},\mathcal{V})$, and $\eta_1,\dots,\eta_{q+1} \in \mathfrak{M}(\mathcal{U})$. In particular, the mapping $\delta^q = \delta|_{C^q(\mathcal{U},\mathcal{V})}\colon C^q(\mathcal{U},\mathcal{V}) \to C^{q+1}(\mathcal{U},\mathcal{V})$ is defined.

**Theorem 3.** *Let a multiplier $\kappa \in C^1(\mathcal{U},\mathcal{V})$ be fixed. Then*

- *the endomorphism $\delta$ is an exterior derivation of the tensor* $\mathrm{cen}\,\widehat{\mathcal{A}}$*-algebra $C(\mathcal{U},\mathcal{V})$; i.e., $\delta(c'\otimes c'') = (\delta c')\otimes c'' + (-1)^q c'\otimes(\delta c'')$ for all $c' \in C^q(\mathcal{U},\mathcal{V})$ and $c'' \in C(\mathcal{U},\mathcal{V})$;*
- *if the curvature $F(\kappa)$ vanishes, then $\delta\circ\delta = 0$ and a differential complex $\{C^q(\mathcal{U},\mathcal{V}),\delta^q \mid q\in\mathbb{Z}\}$ is defined, with the cohomology spaces $H^q(\mathcal{U},\mathcal{V}) = \mathrm{Ker}\,\delta^q/\,\mathrm{Im}\,\delta^{q-1}$, where $F(\kappa) \in C^2(\mathcal{U},\mathcal{V})$ and*

$$F(k)(\eta',\eta'') = \kappa\eta'\circ\kappa\eta'' - \kappa(\eta'\circ\eta'') \qquad \text{for all} \quad \eta',\eta'' \in \mathfrak{M}(\mathcal{U}).$$

**Proof.** The proof is a direct verification. □

**Remark 3.** We have slightly changed the notation here.

## 6. DERIVATIONS IN ALGEBRAS AND MODULES

**6.1. Derivations in algebras.** Let $\mathcal{A}$ be an algebra.

**Definition 10.** A linear mapping $X \in \mathrm{End}_{\mathbb{F}}(\mathcal{A})$ is called a *derivation of the algebra* $\mathcal{A}$ if the *Leibniz rule* holds:

$$X(A\cdot B) = XA\cdot B + A\cdot XB \qquad \text{for all} \quad A,B\in\mathcal{A}.$$

Let $\mathfrak{D}(\mathcal{A})$ be the set of all derivations of $\mathcal{A}$.

The set $\mathfrak{D}(\mathcal{A})$ has the natural structure of a Lie algebra with the commutator $[X,Y] = X\circ Y - Y\circ X$, $X,Y\in\mathfrak{D}(\mathcal{A})$, as a Lie bracket.

**Proposition 10.** *The following statements hold*:

- *the associated Lie algebra $\mathfrak{gl}\mathcal{A}$ is defined, with $\mathfrak{gl}\mathcal{A} = \mathcal{A}$ as linear spaces and with the Lie bracket $[\cdot,\cdot]$ given by the* commutator rule

$$[A,B] = A\cdot B - B\cdot A, \qquad A,B\in\mathcal{A};$$

- *the associated action* $\mathrm{as} \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{gl}\mathcal{A};\mathfrak{D}(\mathcal{A}))$ *is defined by the rule $A \mapsto \mathrm{as}_A\colon \mathcal{A}\to\mathcal{A}$, $B\mapsto \mathrm{as}_A B = [A,B]$; the derivations $\mathrm{as}_A$, $A\in\mathcal{A}$, are called* inner derivations *of the algebra $\mathcal{A}$*;
- *the set $\mathfrak{D}_{\mathrm{inn}}(\mathcal{A})$ of all inner derivations of the algebra $\mathcal{A}$ is an ideal of the Lie algebra $\mathfrak{D}(\mathcal{A})$, because by the Jacobi identity $[X,\mathrm{as}_A] = \mathrm{as}_{XA}$ for all $X\in\mathfrak{D}(\mathcal{A})$ and $A\in\mathcal{A}$*;
- *the sequence*

$$0 \to \mathfrak{gl}\,\mathrm{cen}\,\mathcal{A} \to \mathfrak{gl}\mathcal{A} \xrightarrow{\mathrm{as}} \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathcal{A})/\mathfrak{D}_{\mathrm{inn}}(\mathcal{A}) \to 0$$

  *of Lie algebras is exact.*

In particular, the quotient Lie algebra $\boldsymbol{\mathfrak{D}}(\mathcal{A}) = \mathfrak{D}(\mathcal{A})/\mathfrak{D}_{\mathrm{inn}}(\mathcal{A})$ of *proper derivations* of the algebra $\mathcal{A}$ is defined.

