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Pontryagin's Direct Method for Optimization Problems with Differential Inclusion

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Abstract—We develop Pontryagin's direct variational method, which allows us to obtain necessary conditions in the Mayer extremal problem on a fixed interval under constraints on the trajectories given by a differential inclusion with generally unbounded right-hand side. The established necessary optimality conditions contain the Euler–Lagrange differential inclusion. The results are proved under maximally weak conditions, and very strong statements compared with the known ones are obtained; moreover, admissible velocity sets may be unbounded and nonconvex under a general hypothesis that the right-hand side of the differential inclusion is pseudo-Lipschitz. In the statements, we refine conditions on the Euler–Lagrange differential inclusion, in which neither the Clarke normal cone nor the limiting normal cone is used, as is common in the works of other authors. We also give an example demonstrating the efficiency of the results obtained.

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INTRODUCTION

In the famous work [24], Pontryagin and his colleagues used a very efficient direct variational method in order to prove necessary optimality conditions, which were later called the "Pontryagin maximum principle." This method is based on the linearization of a nonlinear controlled dynamical system near an optimal trajectory. As a result, for the arising linear system of variational differential equations, an adjoint system of differential equations was constructed, and it was proved that under certain boundary conditions the corresponding solution of the adjoint system is a normal vector to the attainability set of the original control system at the points of the optimal trajectory at any instant of time. This was analytically expressed as the Pontryagin maximum principle, transversality conditions, and some other properties of the adjoint system.

Subsequently, other well-known authors (see, for example, [7, 4, 12]) solved more general optimization problems (for instance, those where the right-hand side of a control system is not smooth with respect to the state variable) and applied different methods in order to prove necessary optimality conditions. These methods are based on various types of approximations of the original problems.

The first attempts to develop Pontryagin's direct variational method for optimization problems in the case when the control system is not smooth were made in [25, 2] for a controlled dynamical system represented as a differential inclusion. In [2], Blagodatskikh obtained necessary optimality conditions in terms of support functions under the condition that the multivalued right-hand side of the differential inclusion takes convex compact values and its support function is Lipschitz continuous with respect to the state variable.

In [22, 23], the present author and Smirnov generalized Pontryagin's direct variational method and obtained necessary optimality conditions in optimization problems with differential inclusion in

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the case when the right-hand side of the differential inclusion takes compact (possibly, nonconvex) values and satisfies the Lipschitz condition in the Hausdorff metric.

In the present paper, continuing the studies of [16, 17, 19, 20], we generalize Pontryagin's direct variational method to the case of Mayer optimization problems on an interval with a differential inclusion whose multivalued right-hand side takes unbounded nonconvex values, depends measurably on time, and is pseudo-Lipschitz with respect to the state variable, in the presence of state constraints at the initial and terminal points.

The development of necessary conditions for extremal problems with bounded differential inclusions satisfying the Lipschitz conditions in the Hausdorff metric can be found, for example, in [4, 6, 8, 12, 13, 26, 27]. However, in the case when the right-hand side F of the differential inclusion may take unbounded values, the Lipschitz condition on the multivalued mapping F becomes too burdensome. Nevertheless, the unboundedness of the values of the right-hand side $F(t,x)$ of the differential inclusion is a natural property of differential inclusions that arises in optimal control problems. For example, it arises when one deals with the Mayer problem obtained by reformulating an extremal problem with an integral functional.

Our direct variational method for proving necessary conditions in the Mayer problem consists of the following elements:

(1) proof of the continuous dependence of the set of trajectories of the differential inclusion on some parameters (Proposition 5);

(2) construction of a continuous pseudolinearization of the differential inclusion near some (optimal) trajectory (Proposition 6);

(3) description of the set of trajectories of the adjoint convex process, i.e., calculation of the polar cone to the set of trajectories of the convex process (Proposition 7);

(4) description of the properties of boundary trajectories of the differential inclusion and the dynamics of the normals to the attainability sets at the points of a boundary trajectory (Theorem 1);

(5) description of necessary conditions for solving the initial extremal Mayer problem with differential inclusion (Theorem 2).

Eventually, we prove necessary conditions that also contain the Euler–Lagrange differential inclusion whose graph is a normal cone. However, we use a narrower normal cone than the Clarke normal cone or even a partially convexified limiting normal cone (cf. [5, Theorem 2.2.3]). In conclusion, we give an example of a problem in which the necessary optimality conditions obtained in the present study are more precise compared with other necessary conditions known in the literature (see, for example, [5, 27, 13]).

On the other hand, we should note that our result is somewhat weaker that the assertion of Theorem 2.2.3 in [5], because we have obtained a more general condition of Euler–Lagrange type, but the maximum condition, which generalizes the Weierstrass conditions or the Pontryagin maximum principle, has been established only for a particular case when the right-hand side of the original differential inclusion has an additional property of local convexity (see Corollary 2).

1. MAIN NOTIONS AND DEFINITIONS

Denote by $T := [t_0, t_1]$ a closed interval on the line $\mathbb{R}^1$ with the σ-algebra $\mathcal{L}$ of all measurable subsets of T with respect to the Lebesgue measure μ. Let E be a real reflexive separable Banach space and $\mathbb{R}^n$ be the Euclidean space of dimension n. We also introduce the notation $B_r(a) := \{x \in E \mid \|x-a\| < r\}$ for an open ball and $\varrho(x, A) := \inf\{\|x-y\| \mid y \in A\}$ for the distance function. The closure of a set A is denoted by $\overline{A}$, and its convex hull, by $\operatorname{co} A$. The *set of all absolutely continuous functions* (or, briefly, *arcs*) from T to $\mathbb{R}^n$ forms a *Banach space* $\mathrm{AC}(T, \mathbb{R}^n)$ with the norm

$\|f\|_{\mathrm{AC}} := \|f(t_0)\|_{\mathbb{R}^n} + \|f'\|_{L^1}$. Denote by $\mathrm{AC}^\infty(T,\mathbb{R}^n)$ the subset in $\mathrm{AC}(T,\mathbb{R}^n)$ of all arcs f such that $f' \in L^\infty(T,\mathbb{R}^n)$. The norm in $\mathrm{AC}^\infty(T,\mathbb{R}^n)$ is defined as $\|f\|_{\mathrm{AC}^\infty} := \|f(t_0)\|_{\mathbb{R}^n} + \|f'\|_{L^\infty}$.

Denote by $\mathcal{P}(E)$ the set of all subsets of the Banach space E and by $\mathcal{F}(E)$ the set of all nonempty closed subsets of E. Recall (see [1, Sect. 4.1] or [23, Definition 2]) that the *lower tangent cone* (also known as the *simple tangent cone* or the *adjacent cone*) to a set $A \subset E$ at a point $a \in \overline{A}$ is a cone of the form

$$T_{\mathrm{L}}(A;a) := \Big\{v \in E \mid \lim_{\lambda\downarrow 0} \varrho(v, \lambda^{-1}(A-a)) = 0\Big\}. \tag{1.1}$$

In addition to the lower tangent cone, we will use the *asymptotic lower tangent cone* $T_{\mathrm{AL}}(A,a)$ to a set $A \subset E$ at a point $a \in \overline{A}$. It is defined as

$$T_{\mathrm{AL}}(A,a) := T_{\mathrm{L}}(A,a) \stackrel{*}{-} T_{\mathrm{L}}(A,a), \tag{1.2}$$

where $\stackrel{*}{-}$ is the Minkowski difference of two sets A and B, i.e., $A \stackrel{*}{-} B := \{x \in E \mid x + B \subset A\}$. It is known (see, for example, [1, Sect. 4.5] or [23]) that the asymptotic lower tangent cone is always a convex closed cone containing the Clarke tangent cone (for a definition of the latter see, for example, [4, Sect. 2.4]).

A closed convex cone $K_0 \subset E$ is called a *Boltyanskii tent* to a set $A \subset E$ at a point $a \in \overline{A}$ (see [3, § 3]) if there exists a continuous mapping $q\colon K_0 \cap B_1(0) \to E$ such that $a + v + q(v) \in A$ for all $v \in K_0 \cap B_1(0)$ and the equality $\lim_{v\to 0}(\|q(v)\|/\|v\|) = 0$ holds. The *upper Dini derivative* of a locally Lipschitz function $f\colon E \to \mathbb{R}^1$ is defined as

$$D^+_{\mathrm{L}} f(x_0)(u) = \limsup_{\lambda\downarrow 0} \frac{f(x_0+\lambda u) - f(x_0)}{\lambda}. \tag{1.3}$$

Recall that the *graph* of a mapping $F\colon T \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is the set $\operatorname{Graph} F := \{(t,x,y) \in T \times \mathbb{R}^n \times \mathbb{R}^n \mid y \in F(t,x)\}$. The *cross-section of the graph* $\operatorname{Graph} F$ *at a point* $t \in T$ is the set $\operatorname{Graph} F(t,\cdot) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in F(t,x)\}$.

Given a mapping $F\colon T \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, consider a *differential inclusion*

$$x'(t) \in F(t,x(t)) \qquad \text{for a.e. } \ t \in T. \tag{1.4}$$

For any $C_0 \subset \mathbb{R}^n$, denote by $\mathcal{R}_T(F,C_0)$ the set of all F-trajectories $x \in \mathrm{AC}(T,\mathbb{R}^n)$ of the differential inclusion (1.4) on the interval T under the condition that $x(t_0) \in C_0$. Denote by $\mathcal{R}^\infty_T(F,C_0)$ the subset of all F-trajectories from $\mathcal{R}_T(F,C_0)$ that belong to the space $\mathrm{AC}^\infty(T,\mathbb{R}^n)$.

