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The Homotopy Types of Sp(2)-Gauge Groups over Closed Simply Connected Four-Manifolds

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Abstract—We determine the number of distinct homotopy types for the gauge groups of principal Sp(2)-bundles over a closed simply connected four-manifold.

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1. INTRODUCTION

Let X be a pointed topological space, G a topological group and $P\to X$ a principal G-bundle. The *gauge group* $\mathcal{G}(P)$ of P is the group of G-equivariant automorphisms of P that fix X. The topology of gauge groups is of interest due to its connections with various moduli spaces [1, 2] and Donaldson theory [8].

Considerable effort has gone into determining the homotopy types of gauge groups for specific groups G and spaces X. Typically, G and X are chosen because of their interest in geometry or physics. In general, Crabb and Sutherland [5] showed that if G is a compact connected Lie group and X is a connected finite CW-complex then, even if there are infinitely many inequivalent classes of principal G-bundles P over X, there are only finitely many distinct homotopy types for their gauge groups. Precise enumerations of the homotopy types have been made in the following cases: SU(2)-bundles over S^4 (see [19]) or over a closed simply connected four-manifold (see [20]); SU(3)-bundles over S^4 (see [10]) or over a closed simply connected four-manifold (see [29]); SU(5)-bundles over S^4 (see [30]); SO(3)-bundles over S^4 (see [15]); PU(3)-bundles over S^4 (see [11]); and Sp(2)-bundles over S^4 (see [28]). Substantial but incomplete information has also been obtained for many other cases of principal G-bundles over S^4 (see [6, 7, 11, 12, 16–18, 31]).

In this paper we consider the homotopy types of the gauge groups of principal Sp(2)-bundles over a closed simply connected four-manifold. It is well known that the principal Sp(2)-bundles over a closed simply connected four-manifold M are classified by their second Chern class, which can take any integer value. Let $P_k\to M$ be the principal Sp(2)-bundle classified by the integer k, and let $\mathcal{G}_k(M)$ be its gauge group. For integers a and b let (a,b) be their greatest common denominator.

Theorem 1.1. *Let M be a closed simply connected four-manifold.*

(a) *Suppose that M is Spin. If $\mathcal{G}_k(M)$ is homotopy equivalent to $\mathcal{G}_\ell(M)$ then $(40,k)=(40,\ell)$, and conversely, if $(40,k)=(40,\ell)$ then $\mathcal{G}_k(M)$ is homotopy equivalent to $\mathcal{G}_\ell(M)$ when localized rationally or at any prime.*

(b) *Suppose that M is non-Spin. If $\mathcal{G}_k(M)$ is homotopy equivalent to $\mathcal{G}_\ell(M)$ then $(20,k)=(20,\ell)$, and conversely, if $(20,k)=(20,\ell)$ then $\mathcal{G}_k(M)$ is homotopy equivalent to $\mathcal{G}_\ell(M)$ when localized rationally or at any prime.*

It is notable that precisely the same factor of 2 in the g.c.d. condition occurs in the cases of SU(2)- or SU(3)-bundles over a closed simply connected four-manifold [20, 29]. It is plausible that this holds in general, but the tools for proving this do not yet seem to be available.

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2. REDUCING THE PROBLEM

The main aim of this section is the general result in Theorem 2.5, which will allow us to reduce the problem of determining the homotopy types of $\mathcal{G}_k(M)$ to that of $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$. In general, let G be a simply connected, simple compact Lie group and let M be a closed simply connected four-manifold. Since $[M,BG]\cong\mathbb{Z}$, the principal G-bundles over M are classified by the second Chern class of the bundle. Let $P_k\to M$ be a principal G-bundle whose second Chern class is $k\in\mathbb{Z}$. Let $\mathcal{G}_k(M)$ be the gauge group of P_k.

The following decomposition was proved for the Spin case in [27] and for the non-Spin case in [25].

Theorem 2.1. *Let G be a simply connected, simple compact Lie group and let M be a closed simply connected four-manifold with $H^2(M;\mathbb{Z})$ of rank $d\geq 1$.*

(a) *If M is Spin then there is an integral homotopy equivalence*

$$\mathcal{G}_k(M)\simeq\mathcal{G}_k(S^4)\times\prod_{i=1}^{d}\Omega^2G.$$

(b) *If M is non-Spin then there is an integral homotopy equivalence*

$$\mathcal{G}_k(M)\simeq\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)\times\prod_{i=1}^{d-1}\Omega^2G.\qquad\square$$

Further, at odd primes there is another decomposition, proved in [27].

Theorem 2.2. *Let G be a simply connected, simple compact Lie group. Localize spaces and maps rationally or at an odd prime p. Then there is a homotopy equivalence*

$$\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)\simeq\mathcal{G}_k(S^4)\times\Omega^2G.\qquad\square$$

We will show in Theorem 2.5 that Theorems 2.1 and 2.2 together with a recent result of Huang and Wu [13] allow for a reduction in the determination of the p-local homotopy types of $\mathcal{G}_k(M)$ to that of $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$, and in the latter case, to only worrying about the prime 2. This first requires some notation and a preliminary result.

Let $\mathrm{Map}_k(M,BG)$ be the component of the space of continuous unbased maps from M to BG which contains the map inducing P_k. Similarly, let $\mathrm{Map}^*_k(M,BG)$ be the component of the space of continuous pointed maps from M to BG which contains the map inducing P_k. Observe that there is a fibration $\mathrm{Map}^*_k(M,BG)\to\mathrm{Map}_k(M,BG)\xrightarrow{\mathrm{ev}}BG$, where ev evaluates a map at the basepoint of M. Let $B\mathcal{G}_k(M)$ be the classifying space of $\mathcal{G}_k(M)$. By [1, 9], there is a homotopy equivalence $B\mathcal{G}_k(M)\simeq\mathrm{Map}_k(M,BG)$. The evaluation fibration therefore determines a homotopy fibration sequence

$$G\xrightarrow{\overline{\partial}_k}\mathrm{Map}^*_k(M,BG)\longrightarrow B\mathcal{G}_k(M)\xrightarrow{\mathrm{ev}}BG \tag{2.1}$$

which defines the map $\overline{\partial}_k$. According to [26], for any $k\in\mathbb{Z}$, there is a homotopy equivalence $\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2,BG)\simeq\mathrm{Map}^*_0(\mathbb{C}\mathrm{P}^2,BG)$. So we rewrite (2.1) as a homotopy fibration sequence

$$G\xrightarrow{\partial_k}\mathrm{Map}^*_0(M,BG)\longrightarrow B\mathcal{G}_k(M)\xrightarrow{\mathrm{ev}}BG. \tag{2.2}$$

If $M=S^4$, write $\mathrm{Map}^*_0(S^4,BG)$ as Ω^3_0G.

Lemma 2.3. *Let G be a simply connected, simple compact Lie group, but exclude $G=\mathrm{Spin}(4)$.*

(a) *If M is Spin and G is one of $\mathrm{SU}(n)$ for $n\geq 3$, $\mathrm{Spin}(n)$ for $n\geq 6$, G_2, F_4, E_6, E_7 or E_8 then $\mathcal{G}_k(M)$ is path-connected.*

(b) *If M is Spin and G is one of $\mathrm{SU}(2)$, $\mathrm{Sp}(n)$ for $n\geq 1$ or $\mathrm{Spin}(n)$ for $n\in\{3,5\}$ then $\mathcal{G}_k(M)$ has two connected components.*

(c) *In case* (b), *localized at an odd prime* p, $\mathcal{G}_k(M)$ *is path-connected.*

(d) *If* M *is non-Spin then* $\mathcal{G}_k(M)$ *is path-connected for any* G.

Proof. First, the homotopy equivalences in Theorem 2.1 imply that $\pi_0(\mathcal{G}_k(M)) \cong \pi_0(\mathcal{G}_k(S^4))$ if M is Spin and $\pi_0(\mathcal{G}_k(M)) \cong \pi_0(\mathcal{G}_k(\mathbb{C}\mathrm{P}^2))$ if M is non-Spin. Therefore, $\mathcal{G}_k(M)$ is path-connected if and only if $\mathcal{G}_k(S^4)$ or $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ are depending on whether M is Spin or non-Spin, respectively.

Second, when $M = S^4$, consider the homotopy fibration $\Omega^4_0 G \to \mathcal{G}_k(S^4) \to G$ from (2.2). Since G is 2-connected, we immediately obtain $\pi_0(\mathcal{G}_k(S^4)) \cong \pi_0(\Omega^4_0 G) \cong \pi_4(G)$. It is classical that if G is one of $\mathrm{SU}(n)$ for $n \geq 3$, $\mathrm{Spin}(n)$ for $n \geq 6$, G_2, F_4, E_6, E_7 or E_8 then $\pi_4(G) \cong 0$. Hence, in these cases, $\mathcal{G}_k(S^4)$ is path-connected, proving assertion (a). If G is one of $\mathrm{SU}(2)$, $\mathrm{Sp}(n)$ for $n \geq 1$ or $\mathrm{Spin}(n)$ for $n \in \{3, 5\}$ then $\pi_4(G) \cong \mathbb{Z}/2\mathbb{Z}$. Hence, in these cases, $\mathcal{G}_k(S^4)$ has two connected components, proving assertion (b). Further, localized at an odd prime p, $\pi_4(G) \cong 0$, in which case $\mathcal{G}_k(S^4)$ is path-connected, proving assertion (c).