**Proposition 11.** *The following statements hold*:

- *the action $\circ\colon \mathfrak{M}(\mathcal{A})\to\mathrm{End}_{\mathbb{F}}(\mathfrak{D}(\mathcal{A}))$ of algebras is defined by the* composition rule

$$\mathfrak{M}(\mathcal{A}) \ni R \mapsto R\circ(\cdot)\colon\ \mathfrak{D}(\mathcal{A})\to\mathfrak{D}(\mathcal{A}), \qquad X\mapsto R\circ X;$$

- *the action* $[\cdot,\cdot]\colon \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathfrak{M}(\mathcal{A}))$ *of Lie algebras is defined by the commutator rule*

$$\mathfrak{D}(\mathcal{A}) \ni X \mapsto [X,\cdot]\colon\ \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{A}), \qquad R \mapsto [X,R];$$

- *these actions are related by the* matching condition

$$[X, R\circ Y] = [X,R]\circ Y + R\circ [X,Y] \qquad \textit{for all} \quad X,Y \in \mathfrak{D}(\mathcal{A}), \quad R \in \mathfrak{M}(\mathcal{A}).$$

Thus, the set $\mathfrak{D}(\mathcal{A})$ has the structure of a Lie algebra and the structure of an $\mathfrak{M}(\mathcal{A})$-module, which are related by the matching condition. Briefly, $\mathfrak{D}(\mathcal{A})$ is a Lie $\mathfrak{M}(\mathcal{A})$-algebra.

**Proposition 12.** *For every derivation* $X \in \mathfrak{D}(\mathcal{A})$ *the following statements hold*:

- *the kernel* $\operatorname{Ker} X = \{A \in \mathcal{A} \mid XA = 0\}$ *is a subalgebra of the algebra* $\mathcal{A}$, *and the image* $\operatorname{Im} X = \{A = XB \mid B \in \mathcal{A}\}$ *is a* $\widehat{\operatorname{Ker} X}$*-module*;
- *the equality* $X(A\cdot B) = A\cdot XB$ *holds for all* $A \in \widehat{\operatorname{Ker} X}$ *and* $B \in \mathcal{A}$; *i.e.,* $X \in \operatorname{End}_{\widehat{\operatorname{Ker} X}}(\mathcal{A})$;
- *the image* $X(\operatorname{ann}\mathcal{A})$ *belongs to* $\operatorname{ann}\mathcal{A}$.

**6.2. Derivations in modules.** Let $\mathcal{A}$ be an algebra, and let $\mathcal{M}$ be an $\widehat{\mathcal{A}}$-module.

**Definition 11.** A pair $\mathbf{X} = (\nabla_X, X) = \operatorname{End}_{\mathbb{F}}(\mathcal{M}) \times \mathfrak{D}(\mathcal{A})$ is called a *derivation of the* $\widehat{\mathcal{A}}$*-module* $\mathcal{M}$ if the *Leibniz rule* holds:

$$\nabla_X(A\cdot M) = XA\cdot M + A\cdot\nabla_X M \qquad \text{for all} \quad A \in \mathcal{A}, \quad M \in \mathcal{M}.$$

Let $\mathfrak{D}(\mathcal{M})$ be the set of all derivations of the $\widehat{\mathcal{A}}$-module $\mathcal{M}$.

The set $\mathfrak{D}(\mathcal{M})$ has the structure of a Lie algebra with the componentwise commutator $[\mathbf{X},\mathbf{Y}] = ([\nabla_X,\nabla_Y],[X,Y])$ as a Lie bracket.

**Proposition 13.** *Let* $\mathcal{M}$ *be an* $\widehat{\mathcal{A}}$*-bimodule* (*i.e., a two-sided* $\widehat{\mathcal{A}}$*-module*). *Then the following statements hold*:

- *the associated action* $\mathbf{as} \in \operatorname{Hom}_{\mathrm{Lie}}(\mathfrak{gl}\mathcal{A};\mathfrak{D}(\mathcal{M}))$ *is defined by the rule* $A \mapsto \mathbf{as}_A = (\mathrm{as}_A, \mathrm{as}_A)$, *where* $\mathrm{as}_A M = [A,M] = A\cdot M - M\cdot A$ *for all* $A \in \mathcal{A}$ *and* $M \in \mathcal{M}$; *we call such derivations* inner derivations *of the* $\widehat{\mathcal{A}}$*-bimodule* $\mathcal{M}$;
- *the set* $\mathfrak{D}_{\mathrm{inn}}(\mathcal{M})$ *of all inner derivations of the* $\widehat{\mathcal{A}}$*-bimodule* $\mathcal{M}$ *is an ideal of the Lie algebra* $\mathfrak{D}(\mathcal{M})$.

In particular, the quotient Lie algebra $\boldsymbol{\mathfrak{D}}(\mathcal{M}) = \mathfrak{D}(\mathcal{M})/\mathfrak{D}_{\mathrm{inn}}(\mathcal{M})$ of *proper derivations* of the $\widehat{\mathcal{A}}$-bimodule $\mathcal{M}$ is defined.