Given a mapping $F\colon T \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ and an arc $\widehat{x} \in \mathcal{R}_T(F,C_0)$, we can consider, for a.e. $t \in T$, the *lower multivalued derivative* of $F(t,\cdot)$ at a point $(\widehat{x}(t), \widehat{x}'(t)) \in \operatorname{Graph} F(t,\cdot)$ in the direction $u \in \mathbb{R}^n$ (see [23, 1]):

$$F'_{\mathrm{L}}(t,u) := \big\{v \in \mathbb{R}^n \mid (u,v) \in T_{\mathrm{L}}\big(\operatorname{Graph} F(t,\cdot); (\widehat{x}(t), \widehat{x}'(t))\big)\big\}.$$

By analogy with the notation $\mathcal{R}_T(F,C_0)$ for the set of all F-trajectories of the inclusion (1.4), we denote by $\mathcal{R}^\infty_T(F'_{\mathrm{L}},K_0)$ the set of all F'_{L}-trajectories of the *variational differential inclusion*

$$u'(t) \in F'_{\mathrm{L}}(t,u(t)) \qquad \text{for a.e. } \ t \in T \tag{1.5}$$

each of which satisfies the inclusion $u \in \mathrm{AC}^\infty(T,\mathbb{R}^n)$ and the initial condition $u(t_0) \in K_0$.

2. THE MAYER PROBLEM

Our goal is to obtain necessary conditions for the solution of the following Mayer optimization problem on the interval $T := [t_0,t_1]$:

$$\text{Minimize} \big\{\varphi(x(t_1)) \mid x \in \mathcal{R}_T(F,C_0),\ x(t_1) \in C_1,\ C_0, C_1 \subset \mathbb{R}^n\big\}. \tag{2.1}$$

The problem consists in finding a minimum of a given locally Lipschitz function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^1$ on the set of endpoints of all those F-trajectories of the differential inclusion (1.4) with unbounded right-hand side whose initial values $x(t_0)$ are taken from the subset $C_0 \subset \mathbb{R}^n$ and the terminal values $x(t_1)$ are taken from the subset $C_1 \subset \mathbb{R}^n$.

Suppose that an F-trajectory $\widehat{x} \in \mathcal{R}_T(F, C_0)$ is a solution to the extremal problem (2.1). Let us formulate local conditions on the mapping F in problem (2.1) near the arc $\widehat{x}$.

Hypothesis 0 (measurability)**.** The multivalued mapping $F\colon T \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in (2.1) is $(\mathcal{L} \otimes \mathcal{B})$-measurable, and for every $t \in T$ the set $\operatorname{Graph} F(t,\cdot)$ is a closed subset in $\mathbb{R}^n \times \mathbb{R}^n$.

Hypothesis 1 (pseudo-Lipschitz property)**.** There exists a number $\varepsilon \in (0,1)$, a function $l \in L^1(T, \mathbb{R}^1_+)$, $l(t) > 0$ a.e., and a measurable function $R\colon T \to (0, +\infty]$ such that the inclusion

$$G(t, x_1) \subset F(t, x_2) + l(t)\|x_1 - x_2\|\overline{B_1(0)} \tag{2.2}$$

holds for an arbitrary pair of points $(t, x_1), (t, x_2)$ in the tube

$$W := \bigl\{(t,x) \in T \times \mathbb{R}^n \bigm| \|x - \widehat{x}(t)\| \le \varepsilon\bigr\},$$

where, by definition,

$$G(t,x) := F(t,x) \cap (\widehat{x}'(t) + R(t)B_1(0)). \tag{2.3}$$

Hypothesis 2 (nondegeneracy)**.** For the functions l and R defined in Hypothesis 1, there exists a number $\nu \in (0, 1/\varepsilon)$ such that the inequality $l(t) \le \nu R(t)$ holds for a.e. $t \in T$.

Definition 1. A mapping $F\colon T \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be *measurable pseudo-Lipschitz near* the F-trajectory $\widehat{x}$ if there exist numbers $\nu > 0$ and $\varepsilon \in (0, \min\{1, 1/\nu\})$ as well as functions l and R that satisfy Hypotheses 0–2.

We also consider the following assumption.

Hypothesis 3. For the mapping F and the F-trajectory $\widehat{x}$, there exists a (multivalued) measurable mapping $K\colon T \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ whose values are closed convex cones satisfying the following inclusions for a.e. $t \in T$:

$$T_{\mathrm{C}}\bigl(\operatorname{Graph} F(t,\cdot); (\widehat{x}(t), \widehat{x}'(t))\bigr) \subset K(t) \subset T_{\mathrm{L}}\bigl(\operatorname{Graph} F(t,\cdot); (\widehat{x}(t), \widehat{x}'(t))\bigr). \tag{2.4}$$

For a definition of the Clarke tangent cone T_{C}, see [4, Sect. 2.4]; the definition of the lower tangent cone T_{L} is given in (1.1).

As an example of a mapping $K(t)$ satisfying Hypothesis 3, one can consider (for every $t \in T$) the Clarke tangent cone, the Michel–Penot tangent cone (see [11]), or the asymptotic lower tangent cone (see (1.2)) to the set $\operatorname{Graph} F(t,\cdot)$ at the point $(\widehat{x}(t), \widehat{x}'(t))$. We prefer the greatest of the cones listed above, the asymptotic lower tangent cone.

3. AUXILIARY RESULTS

A. First of all, recall some results (see [17, 19]) that will be needed in what follows.

Proposition 1 (see [19, Lemma 1])**.** *Let a function $f\colon \mathbb{R}^n \to \mathbb{R}^1$ be locally Lipschitz near a point $x_0 \in \operatorname{Dom} f$. Then the tangent cone can be calculated by the formula*

$$T_{\mathrm{L}}(\operatorname{Epi} f; (x_0, f(x_0))) = \left\{(u,v) \in \mathbb{R}^n \times \mathbb{R}^1 \;\middle|\; v \ge \limsup_{\lambda \downarrow 0} \frac{f(x_0 + \lambda u) - f(x_0)}{\lambda}\right\}.$$

Proposition 2 (see [19, Lemma 2])**.** *For any set $Q \subset \mathbb{R}^n$ and any point $x_0 \in \overline{Q}$, define the set $A := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^1 \mid y \ge \varrho(x, Q)\}$. Then the following inclusion holds*: $\{(u,v) \in \mathbb{R}^n \times \mathbb{R}^1 \mid v \ge \varrho(u, T_{\mathrm{L}}(Q; x_0))\} \subset T_{\mathrm{L}}(A; (x_0, 0))$.

Proposition 3 (see [17, inequality (46.21)]). *If a mapping $F\colon (0,1)\to \mathcal{F}(\mathbb{R}^n)$ is such that the subset $\liminf_{\lambda\downarrow 0} F(\lambda)$ is nonempty* (*for the notion of the Kuratowski* $\liminf$, *see* [9, § 29]), *then the inequality*

$$\limsup_{\lambda\downarrow 0} \varrho(x, F(\lambda)) \le \varrho\Big(x, \liminf_{\lambda\downarrow 0} F(\lambda)\Big)$$

holds for all $x\in\mathbb{R}^n$.

Proposition 4 (see [17, Theorem 25.2]). *Let a function $\varphi\colon \mathbb{R}^n\to\mathbb{R}^1$ be locally Lipschitz near some point $x_0\in\mathbb{R}^n$ and $\psi\colon \mathbb{R}^n\to\mathbb{R}$ be a positively homogeneous convex function such that the inequality $D^+_{\mathrm{L}}\varphi(x_0)(x)\le\psi(x)<\infty$ holds for all $x\in\mathbb{R}^n$. Then the set* $\operatorname{Epi}\psi$ *is a Boltyanskii tent to the set* $\operatorname{Epi}\varphi$ *at the point $(x_0,\varphi(x_0))$.*

B. Based on the assertions of Lemmas 3.1 and 3.2 and Theorems 4.1–4.3 from [14], which were proved in the Lipschitz case, we generalize them to the pseudo-Lipschitz case (see [16, 19–21]).

Suppose that the mapping F is measurable pseudo-Lipschitz near the arc $\widehat{x}$ with the corresponding numbers $\nu>0$ and $\varepsilon\in(0,\min\{1,1/\nu\})$ and functions l and R (see Definition 1). Below, we will use *two parameters* $a\in(0,\varepsilon]$ and $\gamma\in(0,1]$ and *two functions*

$$b(a) := \min\Big\{3a, \frac{1}{\nu}\Big\}, \qquad m(t) := \int_{t_0}^{t} l(\tau)\,d\tau, \quad t\in T. \tag{3.1}$$

We will also consider a function $\rho_0\in L^1(T,\mathbb{R}^1_+)$ such that $0\le\rho_0(t)\le\gamma b(\varepsilon)e^{-m(t_1)}l(t)/8$ for a.e. $t\in T$. Define the set

$$D_0(F,\rho_0,\gamma) := \Big\{x\in \mathrm{AC}(T,\mathbb{R}^n)\ \Big|\ \|x-\widehat{x}\|_{\mathrm{AC}}\le\frac{\varepsilon}{4},\ \varrho(x'(t),F(t,x(t)))\le\rho_0(t),\\ \|x'(t)-\widehat{x}'(t)\|\le\frac{\gamma}{2}b(\varepsilon)\,l(t),\ t\in T\Big\}. \tag{3.2}$$

One can easily show that the set $D_0(F,\rho_0,\gamma)\setminus\{\widehat{x}\}$ thus defined is nonempty and the graphs of all arcs in this set are contained in the tube W (see Hypothesis 1). Moreover, for every $x_0\in B_\delta(\widehat{x}(t_0))$, $\delta:=\min\{1/(2\nu),\varepsilon/4\}\gamma e^{-m(t_1)}$, there exists an arc $x\in D_0(F,\rho_0,\gamma)$ such that $x(t_0)=x_0$.