Third, when $M = \mathbb{C}\mathrm{P}^2$, consider the homotopy fibration $\Omega\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, BG) \to \mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \to G$ from (2.2). Since G is 2-connected, we immediately obtain $\pi_0(\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)) \cong \pi_0(\Omega\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, BG)) \cong \pi_1(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, BG))$. Consider the homotopy cofibration sequence $S^2 \to \mathbb{C}\mathrm{P}^2 \to S^4 \xrightarrow{\eta} S^3$. Applying $\mathrm{Map}^*(\,\cdot\,, BG)$ we obtain a homotopy fibration sequence $\Omega^2 G \xrightarrow{\eta^*} \Omega^3 G \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, BG) \to \Omega G$. Applying π_1 then gives an exact sequence $\pi_3(G) \xrightarrow{\eta^*} \pi_4(G) \to \pi_1(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, BG)) \to \pi_1(G)$. Since G is simply connected, $\pi_1(G) \cong 0$. From the previous paragraph, either $\pi_4(G) \cong 0$ or $\pi_4(G) \cong \mathbb{Z}/2\mathbb{Z}$. If $\pi_4(G) \cong 0$ then we immediately obtain $\pi_1(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, BG)) \cong 0$. If $\pi_4(G) \cong \mathbb{Z}/2\mathbb{Z}$ then a representative of the generator is given by the composite $S^4 \xrightarrow{\eta} S^3 \hookrightarrow G$, implying that η^* is an epimorphism and hence $\pi_1(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, BG)) \cong 0$. Thus, in all cases, the space $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ is path-connected, proving assertion (d). $\square$

Remark 2.4. If $G = \mathrm{Spin}(4)$ then $\pi_4(G) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, implying that $\mathcal{G}_k(S^4)$ has four connected components and $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ has two connected components.

Theorem 2.5. *Let* G *be a simply connected, simple compact Lie group, but exclude* $G = \mathrm{Spin}(4)$. *Localize spaces and maps at a prime* p.

(a) *If* M *is Spin then there is a homotopy equivalence* $\mathcal{G}_k(M) \simeq \mathcal{G}_\ell(M)$ *if and only if* $\mathcal{G}_k(S^4) \simeq \mathcal{G}_\ell(S^4)$.

(b) *If* M *is non-Spin then there is a homotopy equivalence* $\mathcal{G}_k(M) \simeq \mathcal{G}_\ell(M)$ *if and only if* $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$.

(c) *If* p *is odd then there is a homotopy equivalence* $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ *if and only if* $\mathcal{G}_k(S^4) \simeq \mathcal{G}_\ell(S^4)$.

Proof. Huang and Wu [13] showed that path-connected H-spaces, once localized at a prime p, satisfy a cancellation law. That is, if X and Y are path-connected H-spaces, $X \simeq A \times B$, $Y \simeq A \times C$ and it is known that $X \simeq Y$, then $B \simeq C$. In our case, note that a simply connected, simple compact Lie group is in fact 2-connected, so $\Omega^2 G$ is path-connected. The connectivity statements in Lemma 2.3 together with the decomposition in Theorem 2.1 therefore imply assertion (a) when G is one of $\mathrm{SU}(n)$ for $n \geq 3$, $\mathrm{Spin}(n)$ for $n \geq 6$, G_2, F_4, E_6, E_7 or E_8, and assertions (b) and (c) in all cases.

It remains to consider the cases when $\mathcal{G}_k(M)$ has two connected components. By Lemma 2.3, this occurs when M is Spin and G is one of $\mathrm{SU}(2)$, $\mathrm{Sp}(n)$ for $n \geq 1$ or $\mathrm{Spin}(n)$ for $n \in \{3, 5\}$. Let $\widetilde{G}_k(M)$ be the path component of $\mathcal{G}_k(M)$ containing the identity. Then $\widetilde{G}_k(M)$ is a topological subgroup and $\mathcal{G}_k(M)$ is homeomorphic to $\mathbb{Z}/2\mathbb{Z} \times \widetilde{G}_k(M)$. A homotopy equivalence $\mathcal{G}_k(M) \simeq \mathcal{G}_\ell(M)$ implies that the composite $\widetilde{G}_k(M) \to \mathcal{G}_k(M) \xrightarrow{\simeq} \mathcal{G}_\ell(M) \to \widetilde{G}_\ell(M)$ induces an isomorphism on π_m for all $m \geq 1$. Since $\widetilde{G}_k(M)$ and $\widetilde{G}_\ell(M)$ are path-connected H-spaces, Whitehead's theorem implies that this composite is a homotopy equivalence. Taking connected components in Theorem 2.1(a), we see that the homotopy equivalence $\widetilde{G}_k(M) \simeq \widetilde{G}_\ell(M)$ therefore induces a homotopy equivalence

$$\widetilde{G}_k(S^4) \times \prod_{i=1}^{d} \Omega^2 G \simeq \widetilde{G}_\ell(S^4) \times \prod_{i=1}^{d} \Omega^2 G.$$

Since $\widetilde{G}(S^4)$ and $\widetilde{G}_\ell(S^4)$ are path-connected, the cancellation law implies that $\widetilde{G}_k(S^4) \simeq \widetilde{G}_\ell(S^4)$ when localized at any prime p. But then $\mathcal{G}_k(S^4) \simeq \mathbb{Z}/2\mathbb{Z} \times \widetilde{G}_k(S^4)$ and $\mathcal{G}_\ell(S^4) \simeq \mathbb{Z}/2\mathbb{Z} \times \widetilde{G}_\ell(S^4)$ imply that $\mathcal{G}_k(S^4) \simeq \mathcal{G}_\ell(S^4)$. $\square$

Remark 2.6. If $G = \mathrm{Spin}(4)$ then by Remark 2.4 there are more connected component cases to consider, but exactly the same argument can be made as in the proof of Theorem 2.5 to prove analogous statements.

For the case of interest in this paper, $G = \mathrm{Sp}(2)$, the homotopy types of $\mathcal{G}_k(S^4)$ were determined in [28] rationally or p-locally for any prime p.

Theorem 2.7. *Let $G = \mathrm{Sp}(2)$. The following hold*:

(a) *if $\mathcal{G}_k(S^4) \simeq \mathcal{G}_\ell(S^4)$ then $(40, k) = (40, \ell)$*;

(b) *if $(40, k) = (40, \ell)$ then $\mathcal{G}_k(S^4) \simeq \mathcal{G}_\ell(S^4)$ when localized rationally or at any prime.* $\square$

Theorems 2.5 and 2.7 imply that the only case we need to consider to complete the proof of Theorem 1.1(b) is that of $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ at the prime 2. Resolving this case will take the remainder of the paper.

3. A METHOD FOR COUNTING THE HOMOTOPY TYPES OF GAUGE GROUPS

Return to the case of any simply connected, simple compact Lie group G and let $M = S^4$ or $\mathbb{C}\mathrm{P}^2$. The evaluation fibration (2.1) satisfies a naturality property. The fact that the pinch map $\mathbb{C}\mathrm{P}^2 \xrightarrow{\pi} S^4$ to the top cell induces a bijection $[S^4, BG] \cong [\mathbb{C}\mathrm{P}^2, BG]$ implies that it induces a one-to-one correspondence between the components $\mathrm{Map}_k(S^4, BG)$ and $\mathrm{Map}_k(\mathbb{C}\mathrm{P}^2, BG)$ and the components $\mathrm{Map}^*_k(S^4, BG)$ and $\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, BG)$. Moreover, it is well known that there is a homotopy equivalence $\mathrm{Map}^*_k(S^4, BG) \simeq \mathrm{Map}^*_0(S^4, BG)$ for every k, and by [26] there is also a compatible homotopy equivalence $\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, BG) \simeq \mathrm{Map}^*_0(\mathbb{C}\mathrm{P}^2, BG)$. Therefore, writing $\Omega^3_0 G$ for $\mathrm{Map}^*_0(S^4, BG)$, we obtain a homotopy fibration diagram

$$\begin{array}{ccccccc}
G & \xrightarrow{\partial_k} & \Omega^3_0 G & \longrightarrow & B\mathcal{G}_k(S^4) & \xrightarrow{\mathrm{ev}} & BG \\
\| & & \downarrow{\scriptstyle \pi^*} & & \downarrow & & \| \\
G & \xrightarrow{\overline{\partial}_k} & \mathrm{Map}^*_0(\mathbb{C}\mathrm{P}^2, BG) & \longrightarrow & B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) & \xrightarrow{\mathrm{ev}} & BG
\end{array} \tag{3.1}$$

The key to understanding the homotopy types of $\mathcal{G}_k(S^4)$ and $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ is understanding the homotopy classes of ∂_k and $\overline{\partial}_k$. The adjoint of ∂_k can be identified. Let $i\colon S^3 \to G$ be the inclusion of the bottom cell and let $1\colon G \to G$ be the identity map. Lang [21] identified the homotopy class of the triple adjoint of ∂_k as follows.

Lemma 3.1. *The adjoint of $G \xrightarrow{\partial_k} \Omega^3_0 G$ is homotopic to the Samelson product $S^3 \wedge G \xrightarrow{\langle ki, 1\rangle} G$. Consequently, the linearity of the Samelson product implies that $\partial_k \simeq k \circ \partial_1$.* $\square$

The order of ∂_1 plays an important role in determining the homotopy types of the gauge groups $\mathcal{G}_k(S^4)$. In [28] it was shown that if $G \xrightarrow{\partial_1} \Omega^3_0 G$ has order m then $(m, k) = (m, \ell)$ implies that $\mathcal{G}_k(S^4)$ is homotopy equivalent to $\mathcal{G}_\ell(S^4)$ when localized rationally or at any prime. There is an analogue for $\overline{\partial}_1$, but care is needed as it makes no sense to talk about the order of the map $\overline{\partial}_1$ since $\mathrm{Map}^*_0(\mathbb{C}\mathrm{P}^2, BG)$ need not be an H-space. Instead, the factorization of $\overline{\partial}_1$ through ∂_1 lets us consider the "order" of $\overline{\partial}_1$, by which we mean the least integer n such that the composite

$G \xrightarrow{\partial_1} \Omega_0^3 G \xrightarrow{n} \Omega_0^3 G \to \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, BG)$ is null homotopic. The following proposition, proved in [24], uses the "order" of $\overline{\partial}_1$ to estimate the number of homotopy types of the gauge groups $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$.

Proposition 3.2. *Suppose the map* $G \xrightarrow{\overline{\partial}_1} \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, BG)$ *has "order"* m. *If* $(m,k) = (m,\ell)$ *then* $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)$ *is homotopy equivalent to* $\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ *when localized rationally or at any prime.* □

The converse direction, showing that $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ implies $(m,k) = (m,\ell)$, tends to be more of a direct calculation using homotopy sets and the diagram (3.1). In Sections 4 and 5 we do the direct calculation when $G = \mathrm{Sp}(2)$. This is then used in Sections 6 and 7 to show that the "order" of $\overline{\partial}_1$ in the Sp(2) case is 20.

