ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2019, Vol. 304, pp. 190–204. © Pleiades Publishing, Ltd., 2019.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2019, Vol. 304, pp. 205–220.

Criteria for Convexity of Closed Sets in Banach Spaces

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Received October 2, 2018; revised October 22, 2018; accepted January 17, 2019

Abstract—Criteria for the convexity of closed sets in general Banach spaces in terms of the Clarke and Bouligand tangent cones are proved. In the case of uniformly convex spaces, these convexity criteria are stated in terms of proximal normal cones. These criteria are used to derive sufficient conditions for the convexity of the images of convex sets under nonlinear mappings and multifunctions.

DOI: 10.1134/S0081543819010139

1. INTRODUCTION

We start the discussion of the subject of the present paper by considering the following problem: Under what assumptions on a nonlinear mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is the image $F(C^p_\Delta)$ of a set C^p_Δ convex?

The set $C^p_\Delta \subset \mathbb{R}^n$ is determined by scalar parameters $p > 1$ and $\Delta > 0$:

$$C^p_\Delta := \bigl\{x = [x_1, \dots, x_n]\colon\ \|x\|_p \le \Delta\bigr\}, \tag{1.1}$$

where

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}. \tag{1.2}$$

Note that the set C^p_Δ is strictly convex for any $p > 1$ and positive Δ and represents a ball of radius Δ in the uniformly convex space $\mathbb{R}^n_p$ equipped with the norm $\|\cdot\|_p$ (1.2).

The answer to this question in the case $p = 2$ follows from a general result by Polyak for a Hilbert space [16]. Namely, for any nonlinear mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ whose derivative F' is surjective and Lipschitz continuous, the image $F(C^2_\Delta)$ is convex for all small positive Δ.

However, the question of convexity of the sets $F(C^p_\Delta)$ for general $p > 1$ under nonlinear mapping F, which was asked by Polyak, turned out to be more complicated.

The following example of a nonlinear mapping $F\colon \mathbb{R}^2_p \to \mathbb{R}^2_p$ was presented in [18]:

$$F(x) = [x_1 + \varepsilon x_2^2, x_2]. \tag{1.3}$$

It is obvious that the derivative of F is close to the identical mapping for small ε. Nevertheless, in the case $p > 2$ the image $F(C^p_\Delta)$ is not convex for any small positive Δ, as it was shown in [18].

In this paper we provide a complete answer to Polyak's question by demonstrating that the images $F(C^p_\Delta)$ are convex for all sufficiently small positive Δ for any $p \in (1, 2]$ and nonlinear F with surjective Lipschitz derivative F'.

This and more general results on convexity of the images of convex sets under nonlinear mappings are derived from general criteria for the convexity of sets in general Banach spaces which are obtained in this paper.

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We state these necessary and sufficient conditions for the convexity of closed sets in Banach spaces in terms of tangent and normal cones to sets.

Theorem 1.1. *Let S be a closed subset of a Banach space $\mathbb{X}$. Then the following statements are equivalent*:

(i) *S is convex*;

(ii) *for any $x \in S$*
$$S \subset x + T_S^{\mathrm{B}}(x); \tag{1.4}$$

(iii) *for any $x \in S$*
$$S \subset x + T_S^{\mathrm{C}}(x); \tag{1.5}$$

(iv) *for any $x \in S$*
$$S \subset x + (N_S^{\mathrm{C}}(x))^*; \tag{1.6}$$

(v) *if, in addition, the space $\mathbb{X}$ is uniformly convex, then for any $x \in S$*
$$S \subset x + (N_S^{\mathrm{P}})^*(x). \tag{1.7}$$

Here $T_S^{\mathrm{B}}(x)$ denotes the Bouligand tangent cone, which can be defined in terms of the lower Dini directional derivative of the distance function
$$d_S(z) := \inf_{x \in S} \|z - x\| \tag{1.8}$$
to the set S; namely,
$$T_S^{\mathrm{B}}(x) := \bigl\{v\colon\ \underline{D} d_S(x; v) \le 0\bigr\}.$$
The Clarke tangent cone $T_S^{\mathrm{C}}(x)$ can be defined in terms of the Clarke generalized directional derivative of the distance function:
$$T_S^{\mathrm{C}}(x) := \bigl\{v\colon\ D^{\circ} d_S(x; v) \le 0\bigr\}.$$