Proposition 5 (see [21, Theorem 2]). *In terms of the above definitions and under the above conditions, denote the set $D_0(F,\rho_0,\gamma)$ by S_0. Let $a\in(0,\varepsilon]$ and $\delta_1\in(0,2^{-7}b(a)e^{-m(t_1)})$, and let $d\colon B_{\varepsilon/4}(\widehat{x}(t_0))\to\mathbb{R}^n$ be a continuous function such that $\|d(x)-x\|\le\delta_1$ for every $x\in B_{\varepsilon/4}(\widehat{x}(t_0))$. Then there exists a continuous mapping $r\colon S_0\to\mathcal{R}_T(F,d(B_{\varepsilon/4}(\widehat{x}(t_0))))$ such that the following relations hold for any arc $x\in S_0$:*

$$\|r(x)-x\|_{\mathrm{AC}}\le\int_{t_0}^{t_1} e^{m(t_1)-m(\tau)}\rho_0(\tau)\,d\tau+\frac{\gamma}{8}b(a), \tag{3.3}$$

$$r(x)(t_0)=d(x(t_0)), \qquad \|r(x)-\widehat{x}\|_{\mathrm{AC}}\le\varepsilon,$$

$$\Big\|\frac{d}{dt}r(x)(t)-\widehat{x}'(t)\Big\|\le\gamma b(\varepsilon)\,l(t) \qquad \textit{for a.e. } t\in T.$$

C. Denote by Σ^k the standard simplex in $\mathbb{R}^{k+1}$, i.e.,

$$\Sigma^k := \Big\{\sigma:=(\sigma_1,\dots,\sigma_{k+1})\in\mathbb{R}^{k+1}\ \Big|\ \sigma_m\ge 0,\ \sum_{m=1}^{k+1}\sigma_m=1\Big\}. \tag{3.4}$$

Proposition 6 (see [21, Theorem 3]). *Let $C_0 \subset \mathbb{R}^n$ and $F\colon T \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be a measurable pseudo-Lipschitz mapping near an arc $\widehat{x} \in \mathcal{R}_T(F, C_0)$. Let K_0 be a Boltyanskii tent to the set C_0 at a point $\widehat{x}(t_0)$. Suppose that a finite set of arcs $\{u_m \in \mathcal{R}_T^\infty(F'_{\mathrm{L}}, K_0)\}$, $m \in \overline{1, k+1}$, is given. For every $\sigma \in \Sigma^k$ (see (3.4)), define the arc $u_\sigma := \sum_{m=1}^{k+1} \sigma_m u_m$ and suppose that the inclusion $\{u_\sigma \mid \sigma \in \Sigma^k\} \subset \mathcal{R}_T^\infty(F'_{\mathrm{L}}, K_0)$ is valid. Then, for every number $\gamma \in (0,1]$, there exists a number $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ and any $\sigma \in \Sigma^k$ there exists an F-trajectory $x_{\lambda,\sigma} \in \mathcal{R}_T(F, C_0)$ and an arc $t \mapsto o(\lambda, \sigma, t)$ for which the following relations hold:*

$$\|x_{\lambda,\sigma} - \widehat{x}\|_{\mathrm{AC}} \le \frac{\varepsilon}{4}, \qquad \|x'_{\lambda,\sigma}(t) - \widehat{x}'(t)\| \le \gamma b(\varepsilon) l(t) \quad \text{for a.e. } t \in T,$$

$$x_{\lambda,\sigma}(t) = \widehat{x}(t) + \lambda u_\sigma(t) + o(\lambda, \sigma, t) \qquad \forall t \in T,$$

$$\lim_{\lambda \downarrow 0} \max_{\sigma \in \Sigma^k} \frac{\|o(\lambda, \sigma, \cdot)\|_{\mathrm{AC}}}{\lambda} = 0,$$

and the mappings $\sigma \mapsto o(\lambda, \sigma, \cdot)$ from Σ^k to $\mathrm{AC}(T, \mathbb{R}^n)$ are continuous.

D. Recall that the *polar cone* to a cone $K \subset E$ is the set $K^0 := \{p \in E^* \mid \langle p, x\rangle \le 0 \ \forall x \in K\}$. We can explicitly describe the polar cone to the cone of all trajectories of a differential inclusion whose right-hand side has a convex conic graph.

Proposition 7 (see [15, 18]). *Let $\widetilde{K}$ be a closed convex cone in a separable reflexive Banach space E. Let a mapping $Q\colon [t_0, t_1] \times E \to \mathcal{F}(E)$ be such that $Q(t, x) := \{y \in E \mid (x, y) \in K(t)\}$, where the set $K(t)$ is a closed convex cone in the space $E \times E$ for a.e. $t \in T$ and the mapping $K\colon T \to \mathcal{F}(E \times E)$ is measurable. Suppose that there exists a function $\gamma \in L^\infty(T, \mathbb{R}^1_+)$ such that $Q(t,x) \cap (\gamma(t)\overline{B_1(0)}) \ne \varnothing$ for any $x \in \overline{B_1(0)}$ and a.e. $t \in T$. Let $\widetilde{K}^0$ and $K^0(t)$ be the polar cones to $\widetilde{K}$ and $K(t)$, respectively. Then the cone $(\mathcal{R}_T^\infty(Q, \widetilde{K}))^0$, which is the polar cone to $\mathcal{R}_T^\infty(Q, \widetilde{K})$, consists of pairs of points $b^* \in E^*$ and functions $y^* \in L^1(T, E^*)$ such that for every such pair (b^*, y^*) there exists a function $x^* \in L^1(T, E^*)$ with*

$$b^* - \int_{t_0}^{t_1} x^*(s)\, ds \in \widetilde{K}^0, \tag{3.5}$$

$$\left(x^*(t), y^*(t) - \int_{t}^{t_1} x^*(s)\, ds\right) \in K^0(t) \qquad \text{for a.e. } t \in T. \tag{3.6}$$

4. BOUNDARY TRAJECTORIES

Theorem 1. *Let $C \in \mathcal{F}(\mathbb{R}^n)$, and let a mapping $F\colon T \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be measurable pseudo-Lipschitz near an arc $\widehat{x} \in \mathcal{R}_T(F, C)$ (see Definition 1). Suppose that a mapping $K\colon T \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies Hypothesis 3 and a cone $K \subset \mathbb{R}^n$ is a Boltyanskii tent to the set C at the point $\widehat{x}(t_0)$. Suppose also that $\Lambda\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator such that $\Lambda\widehat{x}(t_1) \notin \operatorname{Int}\Lambda(A)$, where $A := \{x(t_1) \mid x \in \mathcal{R}_T(F, C)\}$. Then there exists a vector $q \in \mathbb{R}^m$, $q \ne 0$, and an arc $p \in \mathrm{AC}(T, \mathbb{R}^n)$ such that*

$$p(t_0) \in K^0, \qquad p(t_1) = \Lambda^* q, \qquad (p'(t), p(t)) \in K^0(t) \quad \text{for a.e. } t \in T. \tag{4.1}$$

Proof. The proof consists of four steps.

Step 1. First of all, we perform some transformations of the Mayer problem (2.1). We replace the differential inclusion (1.4) by a differential inclusion with the right-hand side $\widetilde{F}$ defined by the formula $\widetilde{F}(t,x) := F(t, x + \widehat{x}(t)) - \widehat{x}'(t)$ for all $x \in B_\varepsilon(0)$ and a.e. $t \in T$. As a result, every F-trajectory x is transformed into an $\widetilde{F}$-trajectory $y := x - \widehat{x}$. This transformation defines a one-to-one correspondence between F-trajectories and $\widetilde{F}$-trajectories. In particular, the F-trajectory $\widehat{x}$ corresponds to the zero trajectory. Then we introduce a change of the time scale $s = m(t)$, where $m(t) := \int_{t_0}^{t} l(r)\,dr$, $t \in T$ (see (3.1)); thus, any arc y is assigned a new arc z such that $z(s) = y(t) = y(m^{-1}(s))$, $s \in [0, m(t_1)]$. As a result, we obtain a one-to-one correspondence between the arcs y defined on the interval $[t_0, t_1]$ and the arcs z defined on the interval $[0, m(t_1)]$. Moreover, the arc y is an $\widetilde{F}$-trajectory if and only if the arc z is an $\widehat{F}$-trajectory, i.e., a trajectory of a differential inclusion with the right-hand side $\widehat{F}$ defined by the formula

$$\widehat{F}(s,z) := \frac{1}{l(t)}\widetilde{F}(t,z), \qquad t = m^{-1}(s), \quad z \in B_\varepsilon(0), \quad \text{for a.e. } s \in [0, m(t_1)].$$

Accordingly, we should change the subset C of initial points to the subset $\widehat{C} := C - \widehat{x}(t_0)$ in the hypotheses of Theorem 1. As a result, the arc $\widehat{z} \equiv 0$ corresponding to the arc $\widehat{x}$ is a boundary $\widehat{F}$-trajectory for the transformed extremal problem on the interval $[0, m(t_1)]$ in the same sense as $\widehat{x}$ is a boundary F-trajectory for the original problem. It is obvious that the multivalued mapping $\widehat{F}$ is measurable pseudo-Lipschitz near the trajectory $\widehat{z} \equiv 0$ with the parameters $\widehat{l}(s) = 1$ for a.e. $s \in [0, m(t_1)]$ and $\widehat{R}(s) = R(t)/l(t)$ for $t = m^{-1}(s)$ and with the same parameter $\widehat{\nu} = \nu$ (see Definition 1).