4. PRELIMINARY INFORMATION FOR PROVING THAT $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ IMPLIES $(m,k) = (m,\ell)$

Recall that $H_*(\mathrm{Sp}(2);\mathbb{Z}) \cong \Lambda(x_3, x_7)$ where x_i has degree i. So Sp(2) can be given a CW-structure with three cells, one in each of the dimensions 3, 7 and 10. (Here, and in what follows, we omit counting the 0-cell.) Let A be the 7-skeleton of Sp(2), so A has two cells, one each in dimensions 3 and 7. Let

$$s\colon\ A \to \mathrm{Sp}(2)$$

be the skeletal inclusion.

Three sequences of spaces will be used. The first is the homotopy fibration sequence

$$\Omega S^7 \xrightarrow{\delta} S^3 \xrightarrow{j} \mathrm{Sp}(2) \longrightarrow S^7 \tag{4.1}$$

where j is the usual group homomorphism from $S^3 \cong \mathrm{Sp}(1)$ to Sp(2). The second is the homotopy cofibration sequence

$$S^6 \xrightarrow{\nu'} S^3 \xrightarrow{i} A \xrightarrow{q} S^7 \xrightarrow{\Sigma\nu'} S^4 \tag{4.2}$$

where ν' is the attaching map for the top cell of A, i is the inclusion of the bottom cell and q is the pinch map to the top cell. Toda's notation is used for ν'; it represents a generator of $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$. Applying $\mathrm{Map}^*(\,\cdot\,, B\mathrm{Sp}(2))$ to the homotopy cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{C}\mathrm{P}^2 \longrightarrow S^4 \xrightarrow{\Sigma\eta} S^3$$

gives the third sequence of spaces, the homotopy fibration sequence

$$\Omega^2\mathrm{Sp}(2) \xrightarrow{(\Sigma\eta)^*} \Omega^3\mathrm{Sp}(2) \longrightarrow \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \longrightarrow \Omega\mathrm{Sp}(2) \xrightarrow{\eta^*} \Omega^2\mathrm{Sp}(2). \tag{4.3}$$

To simplify notation, let $N = \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$. Applying $[A, \cdot\,]$ to the homotopy fibration sequence (4.3) gives an exact sequence of pointed sets

$$[A, \Omega^2\mathrm{Sp}(2)] \xrightarrow{(\Sigma\eta)^*} [A, \Omega^3\mathrm{Sp}(2)] \longrightarrow [A, N] \longrightarrow [A, \Omega\mathrm{Sp}(2)] \xrightarrow{\eta^*} [A, \Omega^2\mathrm{Sp}(2)]. \tag{4.4}$$

Note that, from left to right, the first two terms and the last two terms in this exact sequence are groups and the maps $(\Sigma\eta)^*$ and η^* are group homomorphisms. But $[A,N]$ is only a set since $N = \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ may not be an H-space.

In Proposition 4.15, $[A,N]$ will be calculated 2-locally and we will show that it does in fact have a group structure (although not one coming from the ingredient spaces) and it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. From now on, assume that all spaces, maps and modules have been localized at 2. The starting point is the following result of Choi, Hirato and Mimura [3].

Lemma 4.1. *There is an isomorphism* $[A, \Omega^3\mathrm{Sp}(2)] \cong \mathbb{Z}/8\mathbb{Z}$, *and a representative of the generator is the composite* $A \xrightarrow{s} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Sp}(2)$. □

Given Lemma 4.1, to calculate $[A,N]$ in (4.4) we determine the image of $(\Sigma\eta)^*$ and the kernel of η^*. To do so we first collect some information on the homotopy groups of S^3, S^7 and $\mathrm{Sp}(2)$. For $m\geq 3$, let $\eta_m\colon S^{m+1}\to S^m$ be $\Sigma^{m-2}\eta$.

Lemma 4.2 (Toda [32]). *In the relevant dimensions the groups $\pi_i(S^3)$ have the following 2-components and generators*:

i:	5	6	8	9	10
2-component:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
Generator:	η_3^2	ν'	$\nu'\eta_6^2$		

Further, by [32, Lemma 5.7] *the composite* $S^8\xrightarrow{\Sigma^2\nu'}S^5\xrightarrow{\eta_3^2}S^3$ *is null homotopic.*

In the relevant dimensions the groups $\pi_i(S^7)$ have the following 2-components and generators:

i:	7	10	11
2-component:	$\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$	0
Generator:	ι_7	ν_7	

Further, $2\cdot\nu_7\simeq\Sigma^4\nu'$. □

Lemma 4.3 (Mimura and Toda [23]). *In the relevant dimensions, the groups $\pi_i(\mathrm{Sp}(2))$ have the following 2-components*:

i:	4	5	6	7	8	9	10	13
2-component:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$

Further, the map $S^5\xrightarrow{\eta_4}S^4$ *induces an isomorphism* $\pi_4(\mathrm{Sp}(2))\xrightarrow{\cong}\pi_5(\mathrm{Sp}(2))$. □

We first aim towards Proposition 4.6, which describes the map η^* in (4.4).

Lemma 4.4. *The map* $S^3\xrightarrow{i}A$ *induces an isomorphism* $[A,\Omega\mathrm{Sp}(2)]\xrightarrow{i^*}[S^3,\Omega\mathrm{Sp}(2)]$.

Proof. The homotopy cofibration sequence (4.2) induces an exact sequence of groups

$$[S^7,\Omega\mathrm{Sp}(2)]\xrightarrow{q^*}[A,\Omega\mathrm{Sp}(2)]\xrightarrow{i^*}[S^3,\Omega\mathrm{Sp}(2)]\xrightarrow{(\nu')^*}[S^6,\Omega\mathrm{Sp}(2)].$$

Identifying the first, third and fourth terms using Lemma 4.3, we obtain an exact sequence

$$0\xrightarrow{q^*}[A,\Omega\mathrm{Sp}(2)]\xrightarrow{i^*}\mathbb{Z}/2\mathbb{Z}\xrightarrow{(\nu')^*}\mathbb{Z}.$$

Since $\mathbb{Z}$ is torsion-free, $(\nu')^*$ must be the zero map. The lemma now follows. □

Lemma 4.5. *The map* $S^3\xrightarrow{i}A$ *induces an isomorphism* $[A,\Omega^2\mathrm{Sp}(2)]\xrightarrow{i^*}[S^3,\Omega^2\mathrm{Sp}(2)]$.

Proof. The homotopy cofibration (4.2) induces an exact sequence of groups

$$[S^7,\Omega^2\mathrm{Sp}(2)]\xrightarrow{q^*}[A,\Omega^2\mathrm{Sp}(2)]\xrightarrow{i^*}[S^3,\Omega^2\mathrm{Sp}(2)]\xrightarrow{(\nu')^*}[S^6,\Omega^2\mathrm{Sp}(2)].$$

Identifying the first, third and fourth terms using Lemma 4.3, we obtain an exact sequence

$$0\xrightarrow{q^*}[A,\Omega^2\mathrm{Sp}(2)]\xrightarrow{i^*}\mathbb{Z}/2\mathbb{Z}\xrightarrow{(\nu')^*}0.$$

Therefore, i^* is an isomorphism. □

Proposition 4.6. *The map* $\Omega\mathrm{Sp}(2)\xrightarrow{\eta^*}\Omega^2\mathrm{Sp}(2)$ *induces an isomorphism* $[A,\Omega\mathrm{Sp}(2)]\xrightarrow{\eta^*}[A,\Omega^2\mathrm{Sp}(2)]$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} [A,\Omega\mathrm{Sp}(2)] & \xrightarrow{a} & [S^3,\Omega\mathrm{Sp}(2)] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\ [A,\Omega^2\mathrm{Sp}(2)] & \xrightarrow{c} & [S^3,\Omega^2\mathrm{Sp}(2)] \end{array} \tag{4.5}$$

Here, a and c are induced by i while b and d are induced by η^*. By Lemma 4.4, a is an isomorphism; by Lemma 4.5, c is an isomorphism. Observe that d is the map $\pi_4(\mathrm{Sp}(2)) \to \pi_5(\mathrm{Sp}(2))$ induced by precomposing with $S^5 \xrightarrow{\eta_4} S^4$, which is an isomorphism by Lemma 4.3. Thus, d is an isomorphism. The commutativity of (4.5) therefore implies that b is an isomorphism as well, proving the proposition. $\square$

Next, we aim towards Proposition 4.14, which describes the map $(\Sigma\eta)^*$ in (4.4).