The proximal normal cone $N_S^{\mathrm{P}}(x)$, $x \in S$, for a closed subset S of a uniformly convex space $\mathbb{X}$ consists of all positive multiples $\alpha\zeta$, $\alpha > 0$, where for some $z \in \mathbb{X} \setminus S$ we have $\zeta \in B^*$, $\langle \zeta, z - x\rangle = \|z - x\|$, and x is the closest point to z:
$$\|z - x\| = d_S(z).$$
Note that we use the notation K^* for the dual cone of a set K:
$$K^* := \bigl\{\zeta \in \mathbb{X}^*\colon\ \langle \zeta, k\rangle \le 0\ \ \forall\, k \in K\bigr\}.$$

Among a variety of concepts of tangent cones (starting with the largest Bouligand tangent cone), the Clarke tangent cone has a prominent place with examples of its numerous successful applications in different fields (including well-known applications to optimization and dynamic optimization problems [5–7] and less-known ones to problems of existence of fixed points and equilibria in nonconvex sets [8, 9, 1]).

In this paper we demonstrate yet another application of the Clarke tangent cones to deriving the tangential criteria (1.5) for the convexity of a set S in a general Banach space.

Note that the classical tangent cone $\operatorname{cone}(S - x)$ to a convex set S coincides with the Clarke tangent cone (see [5]):
$$\operatorname{cone}(S - x) = T_S^{\mathrm{C}}(x) \qquad \forall\, x \in S. \tag{1.9}$$

This fact implies that the tangential conditions (1.5) are necessary for the convexity of the set S. It is shown here that conditions (1.5) are also sufficient for the convexity of S. To the best of our knowledge, the question of validity of the tangential conditions (1.5) as a criterion for the convexity of a set S in a general Banach space has been open until now.

The Clarke normal cone $N_S^{\mathrm{C}}(x)$ at $x \in S$ is defined as a dual cone to the Clarke tangent cone:

$$N_S^{\mathrm{C}}(x) := \bigl\{\zeta \in \mathbb{X}^*\colon\ \langle \zeta, v\rangle \le 0\ \ \forall\, v \in T_S^{\mathrm{C}}(x)\bigr\}. \tag{1.10}$$

In the case of a uniformly convex space $\mathbb{X}$, it was established that the Clarke normal cone can be represented in terms of proximal normal cones:

$$N_S^{\mathrm{C}}(x) = \operatorname{co}\Bigl\{\mathop{\text{w*-lim}}_{k\to\infty} \zeta_k\colon\ \zeta_k \in N_S^{\mathrm{P}}(x_k),\ x_k \in S,\ \lim_{k\to\infty} x_k = x\Bigr\}, \tag{1.11}$$

where $\text{w*-lim}_{k\to\infty} \zeta_k$ is the weak* limit of the sequence $\zeta_k \in \mathbb{X}^*$. This proximal normal formula was established in [3, 4, 15] in the case of more general Banach spaces than the uniformly convex ones.

The proximal normal formula (1.11) for such spaces and the dual definition of the Clarke normal cone imply that the normal condition (1.7) is necessary and sufficient for the convexity of S.

It is interesting to note that for arbitrary closed sets in a Hilbert space the normal convexity criterion (1.7) was stated in the book [10, p. 63] in the form of Exercise 11.3.

In this paper we provide a straightforward proof of this proximal normal criterion for the convexity of sets in the case of more general uniformly convex Banach spaces without using the proximal normal formula.

One of the original stimuli for this paper came from a brief note by S. Vakhrameev [22], where tangential convexity criteria in the form of equality (1.9) were stated for the case of closed sets with nonempty interior in the finite-dimensional space $\mathbb{X} = \mathbb{R}^n$.

Another one was a series of papers by Polyak on his *convexity principle* [16, 17]. Namely, Polyak demonstrated that for a nonlinear $C^{1,1}$ mapping $F\colon \mathbb{X} \to \mathbb{Y}$ between Hilbert spaces whose derivative $F'(x_0)$ is surjective at $x_0 \in \mathbb{H}$, the image $F(x_0 + rB)$ of a ball $x_0 + rB$ of sufficiently small radius r is convex.