**Proposition 14.** *The following statements hold*:

- *the action* $\circ\colon \mathfrak{M}(\mathcal{M}) \to \operatorname{End}_{\mathbb{F}}(\mathfrak{D}(\mathcal{M}))$ *of algebras is defined by the componentwise composition rule*

$$\mathfrak{M}(\mathcal{M}) \ni \mathbf{R} \mapsto \mathbf{R}\circ(\cdot)\colon\ \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{M}), \qquad \mathbf{X} \mapsto \mathbf{R}\circ\mathbf{X},$$

*where* $\mathbf{R} = (\Delta_R, R)$, $\mathbf{X} = (\nabla_X, X)$, *and* $\mathbf{R}\circ\mathbf{X} = (\Delta_R\circ\nabla_X, R\circ X)$;

- *the action* $[\cdot,\cdot]\colon \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathfrak{M}(\mathcal{M}))$ *of Lie algebras is defined by the componentwise commutator rule*

$$\mathfrak{D}(\mathcal{M}) \ni \mathbf{X} \mapsto [\mathbf{X},\cdot]\colon\ \mathfrak{M}(\mathcal{M}) \to \mathfrak{M}(\mathcal{M}), \qquad \mathbf{R} \mapsto [\mathbf{X},\mathbf{R}],$$

*where* $\mathbf{X} = (\nabla_X, X)$, $\mathbf{R} = (\Delta_R, R)$, *and* $[\mathbf{X},\mathbf{R}] = ([\nabla_X,\Delta_R],[X,R])$;

- *these actions are related by the* matching condition

$$[\mathbf{X}, \mathbf{R} \circ \mathbf{Y}] = [\mathbf{X}, \mathbf{R}] \circ \mathbf{Y} + \mathbf{R} \circ [\mathbf{X}, \mathbf{Y}]$$

*for all* $\mathbf{X}, \mathbf{Y} \in \mathfrak{D}(\mathcal{M})$ *and* $\mathbf{R} \in \mathfrak{M}(\mathcal{M})$.

Thus, the set $\mathfrak{D}(\mathcal{M})$ has the structure of a Lie algebra and the structure of an $\mathfrak{M}(\mathcal{M})$-module, which are related by the matching condition. Briefly, $\mathfrak{D}(\mathcal{M})$ is a Lie $\mathfrak{M}(\mathcal{M})$-algebra.

**Proposition 15.** *For every derivation* $\mathbf{X} = (\nabla_X, X) \in \mathfrak{D}(\mathcal{M})$ *the following statements hold*:

- *the kernel* $\operatorname{Ker} \nabla_X = \{M \in \mathcal{M} \mid \nabla_X M = 0\}$ *and the image* $\operatorname{Im} \nabla_X = \{M = \nabla_X N \mid N \in \mathcal{M}\}$ *are* $\widehat{\operatorname{Ker} X}$*-modules*;
- *the equality* $\nabla_X(A \cdot M) = A \cdot \nabla_X M$ *holds for all* $A \in \operatorname{Ker} X$ *and* $M \in \mathcal{M}$; *i.e.*, $\nabla_X \in \operatorname{End}_{\widehat{\operatorname{Ker} X}}(\mathcal{M})$;
- *the image* $\nabla_X(\operatorname{ann}_{\mathcal{M}} \mathcal{A})$ *belongs to* $\operatorname{ann}_{\mathcal{M}} \mathcal{A}$, *and the image* $X(\operatorname{ann}_{\mathcal{A}} \mathcal{M})$ *belongs to* $\operatorname{ann}_{\mathcal{A}} \mathcal{M}$.

**Proposition 16.** *There is a natural Lie* $\widehat{\mathcal{A}}$*-algebra injection*

$$\mathrm{I}\colon\ \mathfrak{gl}_{\mathcal{A}}(\mathcal{M}) \to \mathfrak{D}(\mathcal{M}), \qquad \rho \mapsto (\rho, 0) \qquad (i.e.,\ \nabla_0 = \rho).$$

*Moreover, the image* $\operatorname{Im} \mathrm{I} = \{\mathbf{X} = (\rho, 0) \mid \rho \in \operatorname{End}_{\widehat{\mathcal{A}}}(\mathcal{M})\}$ *is an ideal of the Lie* $\mathfrak{M}(\mathcal{M})$*-algebra* $\mathfrak{D}(\mathcal{M})$; *thus the short exact sequence of Lie algebras*

$$0 \to \mathfrak{gl}_{\mathcal{A}}(\mathcal{M}) \xrightarrow{\mathrm{I}} \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{M}) / \operatorname{Im} \mathrm{I} \to 0$$

*is defined.*

In particular, the quotient Lie algebra $\mathfrak{D}_{\mathrm{I}}(\mathcal{M}) = \mathfrak{D}(\mathcal{M}) / \operatorname{Im} \mathrm{I}$ is defined.