Lemma 4.7. *The map $S^3 \xrightarrow{i} A$ induces an isomorphism $[A,\Omega^2S^3] \xrightarrow{i^*} [S^3,\Omega^2S^3] \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. The homotopy cofibration (4.2) induces an exact sequence of groups

$$[S^7,\Omega^2S^3] \xrightarrow{q^*} [A,\Omega^2S^3] \xrightarrow{i^*} [S^3,\Omega^2S^3] \xrightarrow{(\nu')^*} [S^6,\Omega^2S^3].$$

Identifying the first, third and fourth terms using Lemma 4.2, we obtain an exact sequence

$$0 \xrightarrow{q^*} [A,\Omega^2S^3] \xrightarrow{i^*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\nu')^*} \mathbb{Z}/2\mathbb{Z}.$$

The generator of $\pi_5(S^3)$ is η_3^2, so $(\nu')^*(\eta_3^2)$ is the composite $S^8 \xrightarrow{\Sigma^2\nu'} S^5 \xrightarrow{\eta_3^2} S^3$, which is null homotopic by Lemma 4.2. Thus, $(\nu')^*$ is the zero map. The lemma now follows. $\square$

Lemma 4.8. *The map $S^3 \xrightarrow{i} A$ induces multiplication by 4 in the map $\mathbb{Z} \cong [A,\Omega^4S^7] \xrightarrow{i^*} [S^3,\Omega^4S^7] \cong \mathbb{Z}$.*

Proof. The homotopy cofibration (4.2) induces an exact sequence of groups

$$[S^7,\Omega^4S^7] \xrightarrow{q^*} [A,\Omega^4S^7] \xrightarrow{i^*} [S^3,\Omega^4S^7] \xrightarrow{(\nu')^*} [S^6,\Omega^4S^7].$$

Identifying the first, third and fourth terms using Lemma 4.2, we obtain an exact sequence

$$0 \xrightarrow{q^*} [A,\Omega^4S^7] \xrightarrow{i^*} \mathbb{Z} \xrightarrow{(\nu')^*} \mathbb{Z}/8\mathbb{Z}.$$

Thus, $[A,\Omega^4S^7]$ is isomorphic to the kernel of $(\nu')^*$. To determine this kernel, observe that the generator of $[S^3,\Omega^4S^7] \cong \pi_7(S^7) \cong \mathbb{Z}$ is ι_7 while the generator of $[S^6,\Omega^4S^7] \cong \mathbb{Z}/8\mathbb{Z}$ is ν_7. By Lemma 4.2, $2\nu_7 \simeq \Sigma^4\nu'$ so we obtain $(\Sigma\nu')^*(\iota_7) = \iota_7 \circ \Sigma^4\nu' \simeq \Sigma^4\nu' \simeq 2\nu_7$. Therefore the kernel of $(\nu')^*$ is isomorphic to $\mathbb{Z}$ and is generated by $4\iota_7$. $\square$

Lemma 4.9. *The map $S^3 \xrightarrow{j} \mathrm{Sp}(2)$ induces an isomorphism $[A,\Omega^2S^3] \xrightarrow{j_*} [A,\Omega^2\mathrm{Sp}(2)]$.*

Proof. The maps $S^3 \xrightarrow{i} A$ and $S^3 \xrightarrow{j} \mathrm{Sp}(2)$ induce a commutative diagram

$$\begin{array}{ccc} [A,\Omega^2S^3] & \xrightarrow{a} & [A,\Omega^2\mathrm{Sp}(2)] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\ [S^3,\Omega^2S^3] & \xrightarrow{c} & [S^3,\Omega^2\mathrm{Sp}(2)] \end{array} \tag{4.6}$$

By Lemma 4.7, the map b is an isomorphism. By [23], j induces an isomorphism $\pi_5(S^3) \to \pi_5(\mathrm{Sp}(2))$, so the map c is an isomorphism. By Lemma 4.5, d is an isomorphism. Hence in (4.6) the maps b, c and d are isomorphisms, implying that a is as well. □

Lemma 4.10. *The fibration connecting map $\Omega S^7 \xrightarrow{\delta} S^3$ induces the zero map on the homotopy sets $[A, \Omega^4 S^7] \xrightarrow{\delta_*} [A, \Omega^3 S^3]$.*

Proof. The maps $S^3 \xrightarrow{i} A$ and $\Omega S^7 \xrightarrow{\delta} S^3$ induce a commutative diagram

$$\begin{array}{ccccccc} & & [A, \Omega^4 S^7] & \xrightarrow{a} & [S^3, \Omega^4 S^7] & & \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} & & \\ [S^7, \Omega^3 S^3] & \longrightarrow & [A, \Omega^3 S^3] & \xrightarrow{c} & [S^3, \Omega^3 S^3] & \longrightarrow & [S^6, \Omega^3 S^3] \end{array} \qquad (4.7)$$

The composite $d \circ a$ is null homotopic since a is multiplication by 4 by Lemma 4.8 and $[S^3, \Omega^3 S^3] \cong \mathbb{Z}/4\mathbb{Z}$ by Lemma 4.2. Also, the first and the last terms in the bottom row are zero by Lemma 4.2, so c is an isomorphism and b is a zero map. □

Corollary 4.11. *The map $S^3 \xrightarrow{j} \mathrm{Sp}(2)$ induces a monomorphism $[A, \Omega^3 S^3] \xrightarrow{j_*} [A, \Omega^3 \mathrm{Sp}(2)]$.*

Proof. The homotopy fibration (4.1) induces an exact sequence $[A, \Omega^4 S^7] \xrightarrow{\delta_*} [A, \Omega^3 S^3] \xrightarrow{j_*} [A, \Omega^3 \mathrm{Sp}(2)]$. By Lemma 4.10, δ_* is the zero map. Hence j_* is a monomorphism. □

Lemma 4.12. *The map $\Omega^2 S^3 \xrightarrow{(\Sigma\eta)^*} \Omega^3 S^3$ induces an injection $[A, \Omega^2 S^3] \xrightarrow{(\Sigma\eta)^*} [A, \Omega^3 S^3]$.*

Proof. The maps $\Omega^2 S^3 \xrightarrow{(\Sigma\eta)^*} \Omega^3 S^3$ and $A \xrightarrow{i} S^3$ induce a commutative diagram

$$\begin{array}{ccc} & & [S^7, \Omega^3 S^3] \\ & & \downarrow \\ [A, \Omega^2 S^3] & \xrightarrow{a} & [A, \Omega^3 S^3] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\ [S^3, \Omega^2 S^3] & \xrightarrow{c} & [S^3, \Omega^3 S^3] \end{array}$$

By Lemma 4.9, b is an isomorphism. In the bottom row, $\pi_5(S^3)$ is generated by η_3^2 and $\pi_6(S^3)$ is generated by ν'. The map c sends η_3^2 to η_3^3, which is $2\nu'$ by [32], so it is injective. In the right column, $\pi_{10}(S^3)$ is trivial by Lemma 4.2, so d is injective. This implies that a is also injective. □

Lemma 4.13. *The map $\Omega^2 \mathrm{Sp}(2) \xrightarrow{(\Sigma\eta)^*} \Omega^3 \mathrm{Sp}(2)$ induces an injection $[A, \Omega^2 \mathrm{Sp}(2)] \xrightarrow{(\Sigma\eta)^*} [A, \Omega^3 \mathrm{Sp}(2)]$.*

Proof. The map $\Omega^2 X \xrightarrow{(\Sigma\eta)^*} \Omega^3 X$ is natural with respect to maps $X \to Y$, so applying this to the case $S^3 \xrightarrow{j} \mathrm{Sp}(2)$ and taking homotopy sets $[A, \cdot\,]$ gives a commutative diagram

$$\begin{array}{ccc} [A, \Omega^2 S^3] & \xrightarrow{a} & [A, \Omega^3 S^3] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\ [A, \Omega^2 \mathrm{Sp}(2)] & \xrightarrow{c} & [A, \Omega^3 \mathrm{Sp}(2)] \end{array} \qquad (4.8)$$

Here, a and c are induced by $(\Sigma\eta)^*$ while b and d are induced by j_*. By Lemma 4.9, b is an isomorphism and, by Corollary 4.11, d is an injection. By Lemma 4.12, a is an injection. Therefore the commutativity of (4.8) implies that c is also an injection. □

Now we identify the injection in Lemma 4.13.

Proposition 4.14. *The injection* $[A,\Omega^2\mathrm{Sp}(2)] \xrightarrow{(\Sigma\eta)^*} [A,\Omega^3\mathrm{Sp}(2)]$ *in Lemma* 4.13 *is the injection* $\mathbb{Z}/2\mathbb{Z}\to\mathbb{Z}/8\mathbb{Z}$.

Proof. By Lemmas 4.7 and 4.9, $[A,\Omega^2\mathrm{Sp}(2)]\cong\mathbb{Z}/2\mathbb{Z}$, and by Lemma 4.1, $[A,\Omega^3\mathrm{Sp}(2)]\cong\mathbb{Z}/8\mathbb{Z}$. Since $[A,\Omega^2\mathrm{Sp}(2)]\to[A,\Omega^3\mathrm{Sp}(2)]$ is an injection, the proposition follows. □

Finally, we calculate $[A,N]$.

Proposition 4.15. *There is an isomorphism of sets* $[A,N]\cong\mathbb{Z}/4\mathbb{Z}$, *and the map* $\Omega^3_0\mathrm{Sp}(2)\to N$ *induces an epimorphism* $\mathbb{Z}/8\mathbb{Z}\cong[A,\Omega^3\mathrm{Sp}(2)]\to[A,N]\cong\mathbb{Z}/4\mathbb{Z}$. *In particular, this epimorphism induces a group structure on* $[A,N]$.

Proof. By Propositions 4.6 and 4.14, the exact sequence in (4.4) simplifies to an exact sequence

$$0\to\mathbb{Z}/2\mathbb{Z}\to\mathbb{Z}/8\mathbb{Z}\cong[A,\Omega^3\mathrm{Sp}(2)]\to[A,N]\to 0.$$

The proposition follows immediately. □

5. THE PROOF THAT $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)\simeq\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ IMPLIES $(20,k)=(20,\ell)$

A key ingredient is Theorem 5.5. To simplify notation, let $N_k=\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2,B\mathrm{Sp}(2))$. Applying $[A,\cdot\,]$ to (3.1) in the $G=\mathrm{Sp}(2)$ case we obtain a commutative diagram

$$\begin{array}{ccccccc}
[A,\mathrm{Sp}(2)] & \xrightarrow{(\partial_k)_*} & [A,\Omega^3_0\mathrm{Sp}(2)] & \longrightarrow & [A,B\mathcal{G}_k(S^4)] & \longrightarrow & [A,B\mathrm{Sp}(2)]\\
\Vert & & \downarrow & & \downarrow & & \Vert\\
[A,\mathrm{Sp}(2)] & \xrightarrow{(\overline{\partial}_k)_*} & [A,N_0] & \longrightarrow & [A,B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] & \longrightarrow & [A,B\mathrm{Sp}(2)]
\end{array}\tag{5.1}$$

Since A is connected, there is an isomorphism $[A,N_0]\cong[A,N]$, where $N=\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2,B\mathrm{Sp}(2))$ and $[A,N]$ was calculated in Proposition 4.15. By [26], $N_k\simeq N_0$ for every $k\in\mathbb{Z}$, so $[A,N_k]\cong[A,N]$ as well.

Lemma 5.1. *The image of* $(\overline{\partial}_k)_*$ *is* $\mathbb{Z}/(4/(4,k))\,\mathbb{Z}$.