Let us show that the assertion of the theorem for the transformed problem implies the assertion of the theorem for the original problem. Obviously, the equality

$$\operatorname{Graph} F(t,\cdot) - (\widehat{x}(t), \widehat{x}'(t)) = \begin{pmatrix} 1 & 0 \\ 0 & l(t) \end{pmatrix} \operatorname{Graph} \widehat{F}(s,\cdot), \qquad s = m(t),$$

holds for a.e. $t \in T$. Hence

$$T_{\mathrm{L}}\bigl(\operatorname{Graph} F(t,\cdot); (\widehat{x}(t), \widehat{x}'(t))\bigr) = \begin{pmatrix} 1 & 0 \\ 0 & l(t) \end{pmatrix} T_{\mathrm{L}}\bigl(\operatorname{Graph} \widehat{F}(s,\cdot); (0,0)\bigr).$$

In turn, Hypothesis 3 takes the form

$$T_{\mathrm{C}}\bigl(\operatorname{Graph} \widehat{F}(s,\cdot); (0,0)\bigr) \subset \widehat{K}(s) \subset T_{\mathrm{L}}\bigl(\operatorname{Graph} \widehat{F}(s,\cdot); (0,0)\bigr) \qquad \text{for a.e. } s \in [0, m(t_1)],$$

where

$$\widehat{K}(s) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{l(t)} \end{pmatrix} K(t), \qquad t = m^{-1}(s).$$

Based on the definition of the polar cone, we obtain the following relation between the polar cones $\widehat{K}^0(s)$ and $K^0(t)$:

$$\widehat{K}^0(s) = \begin{pmatrix} 1 & 0 \\ 0 & l(t) \end{pmatrix} K^0(t), \qquad t = m^{-1}(s). \tag{4.2}$$

If we prove Theorem 1 for the transformed problem, then we obtain the inclusion

$$(\widehat{p}'(s), \widehat{p}(s)) \in \widehat{K}^0(s) \qquad \text{for a.e. } s \in [0, m(t_1)]. \tag{4.3}$$

Define an arc $p(t) := \widehat{p}(m(t)) = \widehat{p}(s)$. Differentiating this function with respect to t, we have $p'(t) = \widehat{p}'(s) l(t)$. Hence, using (4.2) and (4.3), we arrive at the inclusion

$$(p'(t), p(t)) \in l(t) K^0(t) = K^0(t) \qquad \text{for a.e. } t \in T.$$

Thus, we have shown that without loss of generality it suffices to prove Theorem 1 only for the transformed problem.

Step 2. In view of step 1, in the further proof of the theorem (steps 2–4) we assume that $0 \in C$ and denote the interval $[0, m(t_1)]$ by T for simplicity. The mapping $F\colon T \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ is measurable pseudo-Lipschitz near the trajectory $\widehat{x} \equiv 0$ (and $0 \in F(t, 0)$) in the tube $W := T \times \overline{B_\varepsilon(0)}$ with parameters $\varepsilon, \nu > 0$, $\varepsilon\nu < 1$, with the pseudo-Lipschitz function $l(t) = 1$ a.e., and with a function $R(t) \geq 1/\nu$. Let $K\colon T \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ be a measurable mapping such that its values $K(t)$ are convex cones satisfying the inclusions $T_\mathrm{C}(\operatorname{Graph} F(t, \cdot); (0, 0)) \subset K(t) \subset T_\mathrm{L}(\operatorname{Graph} F(t, \cdot); (0, 0))$ (see Hypothesis 3). Define the norm in the space $\mathbb{R}^n \times \mathbb{R}^1$ by the formula $\|(x, x^0)\|_{\mathbb{R}^n \times \mathbb{R}^1} := \|x\| + |x^0|$.

Lemma 1. *Define mappings* $Q, P\colon T \times \mathbb{R}^n \times \mathbb{R}^1 \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^1_+)$ *as follows:*

$$Q(t, u, u^0) := \bigl\{(v, v^0) \in \mathbb{R}^n \times \mathbb{R}^1 \bigm| v^0 \geq \varrho(v, F(t, u))\bigr\}, \tag{4.4}$$

$$P(t, u, u^0) := \bigl\{(v, v^0) \in \mathbb{R}^n \times \mathbb{R}^1 \bigm| v^0 \geq 2\varrho((u, v), K(t))\bigr\}. \tag{4.5}$$

Then

(i) *the mapping* $Q\colon T \times (\overline{B_\varepsilon(0)} \times \mathbb{R}^1) \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^1)$ *is measurable pseudo-Lipschitz near the arc* $(0, 0) \in \mathcal{R}_T(Q, (C \times \{0\}))$ *with the previous parameters* $l(t) = 1$ *and* $R(t) \geq 1/\nu$;

(ii) *for a.e.* $t \in T$ *and every* $(u, u^0) \in B_1((0, 0)) \subset \mathbb{R}^n \times \mathbb{R}^1$, *the set* $P(t, u, u^0) \cap \overline{B_2((0, 0))}$ *is nonempty, closed, and convex, and the mapping* $P(t, u, u^0)$ *is measurable with respect to* $t \in T$ *and Lipschitz with respect to* (u, u^0) *in the Pompeiu–Hausdorff metric*;

(iii) *the inclusion*

$$P(t, u, u^0) \subset Q'_\mathrm{L}(t, u, u^0) \qquad \text{for a.e. } t \in T, \quad (u, u^0) \in \mathbb{R}^n \times \mathbb{R}^1, \tag{4.6}$$

is valid, where the set $Q'_\mathrm{L}(t, u, u^0)$ *is defined as*

$$Q'_\mathrm{L}(t, u, u^0) := \bigl\{(v, v^0) \in \mathbb{R}^n \times \mathbb{R}^1 \bigm| (u, u^0, v, v^0) \in T_\mathrm{L}(\operatorname{Graph} Q(t, \cdot); (0, 0, 0, 0))\bigr\}.$$

Proof. (i) Take arbitrary points $(t, x_1, x_1^0), (t, x_2, x_2^0) \in T \times \overline{B_\varepsilon}(0) \times [0, \varepsilon]$. Since the intersection $F(t, x_1) \cap R(t)B_1(0)$ is nonempty by the hypothesis, it follows that $Q(t, x_1, x_1^0) \cap R(t)B_1((0, 0))$ is also nonempty. Take an arbitrary point $(v_1, v_1^0) \in Q(t, x_1, x_1^0) \cap R(t)B_1((0, 0))$. According to (4.4), this means that $v_1^0 \geq \varrho(v_1, F(t, x_1))$, as well as $\|v_1\| + |v_1^0| < R(t)$. As a result, there exists a point $\widetilde{v}_1 \in F(t, x_1)$ such that $\|v_1 - \widetilde{v}_1\| \leq v_1^0$. This means that $\|\widetilde{v}_1\| < R(t)$, i.e., $\widetilde{v}_1 \in G(t, x_1)$ (see (2.3) with $\widehat{x} = 0$). From the inclusion (2.2) (with $l(t) = 1$) we obtain $\widetilde{v}_1 \in F(t, x_2) + \|x_1 - x_2\|\overline{B_1(0)}$; i.e., there exists a point $\widetilde{v}_2 \in F(t, x_2)$ such that $\|\widetilde{v}_1 - \widetilde{v}_2\| \leq \|x_1 - x_2\|$. Define $v_2 := \widetilde{v}_2 - \widetilde{v}_1 + v_1$. Then we have $(v_2, v_1^0) \in Q(t, x_2, x_2^0)$ and $(v_1, v_1^0) \in Q(t, x_2, x_2^0) + \|x_1 - x_2\|\overline{B_1((0, 0))}$; i.e., an analog of the inclusion (2.2) for the mapping Q is valid.

(ii) Since the set $K(t)$ is a closed convex cone for a.e. $t \in T$, the distance function $(u, v) \mapsto 2\varrho((u, v), K(t))$ is a convex Lipschitz function. Therefore, the sets $P(t, u, u^0)$ are convex and closed, the multivalued mapping P is measurable with respect to $t \in T$, and the set $\operatorname{Graph} P(t, \cdot)$ is a convex cone. Since $0 \in K(t)$, the inclusion $(0, 2\|u\|) \in P(t, u, u^0)$ follows from (4.5) for any point $(u, u^0) \in \mathbb{R}^n \times \mathbb{R}^1$; i.e., the set $P(t, u, u^0) \cap \overline{B_2((0, 0))}$ is nonempty for all $(u, u^0) \in B_1((0, 0))$. This implies that the mapping P satisfies the Lipschitz condition with respect to (u, u^0) (see, for example, [17, Corollary 7.1]).