**Proposition 17.** *Let* $\mathcal{A}$ *be an algebra, and let* $\mathcal{M}$ *be a free* $\widehat{\mathcal{A}}$*-module with an* $\widehat{\mathcal{A}}$*-basis* $\mathbf{b} = \{b_i \in \mathcal{M} \mid i \in I\}$ *indexed by a set* $I$, *so that* $\mathcal{M} = \{M = M^i \cdot b_i \mid M^i \in \widehat{\mathcal{A}}\}$. *Then a Lie algebra injection* $\mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathcal{M})$ *is defined by the componentwise action,*

$$\mathfrak{D}(\mathcal{A}) \ni X \mapsto \mathbf{X} = (\nabla_X, X) \in \mathfrak{D}(\mathcal{M}), \qquad \nabla_X(M^i \cdot b_i) = X M^i \cdot b_i.$$

There is another way to treat derivations of an $\widehat{\mathcal{A}}$-module $\mathcal{M}$. Namely, consider the mapping

$$\Pi\colon\ \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{A}), \qquad \mathbf{X} = (\nabla_X, X) \mapsto X.$$

**Proposition 18.** *The following diagram is commutative*:

$$\begin{array}{ccc} \mathfrak{M}(\mathcal{M}) \times \mathfrak{D}(\mathcal{M}) & \xrightarrow{\ \circ\ } & \mathfrak{D}(\mathcal{M}) \\ \downarrow \pi \qquad \downarrow \Pi & & \downarrow \Pi \\ \mathfrak{M}(\mathcal{A}) \times \mathfrak{D}(\mathcal{A}) & \xrightarrow{\ \circ\ } & \mathfrak{D}(\mathcal{A}) \end{array}$$

*where* $\circ\colon \mathfrak{M}(\mathcal{M}) \times \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{M})$, $(\mathbf{R}, \mathbf{X}) \mapsto \mathbf{R} \circ \mathbf{X}$.

Clearly, $\Pi \in \operatorname{Hom}_{\mathrm{Lie}}(\mathfrak{D}(\mathcal{M}); \mathfrak{D}(\mathcal{A})) \cap \operatorname{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{M}); \mathfrak{D}(\mathcal{A}))$, which can be briefly written as $\Pi \in \operatorname{Hom}_{\text{Lie-cen}\,\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{M}); \mathfrak{D}(\mathcal{A}))$.

**Theorem 4.** *For every* $X \in \mathfrak{D}(\mathcal{A})$ *the inverse image* $\Pi^{-1}(X)$ *is an* $\widehat{\mathcal{A}}$*-affine space over the* $\widehat{\mathcal{A}}$*-module* $\operatorname{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$. *Hence, the triple* $\Pi\colon \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{A})$ *is an* $\widehat{\mathcal{A}}$*-affine fiber bundle over the* cen $\widehat{\mathcal{A}}$*-module* $\operatorname{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$.

**Proof.** Indeed, let $X \in \mathfrak{D}(\mathcal{A})$ and $\mathbf{X}' = (\nabla'_X, X), \mathbf{X}'' = (\nabla''_X, X) \in \Pi^{-1}(X)$. Then for the difference $\nabla'_X - \nabla''_X$ we have

$$(\nabla'_X - \nabla''_X)(A \cdot M) = (X - X)A \cdot M + A \cdot (\nabla'_X - \nabla''_X)M = A \cdot (\nabla'_X - \nabla''_X)M$$

for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$, i.e., $\nabla'_X - \nabla''_X \in \mathrm{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$. On the other hand, let $\mathbf{X} = (\nabla_X, X) \in \Pi^{-1}(X)$ and $\rho \in \mathrm{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$. Then $(\nabla_X, X) + (\rho, 0) = (\nabla_X + \rho, X) \in \Pi^{-1}(X)$, because $\Pi(\nabla_X + \rho, X) = X$ and

$$\begin{aligned}(\nabla_X + \rho)(A \cdot M) &= \nabla_X(A \cdot M) + \rho(A \cdot M) = XA \cdot M + A \cdot \nabla_X M + A \cdot \rho M \\ &= XA \cdot M + A \cdot (\nabla_X + \rho)M\end{aligned}$$

for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$. □

Consider the set of all $*$-*sections* of the bundle $\Pi$,

$$\mathcal{S}_*(\Pi) = \big\{ \boldsymbol{\nabla} \in \mathrm{Hom}_*(\mathfrak{D}(\mathcal{A}); \mathfrak{D}(\mathcal{M})) \;\big|\; \Pi \circ \boldsymbol{\nabla} = \mathrm{id}_{\mathfrak{D}(\mathcal{A})} \big\}, \qquad * = \mathbb{F}, \mathrm{cen}\,\widehat{\mathcal{A}}, \mathrm{Lie}, \text{Lie-cen}\,\widehat{\mathcal{A}}.$$