Note that here and below B stands for the unit ball in $\mathbb{X}$.

In this paper we apply the convexity criterion (1.4) to obtain sufficient conditions for the convexity of the images of multifunctions

$$G(x) := F(x) + K,$$

where K is a closed convex cone.

Namely, for a set C which is determined as the intersection of sublevel sets of differentiable functions V_i,

$$C := \bigl\{x \in \mathbb{X}\colon\ V_i(x) \le 0,\ i = 1, \dots, m\bigr\},$$

we obtain first-order conditions which imply the convexity of the set

$$S := G(C) := \bigl\{F(x) + k\colon\ x \in C,\ k \in K\bigr\}.$$

As an example of application of this result (see Corollary 4.1 below), we derive a generalization of the convexity principle from [17]. Namely, let $F\colon \mathbb{X} \to \mathbb{Y}$ be a $C^{1,1}$ mapping from a Hilbert space $\mathbb{X}$ to a general Banach space $\mathbb{Y}$ such that

$$F'(x_0)\mathbb{X} + K = \mathbb{Y}. \tag{1.12}$$

Then, for all small $r > 0$, the image $G(x_0 + rB)$ of the ball $x_0 + rB$ is convex.

Finally, we should mention the papers [2, 11] by Bobylev et al., which contain conditions for the convexity of $F(C)$ for a nonlinear mapping $F\colon \mathbb{R}^n \to \mathbb{R}^n$ where a convex set C is a sublevel set of some strictly convex function V.

It seems that by using the tangential and normal criteria obtained in this paper, we answer some open questions which have been raised by Lebourg in his review [13] of Polyak's paper [17]. Let us take a long quote from it on Polyak's convexity principle to list these open questions:

> *Although the emphasis here is on applications, the new principle clearly goes beyond the scope of any particular application, and it raises more interest in itself than in its potential applications. It also sets forth a few unanswered questions. Could this local principle be carried on to uniformly convex Banach spaces? Could it be somehow turned into a global one? What could be said, for instance, of the global image of a convex set—e.g. a ball—under a $C^{1,1}$ or even a C^2 mapping?*
>
> *It is worth spending some time on it, even without any specific application in mind.*

We should mention the recent interesting papers [20, 21], which contain some generalizations of Polyak's convexity principle for uniformly convex spaces and multivalued mappings and which answer some of Lebourg's questions.

The plan of this paper is as follows. The next Section 2 contains the notation which is used in the paper and a description of the results which are used in it. A proof of Theorem 1.1 with tangential and normal convexity criteria is presented in Section 3. Section 4 contains applications of these criteria to the derivation of sufficient conditions for the convexity of images of convex sets under multivalued mappings.

2. NOTATION AND SOME RELEVANT RESULTS

The dual of a Banach space $\mathbb{X}$ is denoted by $\mathbb{X}^*$; the unit closed ball in $\mathbb{X}$ is denoted by $B_{\mathbb{X}}$ or simply B; similarly B^* is the unit closed ball in $\mathbb{X}^*$. For $\zeta \in \mathbb{X}^*$ and $x \in \mathbb{X}$ we have the linear pairing $\langle \zeta, x\rangle$.

For sets A, B and a number α we have the usual algebraic sum $A + B$ and product αA. So $x_0 + rB$ denotes a closed ball of radius r centered at x_0.

The space of differentiable mappings F such that their derivatives F' are locally Lipschitz is denoted by $C^{1,1}$.

For a function $f\colon \mathbb{X} \to \mathbb{R}$, the *lower directional Dini derivative* is defined as

$$\underline{D} f(x; v) := \liminf_{t\to+0,\ w\to v} \frac{f(x+tw) - f(x)}{t}.$$

The *Clarke generalized directional derivative of* f is defined as follows:

$$D^\circ f(x; v) := \limsup_{y\to x,\ w\to v} \frac{f(y+tw) - f(y)}{t}.$$

From the obvious inequality between the Dini and Clarke directional derivatives, we have

$$T^{\mathrm{C}}_S(x) \subset T^{\mathrm{B}}_S(x).$$

The following important inclusion, which is used in this paper, is due to Treiman [19]:

$$\operatorname*{Lim\,inf}_{x' \xrightarrow[S]{} x} T^{\mathrm{B}}_S(x') \subset T^{\mathrm{C}}_S(x); \tag{2.1}$$

here, for a multivalued mapping $G\colon \mathbb{X} \to 2^{\mathbb{Y}}$ such that $G(x)$ are subsets of a Banach space $\mathbb{Y}$,

$$\operatorname*{Lim\,inf}_{x' \xrightarrow[S]{} x} G(x') := \bigcap_{\varepsilon>0} \bigcup_{\delta>0} \bigcap_{x'\in x+\delta B} (G(x') + \varepsilon B_{\mathbb{Y}}). \tag{2.2}$$