(iii) For brevity, denote the lower tangent cone $T_\mathrm{L}(\operatorname{Graph} Q(t, \cdot); (0, 0, 0, 0))$ by $A(t)$. Since the mapping F is pseudo-Lipschitz near the zero arc with the Lipschitz function $l(t) \equiv 1$, for all $v \in \mathbb{R}^n$ and $u \in \overline{B_\varepsilon(0)} \subset \mathbb{R}^n$ we have the inequality $\varrho(v, F(t, u)) \leq 2\varrho((u, v), \operatorname{Graph} G(t, \cdot))$ for a.e. $t \in T$ (see [19, inequality (4.15)]). If we define

$$C(t) := \bigl\{(u, u^0, v, v^0) \in \overline{B_\varepsilon(0)} \times \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^1 \bigm| v^0 \geq 2\varrho((u, v), \operatorname{Graph} G(t, \cdot))\bigr\},$$

then the last inequality implies the inclusion $C(t) \subset \operatorname{Graph} Q(t,\cdot)$. Since the set $C(t)$ is the epigraph of the function $g(t,u,u^0,v) := 2\varrho((u,v), \operatorname{Graph} G(t,\cdot))$, which is Lipschitz continuous with respect to $(u,u^0,v) \in \overline{B_\varepsilon(0)} \times \mathbb{R}^1 \times \mathbb{R}^n$, we obtain the following inclusion by Proposition 1:

$$A(t) \supset T_{\mathrm{L}}(C(t);(0,0,0,0)) = \Big\{(u,u^0,v,v^0) \;\Big|\; v^0 \geq \limsup_{\lambda\to+0} \lambda^{-1}\big(g(t,\lambda u,\lambda u^0,\lambda v) - g(t,0,0,0)\big)\Big\}.$$

Since $g(t,0,0,0)=0$ and $\lambda^{-1}g(t,\lambda u,\lambda u^0,\lambda v) = 2\varrho((u,v),\lambda^{-1}(\operatorname{Graph} G(t,\cdot)))$, Proposition 3 implies the inclusion

$$\begin{aligned} T_{\mathrm{L}}(C(t);(0,0,0,0)) &\supset \Big\{(u,u^0,v,v^0) \mid v^0 \geq 2\varrho\Big((u,v), \liminf_{\lambda\to+0} \lambda^{-1} \operatorname{Graph} G(t,\cdot)\Big)\Big\} \\ &= \big\{(u,u^0,v,v^0) \mid v^0 \geq 2\varrho\big((u,v), T_{\mathrm{L}}(\operatorname{Graph} F(t,\cdot);(0,0))\big)\big\}. \end{aligned}$$

As a result, in view of Hypothesis 3, we arrive at the inclusion

$$A(t) \supset \big\{(u,u^0,v,v^0) \;\big|\; v^0 \geq 2\varrho((u,v),K(t))\big\} = \operatorname{Graph} P(t,\cdot),$$

which implies (4.6). $\square$

Step 3.

Lemma 2. *Under the hypotheses and notation of Lemma* 1, *the cone* $(\operatorname{Graph} P(t,\cdot))^0$, *which is polar to the cone* $\operatorname{Graph} P(t,\cdot)$, *can be calculated by the formula*

$$(\operatorname{Graph} P(t,\cdot))^0 = \Big\{\tau\cdot(u^*,0,v^*,-1) \in \mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n\times\mathbb{R}^1 \;\Big|\; \tau\geq 0,\ (u^*,v^*)\in K^0(t)\cap \overline{B_2^{\mathbb{R}^n\times\mathbb{R}^n}(0)}\Big\}.$$

Proof. Since the cone $\operatorname{Graph} P(t,\cdot)$ is the epigraph of a convex positively homogeneous function $z \mapsto f(t,z)$, where $f(t,z) := 2\varrho((u,v),K(t))$ for $z := (u,u^0,v)$, it follows from the properties of convex functions that the polar cone to this cone can be calculated by the formula

$$(\operatorname{Graph} P(t,\cdot))^0 = \overline{\operatorname{cone}}\big\{(w,-1) \mid w \in \partial_z f(t,0)\big\}.$$

At the same time, the subdifferential of a convex function at zero is given by

$$\partial_z f(t,0) := \big\{z^* \in \mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n \;\big|\; f(t,z) \geq \langle z^*,z\rangle\ \forall z\in\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n\big\},$$

which implies that the point $z^* := (u^*,u^{0*},v^*)$ belongs to the subdifferential $\partial_z f(t,0)$ if and only if $u^{0*}=0$ and $(u^*,v^*) \in K^0(t)\cap\overline{B_2(0)}$. Here the first equality follows from the fact that the function $u^0 \mapsto f(t,u,u^0,v)$ is constant. The second inclusion follows from the inclusion $0\in K(t)$, which means that $f(t,z)\leq 2\|z\|$ for all z; hence, for all $z^*\in\partial_z f(t,0)$ we obtain the inequality $\|z^*\|\leq 2$. In turn, since the equality $f(t,z_0)=0$ holds for all $z_0=(u,u^0,v)$ such that $(u,v)\in K(t)$, it follows that $\langle z^*,z_0\rangle > 0 = f(t,z_0)$ for all $z^*=(u^*,0,v^*)$ such that $(u^*,v^*)\notin K^0(t)$, i.e., $z^*\notin\partial_z f(t,0)$. $\square$

Step 4: *The main part of the proof of Theorem* 1. Under the conditions of step 2, we extend the state space $\mathbb{R}^n$ to the space $\mathbb{R}^n\times\mathbb{R}^1$ and pass to a new problem on the interval $T=[0,m(t_1)]$. To this end, we define the set $\widetilde{C} := C\times\{0\}\subset\mathbb{R}^n\times\mathbb{R}^1$ and the cone $\widetilde{K} := K\times\{0\}\subset\mathbb{R}^n\times\mathbb{R}^1$. In the space $\mathbb{R}^n\times\mathbb{R}^1$, consider the mappings Q and P defined in (4.4) and (4.5), with the properties described in Lemma 1. Consider the sets of trajectories $\mathcal{R}_T(Q,\widetilde{C})$ and $\mathcal{R}_T^\infty(P,\widetilde{K})$. According to (4.4), the inclusion $(0,0)\in\mathcal{R}_T(Q,\widetilde{C})$ is valid (since $\widehat{x}\equiv 0$). Define the set

$$M_\Lambda := \big\{(\Lambda x(m(t_1)), x^0(m(t_1))) \in \mathbb{R}^m\times\mathbb{R}^1 \;\big|\; (x,x^0)\in\mathcal{R}_T^\infty(P,\widetilde{K})\big\}. \tag{4.7}$$

By definition, the set M_Λ is a convex cone in $\mathbb{R}^m\times\mathbb{R}^1_+$, and it is obvious that $(0,1)\in M_\Lambda$. Let us prove that there exists a vector $(q,q^0)\in\mathbb{R}^m\times\mathbb{R}^1$ such that $(q,q^0)\in M_\Lambda^0$ and $q\neq 0$, where M_Λ^0

denotes the polar cone to M_Λ. Suppose the contrary, i.e., that the inclusion $M_\Lambda^0 \subset \{0\} \times \mathbb{R}^1$ holds, with $0 \in \mathbb{R}^m$. This means that $\overline{M_\Lambda} = M_\Lambda^{00} \supset \{(x, 0) \mid x \in \mathbb{R}^m\}$. Since the cone M_Λ is convex, we then obtain

$$M_\Lambda \supset \mathbb{R}^m \times (\mathbb{R}^1_+ \setminus \{0\}). \tag{4.8}$$

In the space $\mathbb{R}^m$, take a simplex Δ with vertices z_i, $i \in \overline{1, m+1}$, and with boundary Γ such that $0 \in \operatorname{Int} \Delta$. Let $b_0 := \min\{\|z\| \mid z \in \Gamma\}$. By construction, $b_0 > 0$. It follows from the definition of the cone M_Λ (4.7) and the inclusion (4.8) that there exist trajectories $(u_i, u_i^0) \in \mathcal{R}_T^\infty(P, \widetilde{K})$, $i \in \overline{1, m+1}$, such that

$$\Lambda u_i(m(t_1)) = z_i, \qquad u_i^0(m(t_1)) \le \frac{b_0}{16(1 + \|\Lambda\|)} \exp(-m(t_1)) \qquad \forall i \in \overline{1, m+1}. \tag{4.9}$$

For every $\sigma := (\sigma_1, \dots, \sigma_{m+1}) \in \Sigma^m$ (see (3.4)), define the arc

$$(u_\sigma, u_\sigma^0) := \sum_{i=1}^{m+1} \sigma_i (u_i, u_i^0) \tag{4.10}$$

in the space $\mathrm{AC}^\infty(T, \mathbb{R}^n \times \mathbb{R}^1)$. It is obvious that for every $\sigma \in \Sigma^m$ the arc (u_σ, u_σ^0) belongs to the convex set of trajectories $\mathcal{R}_T^\infty(P, \widetilde{K})$; moreover, in view of the inclusion (4.6), this arc belongs to the set $\mathcal{R}_T^\infty(Q'_{\mathrm{L}}, \widetilde{K})$.

By Proposition 6 with $\widehat{x} \equiv 0$, $l(t) \equiv 1$, and $\gamma = e^{-m(t_1)}/16$, for the mapping Q (4.4) and the functions (4.10), there exists a number $\lambda_0 \in (0, 1)$ such that for any $\lambda \in (0, \lambda_0]$ and $\sigma \in \Sigma^m$ there exists a trajectory

$$(x_{\lambda,\sigma}, x^0_{\lambda,\sigma}) \in \mathcal{R}_T(Q, \widetilde{C}) \tag{4.11}$$

that can be represented as

$$x_{\lambda,\sigma}(t) = \lambda u_\sigma(t) + o(t, \lambda, \sigma), \qquad x^0_{\lambda,\sigma}(t) = \lambda u^0_\sigma(t) + o^0(t, \lambda, \sigma), \tag{4.12}$$

and every function $\sigma \mapsto (x_{\lambda,\sigma}, x^0_{\lambda,\sigma})$ from Σ^m to $\mathrm{AC}(T, \mathbb{R}^n \times \mathbb{R}^1)$ is continuous.