The set $\mathcal{S}_{\mathbb{F}}(\Pi)$ has the structure of a linear space, $\mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\Pi)$ has the structure of a $\mathrm{cen}\,\widehat{\mathcal{A}}$-module, $\mathcal{S}_{\mathrm{Lie}}(\Pi)$ has the structure of a Lie algebra, and $\mathcal{S}_{\text{Lie-cen}\,\widehat{\mathcal{A}}}(\Pi)$ has the structure of a Lie $\mathrm{cen}\,\widehat{\mathcal{A}}$-algebra, where the algebraic operations are defined pointwise. The elements of the $\mathrm{cen}\,\widehat{\mathcal{A}}$-module $\mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\Pi)$ are called *covariant derivations*.

**Definition 12.** The *curvature* $F(\boldsymbol{\nabla}) \in \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}\big(\bigwedge^2_{\mathrm{cen}\,\widehat{\mathcal{A}}} \mathfrak{D}(\mathcal{A}); \mathrm{End}_{\widehat{\mathcal{A}}}(\mathcal{M})\big)$ of a covariant derivation $\boldsymbol{\nabla} \in \mathcal{S}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\Pi)$ is defined by the *residual rule*

$$[\boldsymbol{\nabla}_X, \boldsymbol{\nabla}_Y] - \boldsymbol{\nabla}_{[X,Y]} = (F(\boldsymbol{\nabla})_{X,Y}, 0) \qquad \text{for all} \quad X, Y \in \mathfrak{D}(\mathcal{A}),$$

where

$$F(\boldsymbol{\nabla})_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

**Proposition 19.** *Let $\boldsymbol{\mathcal{A}}$ be the adjoint $\widehat{\mathcal{A}}$-module of the algebra $\mathcal{A}$. Then*

- *there is a natural Lie algebra injection*

$$\iota_{\mathfrak{D}}\colon \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\boldsymbol{\mathcal{A}}), \qquad X \mapsto \mathbf{X} = (X, X);$$

- *the curvature $F(\iota_{\mathfrak{D}})$ is zero;*
- $\mathfrak{D}(\boldsymbol{\mathcal{A}}) = \mathrm{Im}\,\iota_{\mathfrak{D}} \oplus \mathrm{End}_{\widehat{\mathcal{A}}}(\boldsymbol{\mathcal{A}})$, $\mathbf{X} = (\nabla_X, X) = (X, X) + (\nabla_X - X, 0)$.

## 7. GAUGE TRANSFORM OF DERIVATIONS

Let $\mathcal{A}$ be an algebra, and let $\mathcal{M}$ be an $\widehat{\mathcal{A}}$-module. Let $\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ be the group of all automorphisms of the $\widehat{\mathcal{A}}$-module $\mathcal{M}$.

**Definition 13.** Every mapping $G \in \mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ defines the following *gauge transform* of the Lie $\mathrm{cen}\,\widehat{\mathcal{A}}$-algebra $\mathfrak{D}(\mathcal{M})$:

$$\mathbf{X} = (\nabla_X, X) \mapsto \mathrm{ad}_G\, \mathbf{X} = (\mathrm{ad}_G\, \nabla_X, X), \qquad \mathrm{ad}_G\, \nabla_X = G \circ \nabla_X \circ G^{-1}.$$

**Theorem 5.** *The gauge transforms of the Lie* $\mathrm{cen}\,\widehat{\mathcal{A}}$*-algebra* $\mathfrak{D}(\mathcal{M})$ *define the action*

$$\mathrm{ad}\colon\; \mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M}) \to \mathrm{End}_{\text{Lie-cen}\,\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{M})), \qquad G \mapsto \mathrm{ad}_G\,.$$

**Proof.** Indeed,

$$\begin{aligned}(\mathrm{ad}_G \nabla_X)(A\cdot M) &= (G\circ\nabla_X\circ G^{-1})(A\cdot M) = (G\circ\nabla_X)(A\cdot G^{-1}M)\\ &= G(\nabla_X(A\cdot G^{-1}M)) = G\big(XA\cdot G^{-1}M + A\cdot(\nabla_X\circ G^{-1}M)\big)\\ &= XA\cdot(G\circ G^{-1}M) + A\cdot(G\circ\nabla_X\circ G^{-1})M = XA\cdot M + A\cdot(\mathrm{ad}_G\nabla_X)M;\end{aligned}$$

hence, $\mathrm{ad}_G\,\mathbf{X}\in\mathfrak{D}(\mathcal{M})$ for all $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ and $\mathbf{X}=(\nabla_X,X)\in\mathfrak{D}(\mathcal{M})$. Further,

$$\mathrm{ad}_G(C\nabla_X)M = (G\circ C\nabla_X\circ G^{-1})M = C\cdot(G\circ\nabla_X\circ G^{-1})M = C\cdot\mathrm{ad}_G M$$