Proof. By Lemma 4.1, $[A,\Omega^3_0\mathrm{Sp}(2)]\cong\mathbb{Z}/8\mathbb{Z}$ and the composite $A\xrightarrow{s}\mathrm{Sp}(2)\xrightarrow{\partial_1}\Omega^3_0\mathrm{Sp}(2)$ represents a generator. By Lemma 3.1, $\partial_k\simeq k\cdot\partial_1$, so the image of $(\partial_k)_*$ in (5.1) is $\mathbb{Z}/(8/(8,k))\,\mathbb{Z}$. By Proposition 4.15, the map $[A,\Omega^3_0\mathrm{Sp}(2)]\to[A,N_0]$ is reduction mod 4. Thus the commutativity of the left square in (5.1) implies that the image of $(\overline{\partial}_k)_*$ is $\mathbb{Z}/(4/(4,k))\,\mathbb{Z}$. □

Lemma 5.2. *There is an isomorphism* $[A,B\mathrm{Sp}(2)]\cong 0$.

Proof. The dimension of A is 7 and the map $B\mathrm{Sp}(2)\to B\mathrm{Sp}(\infty)$ induced by the standard inclusion of $\mathrm{Sp}(2)$ into $\mathrm{Sp}(\infty)$ is 10-connected. Therefore, $[A,B\mathrm{Sp}(2)]\cong[A,B\mathrm{Sp}(\infty)]$. But $[A,B\mathrm{Sp}(\infty)]$ is $\widetilde{K}_{\mathrm{Sp}}(A)$, the reduced symplectic K-theory of A. Since $\widetilde{K}_{\mathrm{Sp}}(S^{4m-1})=0$ for every $m\ge 1$, applying $\widetilde{K}_{\mathrm{Sp}}$ to the homotopy cofibration $S^3\to A\to S^7$ shows that $\widetilde{K}_{\mathrm{Sp}}(A)\cong 0$. □

Proposition 5.3. *There is an isomorphism of sets* $[A,B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]\cong\mathbb{Z}/(4,k)\,\mathbb{Z}$.

Proof. By Lemma 5.2 the exact sequence in the bottom row of (5.1) is

$$[A,\mathrm{Sp}(2)]\xrightarrow{(\overline{\partial}_k)^*}[A,\Omega^3_0\mathrm{Sp}(2)]\longrightarrow[A,B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]\longrightarrow 0.$$

Thus, $[A,B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ is the cokernel of $(\overline{\partial}_k)^*$. By Lemma 5.1, this cokernel is $\mathbb{Z}/(4,k)\,\mathbb{Z}$. □

What we would like to say is that a homotopy equivalence $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)\simeq\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ implies that there is a bijection of sets $[A,B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]\cong[A,B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)]$. This would imply that $(4,k)=(4,\ell)$ and prove

Theorem 1.1(b). However, since A is not a suspension, and possibly not even a co-H-space, the desired implication is not immediate through adjunction. Instead, a different argument is needed.

Lemma 5.4. *For any $k \in \mathbb{Z}$ we have $\pi_3(B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)) \cong 0$.*

Proof. There is a homotopy fibration

$$\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \to B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \to B\mathrm{Sp}(2) \tag{5.2}$$

and the homotopy cofibration sequence $S^3 \xrightarrow{\eta} S^2 \to \mathbb{C}\mathrm{P}^2 \to S^4$ induces a homotopy fibration sequence

$$\Omega^3_0\mathrm{Sp}(2) \longrightarrow \mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \longrightarrow \Omega\mathrm{Sp}(2) \xrightarrow{\eta^*} \Omega^2\mathrm{Sp}(2). \tag{5.3}$$

Apply π_3 to (5.3). By Lemma 4.3, $\pi_3(\Omega^3_0\mathrm{Sp}(2)) \cong \pi_6(\mathrm{Sp}(2)) \cong 0$, and $\pi_3(\eta^*)$ is the same as the map $\pi_4(\mathrm{Sp}(2)) \xrightarrow{(\Sigma^2\eta)^*} \pi_5(\mathrm{Sp}(2))$, which is an isomorphism. Hence, $\pi_3(\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong 0$. Now apply π_3 to (5.2). By connectivity, $\pi_3(B\mathrm{Sp}(2)) \cong 0$ so as $\pi_3(\mathrm{Map}^*_k(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong 0$ we obtain $\pi_3(B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)) \cong 0$. $\square$

Consider the homotopy cofibration sequence $S^3 \to A \xrightarrow{q} S^7 \xrightarrow{\Sigma\nu'} S^4$. Applying $[\,\cdot\,, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ gives an exact sequence of pointed sets

$$[S^4, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \xrightarrow{(\Sigma\nu')^*} [S^7, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \xrightarrow{q^*} [A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \longrightarrow [S^3, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)].$$

By Lemma 5.4, $\pi_3(B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)) \cong 0$, so we really have an exact sequence of pointed sets

$$[S^4, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \xrightarrow{(\Sigma\nu')^*} [S^7, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \xrightarrow{q^*} [A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \longrightarrow 0. \tag{5.4}$$

Note that $[S^4, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ and $[S^7, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ are groups and $(\Sigma\nu')^*$ is a group homomorphism, but $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ is only a set since A may not be a co-H-space, and q^* is therefore only a map of sets. So exactness and the fact that q^* is an epimorphism imply that there is a bijection between the set $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ and the group $\mathrm{coker}(\Sigma\nu')^*$.

Now we prove Theorem 5.5, which is stated integrally.

Theorem 5.5. *If $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ then $(20, k) = (20, \ell)$.*

Proof. By Theorems 2.5 and 2.7(a) it suffices to prove the 2-components of the g.c.d. conditions. That is, it suffices to prove that if there is a 2-local homotopy equivalence $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ then $(4, k) = (4, \ell)$.

Let $e\colon \mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \to \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ be a homotopy equivalence. Then for any path-connected space X there is an isomorphism $[\Sigma X, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \cong [X, \mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ which is natural for maps $\Sigma X \xrightarrow{\Sigma f} \Sigma X'$. Applying this to (5.4), together with the map e, gives a commutative diagram

$$\begin{array}{ccccccc}
[S^4, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] & \xrightarrow{(\Sigma\nu')^*} & [S^7, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] & \xrightarrow{q^*} & [A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & & & \\
[S^4, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)] & \xrightarrow{(\Sigma\nu')^*} & [S^7, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)] & \xrightarrow{q^*} & [A, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)] & \longrightarrow & 0
\end{array} \tag{5.5}$$

where the top and bottom rows are exact and the vertical isomorphisms are induced by adjunction and applying e_*. In the top row, $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ bijects with $\mathrm{coker}(\Sigma\nu')^*$, and in the bottom row, $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ bijects with $\mathrm{coker}(\Sigma\nu')^*$. Therefore there is a bijection between $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ and $[A, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)]$.

By Proposition 5.3, $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)] \cong \mathbb{Z}/(4,k)\,\mathbb{Z}$ and similarly $[A, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)] \cong \mathbb{Z}/(4,\ell)\,\mathbb{Z}$. Therefore the bijection between $[A, B\mathcal{G}_k(\mathbb{C}\mathrm{P}^2)]$ and $[A, B\mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)]$ implies that $(4, k) = (4, \ell)$. $\square$

6. PRELIMINARY INFORMATION FOR THE "ORDER" OF $\overline{\partial}_1$

As a CW-complex, Sp(2) has three cells and there are homotopy cofibrations

$$S^6 \xrightarrow{\nu'} S^3 \to A, \qquad S^9 \xrightarrow{f} A \to \mathrm{Sp}(2)$$

for some map f. Following Toda's notation [32] for the homotopy groups of spheres, let ν_4 represent the generator of the free part of $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$, and by definition let $\nu_m = \Sigma^{m-4}\nu_4$ for $m \geq 5$. In particular, $\nu_4^2 = \nu_4 \circ \nu_7$. According to James [14] and Mimura [22], Σf factors as the composite $S^{10} \xrightarrow{2\nu_4^2} S^4 \to \Sigma A$.

Lemma 6.1. *There is a homotopy cofibration sequence*

$$S^3 \xrightarrow{j} \mathrm{Sp}(2) \longrightarrow S^7 \vee S^{10} \xrightarrow{\Sigma\nu' - g} S^4,$$

where $g = t\nu_4^2 \in \mathbb{Z}/8\mathbb{Z}$ with $t = 2u$ or $4u$ for some unit $u \in \mathbb{Z}/8\mathbb{Z}$.

Proof. Define the space C and the map δ by the homotopy cofibration sequence

$$S^3 \xrightarrow{j} \mathrm{Sp}(2) \longrightarrow C \xrightarrow{\delta} S^4. \tag{6.1}$$

In [28] it was shown that there is a homotopy equivalence $e\colon S^7 \vee S^{10} \to C$. It remains to identify δ. Restricting (6.1) to 7-skeletons gives the homotopy cofibration sequence $S^3 \xrightarrow{i} A \to S^7 \xrightarrow{\Sigma\nu'} S^4$. Thus, $\delta \circ e$ restricted to S^7 is $\Sigma\nu'$. Let r be the restriction of $\delta \circ e$ to S^{10} so we have a homotopy cofibration sequence

$$S^3 \xrightarrow{j} \mathrm{Sp}(2) \longrightarrow S^7 \vee S^{10} \xrightarrow{\Sigma\nu' + r} S^4.$$

As described in [22, p. 475], $\Sigma\mathrm{Sp}(2) = \Sigma A \cup e^{11}$ where the 11-cell is attached by $\Sigma i \circ 2\nu_4^2$. In particular, $\Sigma\mathrm{Sp}(2) \not\simeq \Sigma A \vee S^{11}$, and as $\Sigma(2\nu_4^2)$ is null homotopic, $\Sigma^2\mathrm{Sp}(2) \simeq \Sigma^2 A \vee S^{12}$. Now consider r. By [32], $\pi_{10}(S^4) \cong \mathbb{Z}/8\mathbb{Z}$ and is generated by ν_4^2. If r generates $\pi_{10}(S^4)$ then, up to multiplication by a unit, it is a multiple of ν_4^2. But ν_4^2 is stable, implying that the homotopy cofibre of $\Sigma\nu' + r$—that is, $\Sigma\mathrm{Sp}(2)$—does not split stably as $\Sigma^2 A \vee S^{12}$, a contradiction. If r is null homotopic then the homotopy cofibre of $\Sigma\nu' + r$—that is, $\Sigma\mathrm{Sp}(2)$—is homotopy equivalent to $\Sigma A \vee S^{11}$, a contradiction. Thus the remaining options are $r \simeq t\nu_4^2$ where $t = 2u'$ or $t = 4u'$ for some unit $u' \in \mathbb{Z}/8\mathbb{Z}$. Taking $g = -r$ with $u = -u'$ then gives the asserted statement. $\square$

By [28], there is a homotopy commutative diagram

$$\begin{array}{ccc}
\mathrm{Sp}(2) & \xrightarrow{\partial_1} & \Omega_0^3\mathrm{Sp}(2) \\
\downarrow & \nearrow_{a+b} & \\
S^7 \vee S^{10} & &
\end{array}$$

where a represents a generator of $\pi_7(\Omega_0^3\mathrm{Sp}(2)) \cong \pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$ and b represents some class in $\pi_{10}(\Omega_0^3\mathrm{Sp}(2)) \cong \pi_{13}(\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since b has order at most 4, we immediately obtain the following.