Since the Clarke generalized directional derivative of a locally Lipschitz function f is a support function of the bounded convex set $\partial_{\mathrm{C}} f(x)$, which is called the Clarke generalized gradient, it follows that the Clarke normal cone at $x \in S$ can be defined as

$$N_S^{\mathrm{C}}(x) = \operatorname{cone} \partial_{\mathrm{C}} d_S(x), \tag{2.3}$$

and due to duality

$$(N_S^{\mathrm{C}}(x))^* = T_S^{\mathrm{C}}(x). \tag{2.4}$$

Finally we use the following inverse multifunction theorem (see, e.g., [14]) for the multifunction $G(x) := F(x) + K$ where $F\colon \mathbb{X} \to \mathbb{Y}$ is a C^1 nonlinear mapping and K is a closed convex cone.

If the infinitesimal condition (1.12) holds and

$$y_0 \in G(x_0),$$

then for any sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y_0 + \varepsilon B_{\mathbb{Y}} \subset G(x_0 + \delta B_{\mathbb{X}}). \tag{2.5}$$

3. PROOF OF THEOREM 1.1

First we prove the equivalence of statements (i)–(iv).

(i) $\Rightarrow$ (ii). Let S be convex. Then, clearly, for any s and x in S the vector $s - x$ is a Bouligand tangent vector,

$$s - x \in T_S^{\mathrm{B}}(x),$$

since $x + t(s - x) \in S$ for all $t \in [0, 1]$. Of course, this implies (1.4).

(ii) $\Rightarrow$ (iii). It follows from (1.4) and the definition (2.2) that

$$S - x = \operatorname*{Lim\,inf}_{x' \xrightarrow[S]{} x} (S - x') \subset \operatorname*{Lim\,inf}_{x' \xrightarrow[S]{} x} T_S^{\mathrm{B}}(x'). \tag{3.1}$$

In view of (2.1) this implies (1.5).

(iii) $\Rightarrow$ (iv). Because of the duality between the Clarke tangent and normal cones (2.4), relation (1.5) implies (1.6).

(iv) $\Rightarrow$ (i). We need to prove that the normal conditions (1.6) imply the convexity of the set S. Assume on the contrary that S is not convex, namely, there exist $x_1, x_2 \in S$ such that for some maximizing $t_0 \in (0, 1)$

$$\Delta := d_S(x(t_0)) = \max_{t \in [0,1]} d_S(x(t)) > 0 \tag{3.2}$$

where $x(t) = (1 - t)x_1 + t x_2$.

Denote $x(t_0)$ by x^0 and for an arbitrary $\varepsilon > 0$ choose $\widetilde{y}_\varepsilon \in S$ such that

$$\|x^0 - \widetilde{y}_\varepsilon\| < d_S(x^0) + \varepsilon^2.$$

By using Ekeland's minimization principle, we can assume that there exists $y_\varepsilon \in S$ such that

$$\|x^0 - y_\varepsilon\| < d_S(x^0) + \varepsilon^2 \tag{3.3}$$

and $y_\varepsilon \in S$ is a minimizer of the function

$$y \to \|x^0 - y\| + \varepsilon \|y - y_\varepsilon\|$$

on the closed set S.

By using Clarke's exact penalization result [5], we see that y_ε is also a minimizer of the function

$$y \to \|x^0 - y\| + \varepsilon\|y - y_\varepsilon\| + 2d_S(y)$$

on the entire space $\mathbb{X}$.