for all $C\in\mathrm{cen}\,\widehat{\mathcal{A}}$ and $\mathbf{X}=(\nabla_X,X)\in\mathfrak{D}(\mathcal{M})$, with $C\mathbf{X}=(C\nabla_X,CX)$. Hence $\mathrm{ad}_G\in\mathrm{End}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{M}))$ for all $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$. Finally, let $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ and $\mathbf{X}'=(\nabla_{X'},X'),\mathbf{X}''=(\nabla_{X''},X'')\in\mathfrak{D}(\mathcal{M})$. Then

$$\begin{aligned}\big[\mathrm{ad}_G\,\mathbf{X}',\mathrm{ad}_G\,\mathbf{X}''\big] &= \big([\mathrm{ad}_G\nabla_{X'},\mathrm{ad}_G\nabla_{X''}],[X',X'']\big)\\ &= \big(\big[G\circ\nabla_{X'}\circ G^{-1},G\circ\nabla_{X''}\circ G^{-1}\big],[X',X'']\big)\\ &= \big(G\circ[\nabla_{X'},\nabla_{X''}]\circ G^{-1},[X',X'']\big) = \mathrm{ad}_G[\mathbf{X}',\mathbf{X}''];\end{aligned}$$

i.e., $\mathrm{ad}_G\in\mathrm{End}_{\mathrm{Lie}}(\mathfrak{D}(\mathcal{M}))$. □

**Remark 4.** The difference $\mathrm{ad}_G\nabla_X-\nabla_X$ belongs to $\mathrm{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$ for all $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ and $\mathbf{X}=(\nabla_X,X)\in\mathfrak{D}(\mathcal{M})$. Indeed, in this case, $\Pi(\mathrm{ad}_G\,\mathbf{X})=\Pi(\mathbf{X})$ (see Theorem 4).

**Definition 14.** Derivations $\mathbf{X}'=(\nabla_{X'},X'),\mathbf{X}''=(\nabla_{X''},X'')\in\mathfrak{D}(\mathcal{M})$ are said to be *gauge equivalent*, $\mathbf{X}'\sim_{\mathfrak{D}}\mathbf{X}''$, if $\mathbf{X}'=\mathrm{ad}_G\,\mathbf{X}''$ for some $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$. One can check that this relation is indeed an equivalence relation.

**Proposition 20.** *The gauge transforms of the* $\mathrm{cen}\,\widehat{\mathcal{A}}$*-algebra* $\mathfrak{D}(\mathcal{M})$ *define a gauge equivalence relation in* $\mathfrak{D}(\mathcal{M})$.

The corresponding quotient space $\mathfrak{D}(\mathcal{M})/\sim_{\mathfrak{D}}$ is called the *moduli space* of $\mathfrak{D}(\mathcal{M})$.

**Proposition 21.** *Let* $G\in\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$, $\mathbf{R}\in\mathfrak{M}(\mathcal{M})$, *and* $\mathbf{X}\in\mathfrak{D}(\mathcal{M})$. *Then*

$$\mathrm{ad}_G(\mathbf{R}\circ\mathbf{X})=\mathrm{ad}_G\,\mathbf{R}\circ\mathrm{ad}_G\,\mathbf{X}\qquad\text{and}\qquad\mathrm{ad}_G[\mathbf{X},\mathbf{R}]=[\mathrm{ad}_G\,\mathbf{X},\mathrm{ad}_G\,\mathbf{R}].$$

**Remark 5.** Clearly, instead of the whole group $\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$ one can consider a suitable subgroup $\mathcal{G}\subset\mathrm{Aut}_{\widehat{\mathcal{A}}}(\mathcal{M})$.

## 8. DE RHAM COHOMOLOGY

Let $\mathcal{A}$ be an algebra, let $\mathcal{K}$ and $\mathcal{M}$ be $\widehat{\mathcal{A}}$-modules, and let $\mathcal{U}=\mathcal{A},\mathcal{K}$ and $\mathcal{V}=\mathcal{A},\mathcal{M}$.

**Definition 15.** The $\mathrm{cen}\,\widehat{\mathcal{A}}$-module $\Omega(\mathcal{U},\mathcal{V})=\bigoplus_{q\in\mathbb{Z}}\Omega^q(\mathcal{U},\mathcal{V})$ of *differential forms over* $\mathcal{U}$ *with coefficients in* $\mathcal{V}$ is defined by the rule

$$\Omega^q(\mathcal{U},\mathcal{V})=\begin{cases}0, & q<0,\\ \mathcal{V}, & q=0,\\ \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}\big(\bigwedge^q_{\mathrm{cen}\,\widehat{\mathcal{A}}}\mathfrak{D}(\mathcal{U});\mathcal{V}\big), & q>0.\end{cases}$$

In particular, the set $\Omega(\mathcal{U},\mathcal{A})$ has the natural structure of an exterior $\mathrm{cen}\,\widehat{\mathcal{A}}$-algebra, and the set $\Omega(\mathcal{U},\mathcal{M})$ has the natural structure of an exterior $\Omega(\mathcal{U},\mathcal{A})$-module.