Lemma 6.2. *There is a homotopy commutative diagram*

$$\begin{array}{ccccc}
\mathrm{Sp}(2) & & \xrightarrow{\quad\partial_1\quad} & & \Omega_0^3\mathrm{Sp}(2) \\
\downarrow & & & & \downarrow{\scriptstyle 4} \\
S^7 \vee S^{10} & \xrightarrow{p_1} & S^7 & \xrightarrow{4a} & \Omega_0^3\mathrm{Sp}(2)
\end{array}$$

where p_1 is the pinch map onto the left wedge summand. $\square$

7. DETERMINING THE "ORDER" OF $\overline{\partial}_1$

Start with the homotopy cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{C}\mathrm{P}^2 \longrightarrow S^4 \xrightarrow{\Sigma\eta} S^3 \tag{7.1}$$

and the homotopy fibration

$$S^3 \to \mathrm{Sp}(2) \to S^7. \tag{7.2}$$

Applying the pointed mapping space functor $\mathrm{Map}^*(\,\cdot\,,\cdot\,)$ with the spaces in the left variable coming from (7.1) and the spaces in the right variable coming from (7.2), and writing $\mathrm{Map}^*(S^t, X)$ as $\Omega^t X$, we obtain a homotopy fibration diagram

$$\begin{array}{ccccccccc} \Omega^3 S^3 & \xrightarrow{(\Sigma\eta)^*} & \Omega^4 S^3 & \longrightarrow & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^3) & \longrightarrow & \Omega^2 S^3 & \xrightarrow{\eta^*} & \Omega^3 S^3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^3 \mathrm{Sp}(2) & \xrightarrow{(\Sigma\eta)^*} & \Omega^4 \mathrm{Sp}(2) & \longrightarrow & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2)) & \longrightarrow & \Omega^2 \mathrm{Sp}(2) & \xrightarrow{\eta^*} & \Omega^3 \mathrm{Sp}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^3 S^7 & \xrightarrow{(\Sigma\eta)^*} & \Omega^4 S^7 & \longrightarrow & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^7) & \longrightarrow & \Omega^2 S^7 & \xrightarrow{\eta^*} & \Omega^3 S^7 \end{array} \tag{7.3}$$

For the remainder of the section, localize all spaces, maps and modules at 2.

Lemma 7.1. *There is an isomorphism* $\pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) \cong \mathbb{Z}$ *and the map* $\Omega^4\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$ *induces* $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ *on* π_3.

Proof. Apply π_3 to (7.3), using the homotopy group information for spheres in [32] and $\mathrm{Sp}(2)$ in [23], to obtain a diagram of exact sequences

$$\begin{array}{ccccccccc} \mathbb{Z}/4\mathbb{Z} & \xrightarrow{(\Sigma\eta)^*} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{a} & \pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^3)) & \xrightarrow{b} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\eta^*} & \mathbb{Z}/4\mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle c} & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d} & \pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{g} & \pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^7)) & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{7.4}$$

where the maps $a, b, \ldots, g$ are to be determined.

First, consider the map a. By [32], $\pi_3(\Omega^3 S^3) \cong \mathbb{Z}/4\mathbb{Z}$ and $\pi_3(\Omega^4 S^3) \cong \mathbb{Z}/2\mathbb{Z}$ are generated by the adjoints of ν' and $\nu' \circ \eta$, respectively. As $(\Sigma\eta)^*$ precomposes with η, this map is onto. Hence $a = 0$. Second, consider the map b. By [32], $\pi_3(\Omega^2 S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the adjoint of η^2, so η^* sends this to the adjoint of η^3. But this equals $2\nu'$, which is non-trivial in $\pi_3(\Omega^3 S^3)$, so the map b is an injection. Hence exactness along the top row in (7.4) implies that $\pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^3)) \cong 0$. This then implies that the map f is an injection. Third, exactness along the bottom row in (7.4) implies that g is an isomorphism.

Finally, consider the middle row in (7.4). By exactness, either $\pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2)))$ is isomorphic to $\mathbb{Z}$ with $d = 2$ or it is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with d being the inclusion of the integral summand. But as f is an injection and $\pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^7)) \cong \mathbb{Z}$, it must be the first possibility that holds. Hence $\pi_3(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) \cong \mathbb{Z}$ and the map $\Omega^4\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$ induces $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ on π_3. $\square$

Lemma 7.2. *There is an isomorphism* $\pi_6(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) \cong \mathbb{Z}/8\mathbb{Z}$, *and the map* $\Omega^4\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$ *induces an isomorphism on* π_6.

Proof. By [23], $\pi_6(\Omega^3\mathrm{Sp}(2)) \cong 0$ and $\pi_6(\Omega^2\mathrm{Sp}(2)) \cong 0$, so when π_6 is applied to the homotopy fibration in the middle row of (7.3) we obtain an isomorphism $\pi_6(\Omega^4\mathrm{Sp}(2)) \to \pi_6(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2)))$. By [23], $\pi_6(\Omega^4\mathrm{Sp}(2)) \cong \mathbb{Z}/8\mathbb{Z}$. $\square$

Lemma 7.3. *The following hold*:

(a) *there is an isomorphism* $\pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) \cong \mathbb{Z}/4\mathbb{Z}$;

(b) *the map* $\Omega^4\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$ *sends the order* 4 *generator of* $\pi_9(\Omega^4\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *to an order* 4 *generator of* $\pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2)))$.

Proof. Apply π_9 to the top two rows of (7.3) and use the homotopy group information for spheres in [32] and Sp(2) in [23] to obtain a diagram of exact sequences

$$\begin{array}{ccccccccc}
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{a} & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{b} & \pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, S^3)) & \xrightarrow{c} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{d} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle e} \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{i} & \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{j} & \pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) & \xrightarrow{k} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\ell} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}
\end{array} \tag{7.5}$$

where the maps $a, b, \dots, \ell$ are to be determined. Note the two maps labelled e are the same, that is, they are both $\pi_9(\Omega^3 S^3) \to \pi_9(\Omega^3\mathrm{Sp}(2))$.

We aim to show that j is onto, it is an injection on the $\mathbb{Z}/4\mathbb{Z}$ summand and it is trivial on the $\mathbb{Z}/2\mathbb{Z}$ summand. These properties then imply that $\pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))) \cong \mathbb{Z}/4\mathbb{Z}$ and that the map $\Omega^4\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$ sends the order 4 generator of $\pi_9(\Omega^4\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to an order 4 generator of $\pi_9(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2)))$.

First, by [23], the map $S^3 \to \mathrm{Sp}(2)$ induces an isomorphism on π_{11} and π_{12}, implying that h and e respectively are isomorphisms. In fact, the generators of $\pi_{11}(\mathrm{Sp}(2))$ and $\pi_{12}(\mathrm{Sp}(2))$ are chosen as the images of those from $\pi_{11}(S^3)$ and $\pi_{12}(S^3)$ respectively, so e and h may be taken to be identity maps. As well, Theorem 5.1 and relation (5.1) of [23] imply that the map f is $2 \oplus \mathrm{id}$, where id is the identity map.

Consider the map a. Explicitly, a is the map $\pi_9(\Omega^3 S^3) \xrightarrow{\eta^*} \pi_9(\Omega^4 S^3)$. Taking adjoints, we find that a sends a map $S^{12} \xrightarrow{t} S^3$ to the composite $S^{13} \xrightarrow{\eta_{12}} S^{12} \xrightarrow{t} S^3$. By [32], the generators for $\pi_{12}(S^3) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are $\eta_3\epsilon_4$ and μ_3 and the generators for $\pi_{13}(S^3) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are ϵ' and $\eta_3\mu_4$, of orders 4 and 2 respectively. Thus, $a(\eta_3\epsilon_4) = \eta_3\epsilon_4\eta_{12}$ and $a(\mu_3) = \mu_3\eta_{12}$, neither of which are immediately generators of $\pi_{13}(S^3)$. In general, two maps $u\colon S^m \to S^n$ and $v\colon S^{m'} \to S^{n'}$ satisfy $\Sigma^n v \circ \Sigma^{m'} u \simeq \pm\Sigma^{n'} u \circ \Sigma^m v$. In our case, we obtain $\epsilon_5\eta_{13} \simeq \eta_5\epsilon_6$ and $\mu_5\eta_{14} \simeq \eta_5\mu_6$, where the signs have disappeared since all composites are of order 2. Thus after two suspensions the image of a is generated by $\eta_5\epsilon_6\eta_{14} = \eta_5^2\epsilon_7$ and $\mu_5\eta_{14} = \eta_5\mu_6$. By [32, Lemma 6.6], $\eta_5^2\epsilon_7 = 2\Sigma^2\epsilon'$, and by the proof of [32, Theorem 7.3], $\eta_5\mu_6$ and $2\Sigma^2\epsilon'$ represent distinct non-zero elements in $\pi_{15}(S^5)$. Thus after two suspensions a induces an injection, implying that a itself must be an injection.