Then we use necessary conditions for the minimizer y_ε in terms of the Clarke generalized gradient to obtain

$$0 \in \partial_{\mathrm{C}}\big(\|x^0 - y\| + \varepsilon\|y - y_\varepsilon\| + 2d_S(y)\big)\big|_{y=y_\varepsilon}.$$

In particular, this implies that

$$0 \in \partial_{\mathrm{C}}(\|x^0 - y\|)\big|_{y=y_\varepsilon} + 2\,\partial_{\mathrm{C}}\, d_S(y_\varepsilon) + \varepsilon B, \tag{3.4}$$

which means that there exist ζ_ε and ζ in $\mathbb{X}^*$ such that

$$-\zeta_\varepsilon \in \partial_{\mathrm{C}}(\|y - x^0\|)\big|_{y=y_\varepsilon}, \qquad \zeta \in \partial_{\mathrm{C}}\, d_s(y_\varepsilon), \qquad \text{and} \qquad \zeta_\varepsilon \in 2\zeta + \varepsilon B. \tag{3.5}$$

We note that the first inclusion implies the important fact that

$$\langle \zeta_\varepsilon, x^0 - y_\varepsilon\rangle = \|x^0 - y_\varepsilon\| \tag{3.6}$$

Remark also that the third inclusion in (3.5) yields

$$\langle \zeta_\varepsilon, x_1 - y_\varepsilon\rangle \le 2\langle \zeta, x_1 - y_\varepsilon\rangle + \varepsilon\|x_1 - y_\varepsilon\|, \qquad \langle \zeta_\varepsilon, x_2 - y_\varepsilon\rangle \le 2\langle \zeta, x_2 - y_\varepsilon\rangle + \varepsilon\|x_2 - y_\varepsilon\|.$$

Note that $x_1 - y_\varepsilon$ and $x_2 - y_\varepsilon$ belong to $S - y_\varepsilon$ and we can use the normal conditions (1.6) at the point $y_\varepsilon \in S$ and the representation (2.3) of the Clarke normal cone to obtain

$$\langle \zeta_\varepsilon, x_1 - y_\varepsilon\rangle \le \varepsilon\|x_1 - y_\varepsilon\|, \qquad \langle \zeta_\varepsilon, x_2 - y_\varepsilon\rangle \le \varepsilon\|x_2 - y_\varepsilon\|. \tag{3.7}$$

Multiplying the first inequality by t_0, the second one by $1 - t_0$, and adding them, we conclude that

$$\langle \zeta_\varepsilon, x^0 - y_\varepsilon\rangle \le \varepsilon\big(t_0\|x_1 - y_\varepsilon\| + (1 - t_0)\|x_2 - y_\varepsilon\|\big) \le \varepsilon\big(\|x_2 - x_1\| + d_S(x^0) + \varepsilon^2\big).$$

We used (3.3) to obtain the last inequality.

In view of (3.2), (3.6) and the definition of y_ε, we see that

$$0 < \Delta \le O(\varepsilon),$$

which leads to a contradiction since $\varepsilon > 0$ can be arbitrary small. Thus, the set S is convex.

We now demonstrate that in the case of a uniformly convex space $\mathbb{X}$ statements (i) and (v) are equivalent.

(i) $\Rightarrow$ (v). Let S be a closed convex set and $\zeta \in N_S^{\mathrm{P}}(x)$. Without loss of generality we can assume that $\|\zeta\| = 1$ and

$$\langle \zeta, z - x\rangle = \|z - x\| = d_S(x) \tag{3.8}$$

for some $z \notin S$. Then, because of the convexity of S, we have

$$\|z - x - t(s - x)\| - \|z - x\| \ge 0$$

for any $s \in S$ and $t \in (0, 1]$. Dividing this inequality by t and taking the limit as $t \to +0$, we obtain

$$\max_{\zeta}\langle \zeta, s - x\rangle \le 0,$$

where the maximum is taken over the set of all ζ satisfying (3.8).

Of course, this implies that $s - x \in (N_S^{\mathrm{P}}(x))^*$, and inclusion (1.7) is proved.