Since the functions $\lambda \mapsto (o(\cdot, \lambda, \sigma), o^0(\cdot, \lambda, \sigma))$ are o-small as $\lambda \to +0$, there exists a number $\lambda_1 \in (0, \lambda_0]$ such that the inequality

$$\max_{\sigma \in \Sigma^m} \lambda^{-1} \big\| \big(o(\cdot, \lambda, \sigma), o^0(\cdot, \lambda, \sigma)\big) \big\|_{\mathrm{AC}} \le \frac{b_0}{16(1 + \|\Lambda\|)} e^{-m(t_1)} \tag{4.13}$$

holds for every $\lambda \in (0, \lambda_1]$. Moreover, for all $\lambda \in (0, \lambda_1]$ and all $\sigma \in \Sigma^m$, we have $\|x_{\lambda,\sigma}\|_{\mathrm{AC}} \le \varepsilon/4$ and $x^{0\prime}_{\lambda,\sigma}(t) \le e^{-m(t_1)} b(\varepsilon)/16$ for a.e. $t \in T$. By the definition of the mapping Q (4.4), from the inclusion (4.11) we obtain

$$x^{0\prime}_{\lambda,\sigma}(t) \ge \varrho\big(x'_{\lambda,\sigma}(t), F(t, x_{\lambda,\sigma}(t))\big) \qquad \forall \lambda \in (0, \lambda_0), \quad \sigma \in \Sigma^m,$$

for a.e. $t \in T$. Thus, we have shown that the inclusion $x_{\lambda,\sigma} \in D_0(F, x^{0\prime}_{\lambda,\sigma}, 1)$ (see (3.2)) is valid. Let us define the number $\lambda_2 := \min\{\lambda_1, (1 + \|\Lambda\|) b(\varepsilon)/b_0\}$. Then, for any $\lambda \in (0, \lambda_2]$, by Proposition 5 (with $a = \lambda b_0/(3(1 + \|\Lambda\|))$ and $d(x) = x$), there exist trajectories $y_{\lambda,\sigma} \in \mathcal{R}_T(F, x_{\lambda,\sigma}(0))$ such that the function $\sigma \mapsto y_{\lambda,\sigma}$ from Σ^m to $\mathrm{AC}(T, \mathbb{R}^n)$ is continuous. Using relations (3.3), (4.12), (4.9), and (4.13), we obtain the following inequalities for the arcs $w_{\lambda,\sigma} := y_{\lambda,\sigma} - x_{\lambda,\sigma}$:

$$\|w_{\lambda,\sigma}\|_{\mathrm{AC}} < e^{m(t_1)} x^0_{\lambda,\sigma}(m(t_1)) + \frac{b(a)}{8} \le \frac{\lambda b_0}{4(1 + \|\Lambda\|)}, \qquad \lambda \in (0, \lambda_2].$$

Fix $\lambda = \lambda_2$. Then it follows from the above inequalities and (4.12) and (4.13) that $\Lambda y_{\lambda_2,\sigma}(m(t_1))$ can be represented as

$$\Lambda y_{\lambda_2,\sigma}(m(t_1)) = \lambda_2 \Lambda u_\sigma(m(t_1)) + \lambda_2 \Lambda\big(\lambda_2^{-1} o(m(t_1), \lambda_2, \sigma) + \lambda_2^{-1} w_{\lambda_2,\sigma}(m(t_1))\big).$$

This means that

$$\Lambda y_{\lambda_2,\sigma}(m(t_1)) = \lambda_2\varphi(\sigma) + \lambda_2 g(\sigma) \qquad \forall \sigma \in \Sigma^m, \tag{4.14}$$

where $\varphi(\sigma) := \sum_{i=1}^{m+1} \sigma_i z_i \in \Delta$ and the function $g\colon \Sigma^m \to \mathbb{R}^m$ is continuous and satisfies the inequality

$$\max_{\sigma\in\Sigma^m} \|g(\sigma)\| < \frac{b_0}{2}. \tag{4.15}$$

Since the function $\varphi\colon \Sigma^m \to \Delta$ provides a one-to-one correspondence between the sets Σ^m and Δ, we can define a function $\widetilde{g}\colon \Delta \to \mathbb{R}^m$ by setting $\widetilde{g}(z) = g(\varphi^{-1}(z))$. We also define the function $f(z) := z + \widetilde{g}(z)$, $z \in \Delta$. These functions are continuous on Δ, and for every $z \in \Delta$ (with the corresponding $\sigma = \varphi^{-1}(z)$) there exists a trajectory $y_z := y_{\lambda_2,\sigma} \in \mathcal{R}_T(F, C)$ such that the equality $\Lambda y_z(m(t_1)) = \lambda_2 f(z)$ is valid in view of (4.14). For any $u_0 \in (1/2)\Delta$, from (4.15) we obtain

$$\|f(z) - z\| < \frac{b_0}{2} \le \|z - u_0\| \qquad \forall z \in \Gamma. \tag{4.16}$$

Hence, by the scholium from [10, Sect. 4.1], the inclusion $u_0 \in f(\Delta)$ holds. This means that there exists a point $z_0 \in \Delta$ such that $u_0 = f(z_0)$. Since the point u_0 is arbitrary, we obtain the inclusion $(1/2)\Delta \subset f(\Delta)$, which, in turn, implies the inclusion $(\lambda_2 b_0/2)B_1(0) \subset \{\Lambda y_z(m(t_1)) \mid z \in \Delta\} \subset \Lambda(A)$. This contradicts the hypothesis of the theorem that $\Lambda\widehat{x}(m(t_1)) \equiv 0 \notin \operatorname{Int}\Lambda(A)$.

Thus, we have proved that there exists a vector $(q, q^0) \in M_\Lambda^0$ such that $q \ne 0$; hence, by the definition of the polar cone, we obtain the inequality

$$\langle q, \Lambda x(m(t_1))\rangle + q^0 x^0(m(t_1)) \le 0 \tag{4.17}$$

for any arc $(x, x^0) \in \mathcal{R}_T^\infty(P, \widetilde{K})$. Note that the conic mapping $\operatorname{Graph} P(t,\cdot)$ satisfies all the hypotheses of Proposition 7 with $\gamma(t) \equiv 2$. Consider a pair $(b^*, y^*) \in (\mathbb{R}^n \times \mathbb{R}^1) \times L^1(T, (\mathbb{R}^n \times \mathbb{R}^1))$ of the form $b^* = y^*(t) \equiv (\Lambda^* q, q^0) \in \mathbb{R}^n \times \mathbb{R}^1$. For any arc $(x, x^0) \in \mathcal{R}_T^\infty(P, \widetilde{K})$, using inequality (4.17), we calculate the bilinear form

$$\begin{aligned}\langle (b^*, y^*), (x, x^0)\rangle &:= \langle b^*, (x(0), x^0(0))\rangle + \int\limits_0^{m(t_1)} \langle y^*(s), (x'(s), x^{0\prime}(s))\rangle\, ds \\ &= \langle b^*, (x(m(t_1)), x^0(m(t_1)))\rangle = \langle \Lambda^* q, x(m(t_1))\rangle + q^0 x^0(m(t_1)) \le 0.\end{aligned}$$

This means that the pair (b^*, y^*) belongs to the cone $(\mathcal{R}_T^\infty(P, \widetilde{K}))^0$. By Proposition 7, for this pair (b^*, y^*) there exists a function $x^* \in L^1(T, \mathbb{R}^n \times \mathbb{R}^1)$ such that the inclusions (3.5) and (3.6) hold. Define an arc $\widetilde{p} \in \mathrm{AC}(T, \mathbb{R}^n \times \mathbb{R}^1)$ by setting $\widetilde{p}(t) := b^* - \int_t^{m(t_1)} x^*(s)\, ds$. Then $\widetilde{p}(m(t_1)) = b^* = (\Lambda^* q, q^0)$; hence, in view of the inclusion (3.5), we have $\widetilde{p}(0) \in \widetilde{K}^0$. In turn, the inclusion (3.6) implies the differential inclusion

$$(\widetilde{p}\,'(t), \widetilde{p}(t)) \in (\operatorname{Graph} P(t,\cdot))^0 \qquad \text{for a.e. } t \in T.$$

Let us write the arc $\widetilde{p}(\cdot)$ componentwise, i.e., $\widetilde{p}(t) = (p^1(t), p^0(t)) \in \mathbb{R}^n \times \mathbb{R}^1$. Then, by Lemma 2, the preceding inclusion reads

$$(p^{1\prime}(t), p^1(t)) \in K^0(t), \qquad p^{0\prime}(t) \equiv 0 \qquad \text{for a.e. } t \in T.$$

Therefore, taking the arc $p := p^1$, we obtain expressions (4.1). This completes the proof of Theorem 1. $\square$

5. THE MAIN RESULT

Lemma 3. *Let a mapping $H\colon \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be pseudo-Lipschitz near a point $(x_0,y_0)\in \operatorname{Graph} H$; i.e., let there exist numbers $\varepsilon>0$, $l>0$, and $R\in(0,+\infty]$ such that the set $G(x):=H(x)\cap B_R(y_0)$ is nonempty and the inclusion $G(x)\subset H(z)+l\|x-z\|\overline{B_1(0)}$ holds for all $x,z\in B_\varepsilon(x_0)$. Then the following inclusion is valid for all $w\in G(x_0)$:*

$$(0,w-y_0)\in T_{\mathrm{C}}(\operatorname{Graph}(\operatorname{co} G);(x_0,y_0)).$$

Proof. Take an arbitrary point $w\in G(x_0)$. Define a point $v_0:=w-y_0$ and a number $\delta:=\|v_0\|$. By construction, $\delta<R$. Consider an arbitrary number sequence $\{\lambda_k>0\}_{k\in\mathbb{N}}$ such that $\lambda_k\to 0$ as $k\to+\infty$. Consider also an arbitrary sequence $\{(x_k,y_k)\in \operatorname{Graph}(\operatorname{co} G)\}_{k\in\mathbb{N}}$ that converges to the point (x_0,y_0) as $k\to+\infty$.