Let x, y be generators for each of the summands in $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since a is an injection and its domain consists of elements of order 2, the image is generated by $2sx + ty$ and $2s'x + t'y$ for $s, s', t, t' \in \mathbb{Z}/2\mathbb{Z}$. The injectivity of a implies that at least one of t or t' is equal to 1. As $f = 2 \oplus \mathrm{id}$, the image of $f \circ a$ is generated by y. The commutativity of the leftmost square in (7.5) then implies that the image of i is generated by y. Therefore the image of i in $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is trivial when projected to $\mathbb{Z}/4\mathbb{Z}$ and onto when projected to $\mathbb{Z}/2\mathbb{Z}$. Thus, exactness along the bottom row of (7.5) implies that j is an injection on the $\mathbb{Z}/4\mathbb{Z}$ summand and is trivial on the $\mathbb{Z}/2\mathbb{Z}$ summand.

Consider the map ℓ. Since h and e are identity maps, it is equivalent to determine d. Explicitly, d is the map $\pi_9(\Omega^2 S^3) \xrightarrow{(\Sigma\eta)^*} \pi_9(\Omega^3 S^3)$. Taking adjoints, we find that d sends a map $S^{11} \xrightarrow{s} S^3$ to the composite $S^{12} \xrightarrow{\eta_{11}} S^{11} \xrightarrow{s} S^3$. By [32], $\pi_{11}(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by ϵ_3, and as above, $\pi_{12}(S^3) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is generated by $\eta_3\epsilon_4$ and μ_3. Observe that $d(\epsilon_3) = \epsilon_3\eta_{11}$, so as before, after two suspensions the image of d is $\epsilon_5\eta_{13} = \eta_5\epsilon_6$, which is nontrivial. Therefore after two suspensions d induces an injection, implying that d itself is an injection. Hence ℓ is an injection, implying that k is the zero map and j is onto. $\square$

In what follows we work with $\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$, and note that the pointed exponential law implies that $\Omega\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \simeq \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, \mathrm{Sp}(2))$. By Lemma 7.1, $\pi_4(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}$. Let $\gamma\colon S^4 \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ represent a generator. The second statement in Lemma 7.1 implies that there is a homotopy commutative diagram

$$\begin{array}{ccc} S^4 & \xrightarrow{\ \theta\ } & \Omega^3\mathrm{Sp}(2) \\ \big\downarrow{\scriptstyle \underline{2}} & & \big\downarrow \\ S^4 & \xrightarrow{\ \gamma\ } & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \end{array} \tag{7.6}$$

where θ represents a generator of $\pi_4(\Omega^3\mathrm{Sp}(2)) \cong \mathbb{Z}$. We will precompose (7.6) with $S^7 \xrightarrow{\nu_4} S^4$.

Lemma 7.4. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} S^7 & \xrightarrow{\ \nu_4\ } & S^4 \\ & {\scriptstyle \nu_4\circ\underline{4}-\Sigma\nu'}\searrow & \big\downarrow{\scriptstyle \underline{2}} \\ & & S^4 \end{array}$$

Proof. Consider the composite $S^6 \xrightarrow{\widetilde{\nu}_4} \Omega S^4 \xrightarrow{\Omega\underline{2}} \Omega S^4$, where $\widetilde{\nu}_4$ is the adjoint of ν_4. By [4, Sect. 4],

$$\Omega\underline{2} \simeq 2 + \Omega[\iota_4, \iota_4] \circ H_2 + \Omega[\iota_4, [\iota_4, \iota_4]] \circ H_3$$

where 2 is the H-space squaring map on ΩS^4, $H_2\colon \Omega S^4 \to \Omega S^7$ and $H_3\colon \Omega S^4 \to \Omega S^{10}$ are the second and third Hilton–Hopf invariants respectively, and $[\iota_4, \iota_4]\colon S^7 \to S^4$ and $[\iota_4, [\iota_4, \iota_4]]\colon S^{10} \to S^4$ are iterated Whitehead products of the identity map ι_4 on S^4 with itself. For dimensional reasons, $H_3 \circ \widetilde{\nu}_4$ is null homotopic. So, precomposing with $\widetilde{\nu}_4$, we are left with

$$\Omega\underline{2} \circ \widetilde{\nu}_4 \simeq 2 \circ \widetilde{\nu}_4 + \Omega[\iota_4, \iota_4] \circ H_2 \circ \widetilde{\nu}_4. \tag{7.7}$$

Since ν_4 is an element of Hopf invariant 1, $H \circ \widetilde{\nu}_4 \simeq E$, where $E\colon S^6 \to \Omega S^7$ is the suspension map. By [32, eq. (5.8)], $[\iota_4, \iota_4] \simeq \nu_4 \circ \underline{2} - \Sigma\nu'$. Thus, $\Omega[\iota_4, \iota_4] \circ E \simeq \widetilde{\nu}_4 \circ \underline{2} - E \circ \nu'$. Hence from (7.7) we obtain

$$\Omega\underline{2} \circ \widetilde{\nu}_4 \simeq 2 \circ \widetilde{\nu}_4 + \widetilde{\nu}_4 \circ \underline{2} - E \circ \nu' \simeq \widetilde{\nu}_4 \circ \underline{2} + \widetilde{\nu}_4 \circ \underline{2} - E \circ \nu' \simeq \widetilde{\nu}_4 \circ \underline{4} - E \circ \nu'.$$

Taking adjoints, we get $\Omega\underline{2} \circ \nu_4 \simeq \nu_4 \circ \underline{4} - \Sigma\nu'$, as asserted. $\square$

Next consider how $\Sigma\nu'$ and ν_4 compose with the map γ in (7.6).

Lemma 7.5. *The composite* $S^7 \xrightarrow{\Sigma\nu'} S^4 \xrightarrow{\gamma} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ *has order* 2.

Proof. First observe that the homotopy fibration

$$\Omega^3\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \to \Omega\mathrm{Sp}(2)$$

implies that there is an exact sequence

$$\pi_5(\Omega^3\mathrm{Sp}(2)) \to \pi_5(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \to \pi_5(\Omega\mathrm{Sp}(2)).$$

By Lemma 4.3, $\pi_5(\Omega^3\mathrm{Sp}(2)) \cong \pi_5(\Omega\mathrm{Sp}(2)) \cong 0$, implying that $\pi_5(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong 0$.

Next, by [32], $2\nu' = \eta_3^3$. Therefore the composite

$$S^7 \xrightarrow{\underline{2}} S^7 \xrightarrow{\Sigma\nu'} S^4 \xrightarrow{\gamma} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$$

factors through the composite $S^5 \xrightarrow{\eta_4} S^4 \xrightarrow{\gamma} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$. The latter is null homotopic since $\pi_5(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong 0$, and hence $\gamma \circ \Sigma\nu' \circ \underline{2}$ is null homotopic. Consequently, $\gamma \circ \Sigma\nu'$ has order at most 2.

By Lemma 5.1, the composite

$$A \xrightarrow{s} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Sp}(2) \xrightarrow{\underline{4}} \Omega_0^3\mathrm{Sp}(2) \longrightarrow \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$$

is null homotopic. Therefore, from the homotopy cofibration $A \to S^7 \xrightarrow{\Sigma\nu'} S^4$, we obtain an extension

$$\begin{array}{ccc} S^7 & \xrightarrow{\quad 4a \quad} & \Omega_0^3\mathrm{Sp}(2) \\ \big\downarrow{\scriptstyle \Sigma\nu'} & & \big\downarrow \\ S^4 & \overset{\xi}{\dashrightarrow} & \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \end{array} \tag{7.8}$$

for some map ξ. We claim that ξ must represent a generator of $\pi_4(\mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}$. Since a represents a generator of $\pi_7(\Omega^3\mathrm{Sp}(2)) \cong \mathbb{Z}/8\mathbb{Z}$, the map $4a$ has order 2. By Lemma 7.2 the map $\Omega_0^3\mathrm{Sp}(2) \to \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ induces an isomorphism on π_7. Thus the upper direction around (7.8) has order 2. Therefore the homotopy commutativity of the diagram implies that $\xi \circ \Sigma\nu'$ has order 2. If ξ did not represent a generator of $\pi_4(\mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}$, then as we are localized at 2, we must have $\xi \simeq 2t \cdot \gamma$ for some integer t. But then $\xi \circ \Sigma\nu' \simeq (t\gamma) \circ \underline{2} \circ \Sigma\nu' \simeq (t\gamma) \circ \Sigma\nu' \circ \underline{2}$, where the right homotopy holds since $\Sigma\nu'$ is a suspension. In the first part of the proof it was shown that $\gamma \circ \Sigma\nu' \circ \underline{2}$ is null homotopic. Therefore, $\xi \circ \Sigma\nu'$ is null homotopic, a contradiction. Hence ξ represents a generator of $\pi_4(\mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}$, so $\xi \simeq \pm\gamma$. Consequently, the lower direction around (7.8) is homotopic to $\pm\gamma \circ \Sigma\nu'$, implying that $\gamma \circ \Sigma\nu'$ has order equal to that of the upper direction around (7.8), which is 2. $\square$

Lemma 7.6. *The composite* $S^7 \xrightarrow{\nu_4} S^4 \xrightarrow{\gamma} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ *has order at most* 4.

Proof. Consider the diagram

$$\begin{array}{ccccc} S^7 & \xrightarrow{\nu_4} & S^4 & \xrightarrow{\quad\theta\quad} & \Omega^3\mathrm{Sp}(2) \\ & {\scriptstyle \nu_4\circ\underline{4}-\Sigma\nu'}\searrow & \big\downarrow{\scriptstyle \underline{2}} & & \big\downarrow{\scriptstyle \pi^*} \\ & & S^4 & \xrightarrow{\gamma} & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \end{array} \tag{7.9}$$

The left triangle homotopy commutes by Lemma 7.4, and the right square homotopy commutes by (7.6), where the vertical map on the right has been labelled as π^* for convenience.