**Definition 16.** For any $\xi\in\mathfrak{D}(\mathcal{U})$ the *interior product* $i_\xi\in\mathrm{End}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\Omega(\mathcal{U},\mathcal{V}))$ is defined by the *contraction rule*

$$(i_\xi\omega)(\xi_1,\dots,\xi_{q-1})=q\omega(\xi,\xi_1,\dots,\xi_{q-1})$$

for all $q\in\mathbb{Z}_+$, $\omega\in\Omega^q(\mathcal{U},\mathcal{V})$, and $\xi_1,\dots,\xi_{q-1}\in\mathfrak{D}(\mathcal{U})$.

**Proposition 22.** *The following statements hold*:

- $i_{\xi'} \circ i_{\xi''} + i_{\xi''} \circ i_{\xi'} = 0$ *for all* $\xi', \xi'' \in \mathfrak{D}(\mathcal{U})$;
- $i_\xi(\phi \wedge \omega) = (i_\xi \phi) \wedge \omega + (-1)^q \phi \wedge (i_\xi \omega)$ *for all* $\xi \in \mathfrak{D}(\mathcal{U})$, $\phi \in \Omega^q(\mathcal{U}, \mathcal{A})$, $q \in \mathbb{Z}_+$, *and* $\omega \in \Omega(\mathcal{U}, \mathcal{V})$; *i.e., the mapping* $i_\xi$ *is an exterior derivation of the exterior algebra* $\Omega(\mathcal{U}, \mathcal{A})$ *and the exterior* $\Omega(\mathcal{U}, \mathcal{A})$*-module* $\Omega(\mathcal{U}, \mathcal{M})$.

**Definition 17.** Let a mapping $\varkappa \in \Omega^1(\mathcal{U}, \mathfrak{D}(\mathcal{V})) = \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{U}); \mathfrak{D}(\mathcal{V}))$ be fixed. For every $\xi \in \mathfrak{D}(\mathcal{U})$ the *Lie derivative* $L_\xi \in \mathrm{End}_{\mathbb{F}}(\mathcal{U}, \mathcal{V})$ is defined by the rule

$$(L_\xi \omega)(\xi_1, \dots, \xi_q) = (\varkappa\xi)(\omega(\xi_1, \dots, \xi_q)) - \sum_{1 \le r \le q} \omega\big(\xi_1, \dots, \xi_{r-1}, [\xi, \xi_r], \xi_{r+1}, \dots, \xi_q\big)$$

for all $q \in \mathbb{Z}_+,\ \omega \in \Omega^q(\mathcal{U}, \mathcal{V}),$ and $\xi_1, \dots, \xi_q \in \mathfrak{D}(\mathcal{U})$.

**Proposition 23.** *The following statements hold*:

- $L_\xi(\phi \wedge \omega) = (L_\xi \phi) \wedge \omega + \phi \wedge (L_\xi \omega)$ *for all* $\phi \in \Omega^q(\mathcal{U}, \mathcal{A})$, $q \in \mathbb{Z}_+$, *and* $\omega \in \Omega(\mathcal{U}, \mathcal{V})$; *i.e., the mapping* $L_\xi$ *is a derivation of the exterior algebra* $\Omega(\mathcal{U}, \mathcal{A})$ *and the exterior* $\Omega(\mathcal{U}, \mathcal{A})$*-module* $\Omega(\mathcal{U}, \mathcal{M})$;
- $[L_{\xi'}, i_{\xi''}] = i_{[\xi', \xi'']}$ *for all* $\xi', \xi'' \in \mathfrak{D}(\mathcal{U})$;
- *if the curvature* $F(\varkappa)$ *vanishes, then* $[L_{\xi'}, L_{\xi''}] = L_{[\xi', \xi'']}$ *for all* $\xi', \xi'' \in \mathfrak{D}(\mathcal{U})$, *where* $F(\varkappa) \in \Omega^2(\mathcal{U}, \mathfrak{D}(\mathcal{V})) = \mathrm{Hom}_{\mathrm{cen}\,\widehat{\mathcal{A}}}\big(\bigwedge^2_{\mathrm{cen}\,\widehat{\mathcal{A}}} \mathfrak{D}(\mathcal{U}); \mathfrak{D}(\mathcal{V})\big)$,
$$F(\varkappa)(\xi', \xi'') = [\varkappa\xi', \varkappa\xi''] - \varkappa[\xi', \xi''] \qquad \textit{for all} \quad \xi', \xi'' \in \mathfrak{D}(\mathcal{U}).$$