Let $\alpha:=\min\{(R-\delta)/l,\varepsilon\}$. There exists a number K_0 such that $x_k\in B_\alpha(x_0)$ and $\lambda_k\in(0,1)$ for all $k>K_0$. Since the mapping H is pseudo-Lipschitz, the inclusion $w\in H(x_k)+l\|x_k-x_0\|\overline{B_1(0)}$ holds for all $k>K_0$; i.e., there exist points $z_k\in H(x_k)$ such that $\|z_k-w\|\le l\|x_k-x_0\|$; i.e., $z_k\to w$ as $k\to+\infty$ and $\|z_k-y_0\|<R$; i.e., $z_k\in G(x_k)$. Define $v_k:=z_k-y_k$ for all $k>K_0$ and $v_k:=0$ for all $k\le K_0$. Obviously, $v_k\to v_0$ as $k\to+\infty$, and the inclusion $y_k+\lambda_k v_k=(1-\lambda_k)y_k+\lambda_k z_k\in\operatorname{co} G(x_k)$ is valid for all $k\in\mathbb{N}$. Thus, for any sequences $\{\lambda_k>0\}_{k\in\mathbb{N}}$ and $\{(x_k,y_k)\in\operatorname{Graph}(\operatorname{co} G)\}$ such that $\lambda_k\to 0$ and $(x_k,y_k)\to(x_0,y_0)$ as $k\to+\infty$, we have found a sequence $\{(0,v_k)\in\mathbb{R}^n\times\mathbb{R}^n\}_{k\in\mathbb{N}}$ that converges to $(0,v_0)\in\mathbb{R}^n\times\mathbb{R}^n$, and for all $k\in\mathbb{N}$ the inclusion $(x_k,y_k)+\lambda_k(0,v_k)\in\operatorname{Graph}(\operatorname{co} G)$ is valid. By the definition of the Clarke tangent cone (see [4]), this means that $(0,v_0)\in T_{\mathrm{C}}(\operatorname{Graph}(\operatorname{co} G);(x_0,y_0))$. ☐

Corollary 1. *Let the hypotheses of Lemma 3 be satisfied. Suppose that a closed convex cone $K\subset\mathbb{R}^n\times\mathbb{R}^n$ satisfies the inclusion $K\supset T_{\mathrm{C}}(\operatorname{Graph}(\operatorname{co} G);(x_0,y_0))$. Then, for any point $(q,p)\in K^0$, where K^0 is the polar cone to K, the following maximum principle holds*:

$$\langle p,y_0\rangle=\max\{\langle p,w\rangle \mid w\in G(x_0)\}.$$

Theorem 2. *Let $T:=[t_0,t_1]$, and let a trajectory $\widehat{x}\in\mathcal{R}_T(F,C_0)$ be a local solution to the original Mayer problem (2.1) in the space $\mathrm{AC}(T,\mathbb{R}^n)$. Suppose that the mapping F is measurable pseudo-Lipschitz near the arc $\widehat{x}$ (see Definition 1). Let the cones $K(t)$ satisfy Hypothesis 3. Let K_0 and K_1 be some Boltyanskii tents to the sets C_0 and C_1 at the points $\widehat{x}(t_0)$ and $\widehat{x}(t_1)$, respectively. Suppose that $\psi\colon\mathbb{R}^n\to\mathbb{R}^1$ is a convex positively homogeneous function satisfying the inequality $D^+_{\mathrm{L}}\varphi(\widehat{x}(t_1))(x)\le\psi(x)<\infty$ for all $x\in\mathbb{R}^n$ (see (1.3)). Then there exists a number $\lambda\ge 0$ and an arc $p\in\mathrm{AC}(T,\mathbb{R}^n)$ such that the following conditions are satisfied*:

(1) *the nontriviality conditions $\lambda+\|p\|_{\mathrm{AC}}>0$;*

(2) *the transversality conditions*

$$p(t_0)\in K_0^0,\qquad -p(t_1)\in K_1^0+\lambda\partial\psi(0);\tag{5.1}$$

(3) *the Euler differential inclusion for the arc p,*

$$(p'(t),p(t))\in K^0(t)\qquad \textit{for a.e. } t\in T.\tag{5.2}$$

Proof. Let us extend the state space $\mathbb{R}^n$ to the following spaces Z and V:

$$Z:=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^1,\qquad V:=\mathbb{R}^1\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n.\tag{5.3}$$

Define a set $\widetilde{C} \subset Z$, a linear operator $\Lambda\colon Z \to V$, and a multivalued mapping $H\colon T \times Z \to \mathcal{P}(Z)$ by the formulas

$$\begin{gathered} \widetilde{C} := \bigl\{z := (z^1, z^2, z^3, z^4, z^5) \bigm| z^1 \in C_0,\ z^3 \in C_1,\ z^5 \ge \varphi(z^4)\bigr\}, \\ \Lambda z := \bigl\{(v^1, v^2, v^3, v^4) \in V \bigm| v^1 = z^5,\ v^2 = z^3 - z^1,\ v^3 = z^4 - z^2,\ v^4 = z^2 - z^1\bigr\}, \\ H(t, z) := \bigl\{w \in Z \bigm| w = (w^1, w^1, 0, 0, 0),\ w^1 \in F(t, z^1)\bigr\}. \end{gathered} \tag{5.4}$$

Define an arc $\widehat{z}\colon T \to Z$ and a set $A \subset Z$ as

$$\widehat{z}(t) := \bigl(\widehat{x}(t), \widehat{x}(t), \widehat{x}(t_1), \widehat{x}(t_1), \varphi(\widehat{x}(t_1))\bigr), \qquad t \in T, \tag{5.5}$$

$$A := \bigl\{z(t_1) \in Z \bigm| z \in \mathcal{R}_T(H, \widetilde{C})\bigr\}. \tag{5.6}$$

Obviously, the arc $\widehat{z}$ is a trajectory of the extended differential inclusion, i.e., $\widehat{z} \in \mathcal{R}_T(H, \widetilde{C})$. Since the mapping F is measurable pseudo-Lipschitz near $\widehat{x}$, one can easily show that the mapping H is also measurable pseudo-Lipschitz near $\widehat{z}$.

Let us show that $\Lambda\widehat{z}(t_1) \notin \operatorname{Int}\Lambda(A)$. Notice that for any $\alpha \ge 0$ and any arc $x \in \mathcal{R}_T(F, C_0)$ such that $x(t_1) \in C_1$, the function

$$u_\alpha(t) := \bigl(x(t), x(t), x(t_1), x(t_1), \varphi(x(t_1)) + \alpha\bigr), \qquad t \in T, \tag{5.7}$$

satisfies the inclusion $u_\alpha \in \mathcal{R}_T(H, \widetilde{C})$. In view of (5.4), the equality $\Lambda u_\alpha(t_1) = (\varphi(x(t_1)) + \alpha, 0, 0, 0)$ is also valid. The converse also holds; i.e., if an arc $z \in \mathcal{R}_T(H, \widetilde{C})$ is such that there exists a number β for which the equality $\Lambda z(t_1) = (\beta, 0, 0, 0)$ is satisfied, then it follows from (5.4) that the trajectory z satisfies the equality $z(t) = (z^1(t), z^1(t), z^1(t_1), z^1(t_1), \beta)$, where $\beta \ge \varphi(z^1(t_1))$, $z^1 \in \mathcal{R}_T(F, C_0)$, and $z^1(t_1) \in C_1$. For the trajectory $\widehat{z}$ (see (5.5)), we have $\Lambda\widehat{z}(t_1) = (\varphi(\widehat{x}(t_1)), 0, 0, 0)$. Suppose that the inclusion $\Lambda\widehat{z}(t_1) \in \operatorname{Int}\Lambda(A)$ is valid. Then there exists a number $\mu > 0$ and an arc $\widetilde{u}_0 \in \mathcal{R}_T(H, \widetilde{C})$ (see (5.7) for $\alpha = 0$) such that $\Lambda\widetilde{u}_0(t_1) = (\varphi(\widehat{x}(t_1)) - \mu, 0, 0, 0)$; hence, $\varphi(\widetilde{x}(t_1)) = \varphi(\widehat{x}(t_1)) - \mu$, which is impossible since the trajectory $\widehat{x}$ is a solution to problem (2.1). Thus, $\Lambda\widehat{z}(t_1) \notin \operatorname{Int}\Lambda(A)$. Define $K_2 := \operatorname{Epi}\psi$. By Proposition 4, the cone K_2 is a Boltyanskii tent to the set $\operatorname{Epi}\varphi$ at the point $(\widehat{x}(t_1), \varphi(\widehat{x}(t_1)))$, and the inclusion $K_2 \subset T_{\mathrm{L}}(\operatorname{Epi}\varphi, (\widehat{x}(t_1), \varphi(\widehat{x}(t_1))))$ holds. Using (5.4), we define cones $\widetilde{K} \subset Z$ and $\widetilde{K}(t) \subset Z \times Z$ (see (5.3)):

$$\begin{gathered} \widetilde{K} := K_0 \times \mathbb{R}^n \times K_1 \times K_2, \\ \widetilde{K}(t) := \bigl\{(z, w) \in Z \times Z \bigm| w^1 = w^2,\ w^3 = w^4 = 0,\ w^5 = 0,\ (z^1, w^1) \in K(t)\bigr\}. \end{gathered}$$