First, it will be shown that $\pi^* \circ \theta \circ \nu_4$ is a map of order 2. By Lemma 7.2, the map $\Omega^3\mathrm{Sp}(2) \xrightarrow{\pi^*} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ induces an isomorphism on π_7, implying that it suffices to show that the map $\theta \circ \nu_4$ has order 2. If $c\colon S^7 \to \mathrm{Sp}(2)$ is the adjoint of θ, it is equivalent to show that $c \circ \nu_7$ has order 2. Consider the composite

$$S^{10} \xrightarrow{\nu_7} S^7 \xrightarrow{c} \mathrm{Sp}(2) \xrightarrow{q} S^7.$$

By [23], q induces an isomorphism on π_{10}, so it suffices to show that $q \circ c \circ \nu_7$ has order 2. By [32], $\pi_{10}(S^7) \cong \mathbb{Z}/8\mathbb{Z}$ is generated by ν_7, and by [23], $q \circ c \simeq \pm 4$. Thus, $q \circ c \circ \nu_7 \simeq \pm 4\nu_7$ has order 2.

By Lemma 7.2, $\pi_7(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}/8\mathbb{Z}$. Let $\zeta\colon S^7 \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ represent a generator. There is a unique element of order 2 in $\mathbb{Z}/8\mathbb{Z}$, so as $\pi^* \circ \theta \circ \nu_4$ has order 2, we have $\pi^* \circ \theta \circ \nu_4 \simeq 4\zeta$. By Lemma 7.5, $\gamma \circ \Sigma\nu'$ also has order 2 so $\gamma \circ \Sigma\nu' \simeq 4\zeta$. The homotopy commutativity of (7.9) then gives

$$\pi^* \circ \theta \circ \nu_4 \simeq \gamma \circ (\nu_4 \circ \underline{4} - \Sigma\nu') \simeq \gamma \circ \nu_4 \circ \underline{4} - \gamma \circ \Sigma\nu' \qquad \Rightarrow \qquad 4\zeta \simeq \gamma \circ \nu_4 \circ \underline{4} - 4\zeta.$$

That is, $8\zeta \simeq \gamma \circ \nu_4 \circ \underline{4}$. But 8ζ is null homotopic, so $\gamma \circ \nu_4 \circ \underline{4}$ is also null homotopic. Hence $\gamma \circ \nu_4$ has order at most 4. $\square$

Recall that the map $S^{10} \xrightarrow{g} S^4$ in Lemma 6.1 is $t\nu_4^2 \in \mathbb{Z}/8\mathbb{Z}$ where $t = 2u$ or $4u$ for some unit $u \in \mathbb{Z}/8\mathbb{Z}$.

Proposition 7.7. *The composite* $S^{10} \xrightarrow{g} S^4 \xrightarrow{\gamma} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ *is null homotopic.*

Proof. Consider the diagram

$$\begin{array}{ccccc} S^{10} & \xrightarrow{2\nu_7} & S^7 & \xrightarrow{\nu_4} & S^4 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\gamma} \\ & & \Omega^3\mathrm{Sp}(2) & \longrightarrow & \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \end{array} \tag{7.10}$$

where λ will be defined momentarily. By Lemma 7.2, the map $\Omega^3\mathrm{Sp}(2) \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ induces an isomorphism on π_7. Thus, $\gamma \circ \nu_4$ lifts to a map $\lambda\colon S^7 \to \Omega^3\mathrm{Sp}(2)$ of the same order as $\gamma \circ \nu_4$, and which makes (7.10) homotopy commute. By Lemma 7.6, $\gamma \circ \nu_4$ has order at most 4, so λ also has order at most 4.

Mimura and Toda chose a generator of $\pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/8\mathbb{Z}$ they called $[\nu_7]$. Let $c\colon S^{10} \to \mathrm{Sp}(2)$ be the triple adjoint of λ, so $c \simeq 2t \cdot [\nu_7]$ for some $t \in \mathbb{Z}/8\mathbb{Z}$. In [23] it is also shown that $[\nu_7] \circ \nu_{10}$ represents the order 4 generator in $\pi_{13}(\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus, $c \circ \nu_{10}$ has order at most 2. Taking adjoints, we find that $\lambda \circ \nu_7$ has order at most 2. Thus, $\lambda \circ 2\nu_7$ is null homotopic. Hence the homotopy commutativity of (7.10) implies that $\gamma \circ 2\nu_4^2$ is null homotopic. Since $g = 2u\nu_4^2$ or $4u\nu_4^2$, this implies that $\gamma \circ g$ is null homotopic. $\square$

Theorem 7.8. *The composite*

$$\mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Sp}(2) \xrightarrow{4} \Omega_0^3\mathrm{Sp}(2) \longrightarrow \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \tag{7.11}$$

is null homotopic.

Proof. By Lemma 6.2 there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(2) & \xrightarrow{\partial_1} & \Omega_0^3\mathrm{Sp}(2) \\ \downarrow & & \downarrow{\scriptstyle 4} \\ S^7 \vee S^{10} \xrightarrow{p_1} S^7 & \xrightarrow{4a} & \Omega_0^3\mathrm{Sp}(2) \end{array} \tag{7.12}$$

We will show that there is also a homotopy commutative diagram

$$\begin{array}{ccc} S^7 \vee S^{10} \xrightarrow{p_1} S^7 & \xrightarrow{4a} & \Omega_0^3\mathrm{Sp}(2) \\ \downarrow{\scriptstyle \Sigma\nu' - g} & & \downarrow{\scriptstyle \pi^*} \\ S^4 & \xrightarrow{\gamma} & \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)) \end{array} \tag{7.13}$$

where π^* is induced by the pinch map $\mathbb{C}\mathrm{P}^2 \xrightarrow{\pi} S^4$. Granting this, juxtapose (7.12) and (7.13). The upper direction around the juxtaposed diagram is the composite (7.11). This is null homotopic since the lower direction around the juxtaposed diagram factors through $\mathrm{Sp}(2) \to S^7 \vee S^{10} \xrightarrow{\Sigma\nu' - g} S^4$, which by Lemma 6.1 is two consecutive maps in a homotopy cofibration and so is null homotopic.

It remains to show that (7.13) exists. By Proposition 7.7, $\gamma \circ (\Sigma\nu' - g) \simeq \gamma \circ \Sigma\nu' - \gamma \circ g \simeq \gamma \circ \Sigma\nu' - *$, and by Lemma 7.5, $\gamma \circ \Sigma\nu'$ has order 2. We have $\pi_7(\mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))) \cong \mathbb{Z}/8\mathbb{Z}$ by Lemma 7.2. Let $\zeta\colon S^7 \to \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ represent a generator. There is a unique element of order 2 in $\mathbb{Z}/8\mathbb{Z}$, so as $\gamma \circ \Sigma\nu'$ has order 2 we obtain $\gamma \circ \Sigma\nu' \simeq 4\zeta$. Thus the lower direction around the diagram (7.13) is homotopic to the composite $S^7 \vee S^{10} \xrightarrow{p_1} S^7 \xrightarrow{4\zeta} \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$. On the other hand, since a represents a generator of $\pi_7(\Omega^3\mathrm{Sp}(2)) \cong \mathbb{Z}/8\mathbb{Z}$, the map $4a$ has order 2. By Lemma 7.2 the map $\Omega_0^3\mathrm{Sp}(2) \xrightarrow{\pi^*} \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ induces an isomorphism on π_7. Thus, $\pi^* \circ 4a$ has order 2, and so $\pi^* \circ 4a \simeq 4\xi$. Hence the upper direction around (7.13) is also homotopic to the composite $S^7 \vee S^{10} \xrightarrow{p_1} S^7 \xrightarrow{4\zeta} \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$. Therefore, (7.13) homotopy commutes. □

Remark 7.9. It should also be noted that the composite

$$\mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Sp}(2) \xrightarrow{2} \Omega_0^3\mathrm{Sp}(2) \longrightarrow \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$$

is nontrivial. For if it were null homotopic then so would be the composite

$$A \xrightarrow{s} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Sp}(2) \xrightarrow{2} \Omega_0^3\mathrm{Sp}(2) \longrightarrow \mathrm{Map}_0^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2)). \tag{7.14}$$

By Lemma 4.1, $[A, \Omega_0^3\mathrm{Sp}(2)] \cong \mathbb{Z}/8\mathbb{Z}$ and $\partial_1 \circ s$ represents a generator. By Lemma 3.1, $\partial_k \simeq k \cdot \partial_1$, so $2 \circ \partial_1 \circ s$ has order 4. Therefore, by Proposition 4.15, the composite (7.14) is nontrivial.

Finally, our results are assembled to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.5, it suffices to prove the statement for the special cases $M = S^4$ and $M = \mathbb{C}\mathrm{P}^2$. The $M = S^4$ case is dealt with by Theorem 2.7, proving assertion (a). For assertion (b), consider the $M = \mathbb{C}\mathrm{P}^2$ case. By Theorem 5.5, if $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$ then $(20, k) = (20, \ell)$. By Theorem 7.8 and Remark 7.9, the "order" of $\mathrm{Sp}(2) \xrightarrow{\overline{\partial}_1} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ is 4 when localized at 2. In general, at odd primes, by [27], the map $\Omega_0^3\mathrm{Sp}(2) \xrightarrow{\pi^*} \mathrm{Map}^*(\mathbb{C}\mathrm{P}^2, B\mathrm{Sp}(2))$ has a left homotopy inverse. Therefore the homotopy commutativity of the left square in (3.1) implies that the "order" of $\overline{\partial}_1$ equals the order of ∂_1. In our case, by [28], the order of $\mathrm{Sp}(2) \to \Omega_0^3\mathrm{Sp}(2)$ is 40, so the "order" of $\overline{\partial}_1$ is 5 when localized at 5 and 1 when localized at any prime $p \notin \{2, 5\}$. Thus, $\overline{\partial}_1$ has "order" 20. Theorem 3.2 therefore implies that if $(20, k) = (20, \ell)$ then $\mathcal{G}_k(\mathbb{C}\mathrm{P}^2) \simeq \mathcal{G}_\ell(\mathbb{C}\mathrm{P}^2)$. □

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