**Definition 18.** The endomorphism $d = d_\varkappa \in \mathrm{End}_{\mathbb{F}}(\Omega(\mathcal{U}, \mathcal{V}))$ is defined by the *Cartan formula*

$$d\omega(\xi_0, \dots, \xi_q) = \frac{1}{q+1} \Bigg\{ \sum_{0 \le r \le q} (-1)^r (\varkappa\xi_r)\big(\omega(\xi_0, \dots \check{\xi}_r \dots, \xi_q)\big) + \sum_{0 \le r < s \le q} (-1)^{r+s} \omega\big([\xi_r, \xi_s], \xi_0, \dots \check{\xi}_r \dots \check{\xi}_s \dots, \xi_q\big) \Bigg\}$$

for all $\omega \in \Omega^q(\mathcal{U}, \mathcal{V})$ and $\xi_0, \dots, \xi_q \in \mathfrak{D}(\mathcal{U})$, where the "checked" arguments are assumed to be omitted; in particular, $d^q = d|_{\Omega^q(\mathcal{U}, \mathcal{V})} \colon \Omega^q(\mathcal{U}, \mathcal{V}) \to \Omega^{q+1}(\mathcal{U}, \mathcal{V})$.

**Theorem 6.** *The following statements hold*:

- $d(\phi \wedge \omega) = d\phi \wedge \omega + (-1)^q \phi \wedge d\omega$ *for all* $\phi \in \Omega^q(\mathcal{U}, \mathcal{A})$, $q \in \mathbb{Z}_+$, *and* $\omega \in \Omega(\mathcal{U}, \mathcal{V})$; *i.e., the mapping* $d$ *is an exterior derivation of the exterior algebra* $\Omega(\mathcal{U}, \mathcal{A})$ *and the exterior* $\Omega(\mathcal{U}, \mathcal{A})$*-module* $\Omega(\mathcal{U}, \mathcal{M})$;
- *if the curvature* $F(\varkappa)$ *vanishes, then* $d \circ d = 0$ *and the differential complex* $\{\Omega^q(\mathcal{K}, \mathcal{M}), d^q \mid q \in \mathbb{Z}\}$ *is defined, with the cohomology spaces* $H^q(\mathcal{K}, \mathcal{M}) = \mathrm{Ker}\, d^q / \mathrm{Im}\, d^{q-1}$, $q \in \mathbb{Z}$.

**Proof.** The proof of the equality $d \circ d = 0$ is based on the assumed property $\varkappa[\xi', \xi''] = [\varkappa\xi', \varkappa\xi'']$, $\xi', \xi'' \in \mathfrak{D}(\mathcal{U})$, of the mapping $\varkappa$ (see [13] for a detailed exposition). The other statements are easy to verify directly. □

**Theorem 7.** *Let the curvature* $F(\varkappa)$ *vanish. Then the* Cartan magic formula

$$L_\xi = d \circ i_\xi + i_\xi \circ d$$

*holds for every* $\xi \in \mathfrak{D}(\mathcal{U})$.

**Proof.** The proof is standard, but the calculations are rather cumbersome. See [13] for full details. □

**Corollary 1.** *The commutator* $[L_\xi, d]$ *vanishes for every* $\xi \in \mathfrak{D}(\mathcal{U})$.

**Theorem 8.** *Let* $\mathcal{M}$ *be an* $\widehat{\mathcal{A}}$*-module, let* $\Omega(\mathcal{M}) = \Omega(\mathcal{M}, \mathcal{M})$, *and let us take* $\varkappa = \mathrm{id}_{\mathfrak{D}(\mathcal{M})} \in \mathrm{End}_{\widehat{\mathcal{A}}}(\mathfrak{D}(\mathcal{M}))$. *Then the complex* $\{\Omega^q(\mathcal{M}), d^q \mid q \in \mathbb{Z}\}$ *is exact*; *i.e., the cohomology spaces* $H^q(\mathcal{M}) = H^q(\mathcal{M}, \mathcal{M})$ *are trivial for all* $q \in \mathbb{Z}$.

**Proof.** Indeed, by Proposition 16, we have $\mathbf{E} = (\mathrm{id}_{\mathcal{M}}, 0) \in \mathfrak{D}(\mathcal{M})$, while $L_{\mathbf{E}} = \mathrm{id}_{\Omega(\mathcal{M})}$. Hence, by the Cartan magic formula, the homotopy formula

$$\mathrm{id}_{\Omega(\mathcal{M})} = i_{\mathbf{E}} \circ d + d \circ i_{\mathbf{E}}$$

holds, implying the claim. □

## 9. CONCLUSIONS

Here we have presented only the basic definitions, constructions, and results of the noncommutative analysis in algebras and modules. We hope that this paper will be a good supplement to the fundamental studies [1, 2, 8]. Possible applications lie in the theory of complex quantum models, quantum calculus, noncommutative geometry, noncommutative partial differential equations, etc. For example, the technique presented above may turn out to be useful in the problems addressed in [3–7, 9–14].

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