One can easily verify that the cone $\widetilde{K}$ is a Boltyanskii tent to the set C at the point $\widehat{z}(t_0)$, the cones $\widetilde{K}(t)$ are convex and closed for a.e. $t \in T$, the mapping $\widetilde{K}\colon T \to Z \times Z$ is measurable, and the following inclusion holds:

$$\widetilde{K}(t) \subset T_{\mathrm{L}}\bigl(\operatorname{Graph} H(t, \cdot); (\widehat{z}(t), \widehat{z}\,'(t))\bigr) \qquad \text{for a.e. } t \in T.$$

The cones $\widetilde{K}^0$ and $\widetilde{K}^0(t)$, which are polar to the cones $\widetilde{K}$ and $\widetilde{K}(t)$, have the form

$$\widetilde{K}^0 = K_0^0 \times \{0\} \times K_1^0 \times K_2^0, \tag{5.8}$$

$$\begin{aligned} \widetilde{K}^0(t) = \bigl\{\bigl(p^1, p^2, p^3, p^4, p^5, v^1, v^2, v^3, v^4, v^5\bigr) \in Z \times Z \bigm| p^2 = p^3 = p^4 = 0,\ p^5 = 0,& \\ (p^1, v^1 + v^2) \in K^0(t)\bigr\}.& \end{aligned} \tag{5.9}$$

According to Theorem 1, there exists an arc $\widetilde{p} := (p^1, p^2, p^3, p^4, p^5) \in \mathrm{AC}(T, Z)$ and a nonzero vector $q := (q^1, q^2, q^3, q^4) \in V$ such that relations (4.1) are valid. From (4.1), (5.4), and (5.8), we obtain the following formulas:

$$\widetilde{p}(t_1) = \Lambda^* q = \left(-q^2 - q^4, -q^3 + q^4, q^2, q^3, q^1\right), \qquad p^1(t_0) \in K_0^0,$$
$$p^2(t_0) = 0, \qquad p^3(t_0) \in K_1^0, \qquad (p^4(t_0), p^5(t_0)) \in K_2^0.$$

Using (4.1) and (5.9), we then obtain

$$(p^{1\prime}(t),\ p^1(t) + p^2(t)) \in K^0(t), \qquad p^{2\prime}(t) = p^{3\prime}(t) = p^{4\prime}(t) = 0, \qquad p^{5\prime}(t) = 0.$$

This means that the functions $p^2(t) \equiv 0$, $p^3(t) \equiv p^3$, $p^4(t) \equiv p^4$, and $p^5(t) \equiv p^5$ are constant; i.e.,

$$-q^3 + q^4 = p^2 = 0, \qquad -p^1(t_1) = q^2 + q^3, \qquad p^3 = q^2, \qquad p^4 = q^3, \qquad p^5 = q^1.$$

The vector (p^4, p^5) is normal to the set $\operatorname{Epi}\varphi$ at the point $(\widehat{x}(t_1), \varphi(\widehat{x}(t_1)))$; therefore, two cases are possible.

(1) $p^5 < 0$. In view of positive homogeneity, it can be normalized; i.e., we assume that $p^5 = -1$. Then we obtain $p^4 \in \partial\psi(0)$; hence, $p^1(t_1) = -p^3 - p^4 \in -K_1^0 - \partial\psi(0)$ (i.e., the case when $\lambda = 1$). In this case, we take the arc $p(t) := p^1(t)$.

(2) $p^5 = 0$. Then $p^4 = 0$, and so $q^1 = q^3 = q^4 = 0$; i.e., $q = (0, q^2, 0, 0)$. Since $q \neq 0$, we have $q^2 \neq 0$; i.e., $p^1(t_1) \neq 0$ and $p^1(t_1) \in -K_1^0$ (i.e., the case of $\lambda = 0$). In this case, we also take the arc $p(t) := p^1(t)$. $\square$

Theorem 2 and Corollary 1 imply

Corollary 2. *In particular, suppose that the sets $F(t,x) \cap (\widehat{x}'(t) + R(t)B_1(0))$ are convex for a.e. $t \in T$ and all $x \in B_\varepsilon(\widehat{x}(t))$. Then it follows from the Euler differential inclusion* (5.2) *that the arc p satisfies the Pontryagin maximum principle, i.e.,*

$$\langle p(t), \widehat{x}'(t)\rangle \geq \langle p(t), y\rangle \qquad \forall y \in F(t, \widehat{x}(t)) \cap (\widehat{x}'(t) + R(t)B_1(0)) \quad \textit{for a.e. } t \in T.$$

6. EXAMPLE

Consider a simple example of the Mayer problem (2.1) in which the necessary optimality conditions involving polar cones to tangent cones can be more precise than those with the Clarke normal cone or the limiting normal cone [5].

Let $n = 1$ and the time interval $[t_0, t_1]$ be $[0, 1]$. Let $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ be a Lipschitz function such that $f(x) = f(-x)$ for $x \in \mathbb{R}^1$ and $f(x) = 1/2$ for $|x| \geq 1$. On the interval $[0, 1]$, the function f is a continuous piecewise affine function with a countable number of segments, and the graph of f is enclosed between the parabola $y = x^2/2$ and the line $y = 0$ with corner points on these two arcs. All segments of this piecewise affine function have inclination angles to the axis Ox equal to $\pm\pi/4$, and when the argument x decreases from 1 to 0, the signs of the angles alternate. Suppose also that a function $l \in L^1([0,1], \mathbb{R}^1_+)$, $l(t) > 0$, is chosen in such a way that it is unbounded on any open interval.

Define the mapping $F(t,x) := l(t)(\{y \in \mathbb{R}^1 \mid f(x) \leq y \leq x^2\} \cup \{y \in \mathbb{R}^1 \mid y \geq 2x^2\})$ for $t \in [0,1]$ and $x \in \mathbb{R}^1$. Then, for any choice of the numbers $\nu > 0$ and $\varepsilon \in (0, \min\{1/4, 1/\nu\})$ and the function $R(t) \geq \nu l(t)$, the mapping F is measurable pseudo-Lipschitz near $\widehat{x}(t) \equiv 0$, $t \in [0,1]$. Consider the problem of minimizing the terminal value $x(1)$ over all trajectories of the differential inclusion $x' \in F(t,x)$ that satisfy the initial condition $x(0) = 0$. It is obvious that $\widehat{x}(t) \equiv 0$ is a solution to this problem.

One can easily verify that the lower tangent cone to the set $\operatorname{Graph} F(t,\cdot)$ at the point 0 has the form $T_{\mathrm{L}}(\operatorname{Graph} F(t,\cdot); 0) = \{(u,v) \in \mathbb{R}^2 \mid v \geq 0\}$. Taking $K(t) := T_{\mathrm{L}}(\operatorname{Graph} F(t,\cdot); 0)$, we calculate its polar cone $K^0(t) = \{(q,p) \in \mathbb{R}^2 \mid q = 0,\, p \le 0\}$. Thus, the necessary conditions (5.2) (the Euler differential inclusion) take a simple form: $p'(t) = 0$ and $p(t) = \mathrm{const} \le 0$. From the transversality conditions we find $\lambda = 1$ and $p(t) \equiv -1$.

One can also easily verify that the limiting normal cone to the set $\operatorname{Graph} F(t,\cdot)$ at the point 0 has the form

$$N^{\mathrm{L}}_{\operatorname{Graph} F(t,\cdot)}(0) = \big\{(q,p) \in \mathbb{R}^2 \bigm| |q| + l(t)p \le 0\big\} \cup \big\{(q,p) \bigm| q = 0,\ p \ge 0\big\}$$

for a.e. $t \in [0,1]$. The necessary conditions from [5, Theorem 3.1.1] in the form of the Euler differential inclusion for the arc p are more complicated: either $|p'(t)| + l(t)p(t) \le 0$ a.e., or $p(t) = \mathrm{const} \ge 0$ a.e.

7. CONCLUSIONS

We have described a direct method for finding necessary conditions in the form of the Euler–Lagrange differential inclusion for the Mayer optimization problem on an interval under constraints on the trajectories given by a differential inclusion with unbounded right-hand side. In [20] we presented a direct method for finding necessary conditions in the Lagrangian form for a time-optimal problem with differential inclusion having an unbounded right-hand side, and in [19] we obtained necessary optimality conditions in the Lagrangian form for the Mayer problem with free right end in which the values of trajectories belong to a reflexive separable Banach space.

Our necessary conditions are obtained by the direct method, which is a development of Pontryagin's direct method, and are expressed in terms of polar cones, which differ from the Clarke normal cone and the limiting normal cone [5]. We have presented a simple example in which polar cones provide a more precise result than the limiting normal cones.

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