(v) ⇒ (i). It is well known that for a closed subset S of a uniformly convex space $\mathbb{X}$ there exists a dense set of points z for each of which there is a unique closest point $x \in S$:

$$\|z - x\| = d_S(z). \tag{3.9}$$

Let us assume that the closed set S satisfies (1.7) but is not convex. This means that there exist x_1 and x_2 in S and $t_0 \in (0,1)$ such that

$$x_0 := t_0 x_1 + (1 - t_0) x_2 \notin S. \tag{3.10}$$

This also means that the distance from x_0 to S is positive:

$$d_S(x_0) > 0. \tag{3.11}$$

Due to the uniform convexity of $\mathbb{X}$, for any $\varepsilon > 0$ there exists a point $z \in x_0 + \varepsilon B$ such that there is a unique closest point $x \in S$ to z satisfying (3.9).

Because of the Lipschitz continuity of the distance function d_S, we have

$$d_S(x_0) - \varepsilon \le d_S(z) \le d_S(x_0) + \varepsilon. \tag{3.12}$$

Note that any vector $\zeta \in B^*$ satisfying (3.8) is a proximal normal to the set S at the point x.

It follows from (1.7) that

$$\langle \zeta, x_1 - x \rangle \le 0, \qquad \langle \zeta, x_2 - x \rangle \le 0.$$

Then from (3.9), (3.8), and the previous inequalities we obtain

$$d_S(z) = \langle \zeta, z - x \rangle \le \langle \zeta, t_0 x_1 + (1 - t_0) x_2 - x \rangle + \|z - x_0\| \le \varepsilon, \tag{3.13}$$

but this implies that

$$d_S(x_0) \le 2\varepsilon$$

for an arbitrary $\varepsilon > 0$. Of course, this inequality contradicts (3.11), and so the set S is convex.

Theorem 1.1 is proved.

4. CONVEXITY OF IMAGES OF NONLINEAR MAPPINGS: FIRST-ORDER CONDITIONS

In this section we use convexity criteria from Section 2 to prove some sufficient conditions for the convexity of the image of a set C under a multivalued mapping $G(x)$.

This multifunction G is defined as follows:

$$G(x) := F(x) + K, \tag{4.1}$$

where $F\colon \mathbb{X} \to \mathbb{Y}$ is a single-valued mapping between Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ and $K \subset \mathbb{Y}$ is a closed convex cone.

The set C is determined as an intersection of sublevel sets of differentiable functions $V_i\colon \mathbb{X} \to \mathbb{R}$:

$$C := \big\{ x \in \mathbb{X}\colon\ V_i(x) \le 0,\ i = 1, \dots, m \big\}. \tag{4.2}$$

It is assumed that C is nonempty.

Here we consider the image

$$S := G(C) := \big\{ F(x) + k\colon\ x \in C,\ k \in K \big\} \tag{4.3}$$

and provide conditions which imply the convexity of S.

In the next subsection we consider conditions which are formulated globally in terms of all points in C. Then we consider conditions which have a more local character and are stated in terms of points near the boundary of C.

4.1. Global conditions for the convexity of images. Let us assume that the mapping F is differentiable and satisfies the following conditions.

Assumption 4.1. For any $x \in C$ there exist positive $\nu(x)$ and $L(x)$ such that

$$\|F'(x) - F'(u)\| \le L(x)\|x - u\| \qquad \forall\, u \in C \tag{4.4}$$

and

$$\|F'^*(x)e\| \ge \nu(x)\|e\| \qquad \forall\, e \in K^*. \tag{4.5}$$

The first part of this assumption is satisfied if we assume that F' is a Lipschitz function inside C. The second part of this assumption is a generalized condition of surjectivity of $F'(x)$, with $\nu(x)$ a modulus of surjectivity.

Let $I := \{1, \dots, m\}$ and for $x \in \operatorname{bdry} C$ a set $I(x)$ is the set of active indices

$$I(x) := \{i \in I\colon\ V_i(x) = 0\}.$$

Let the functions V_i, $i \in I$, satisfy the following assumption.

Assumption 4.2. (i) The functions $V_i\colon \mathbb{X} \to \mathbb{R}$ are differentiable at any $x \in \operatorname{bdry} C$.

(ii) There exists a positive $\alpha(x)$ such that for any $x \in \operatorname{bdry} C$, $u \in C$, and $i \in I(x)$

$$\langle V_i'(x), u - x\rangle + \alpha(x)\|u - x\|^2 \le V_i(u) - V_i(x). \tag{4.6}$$

(iii) There exists a positive $\beta(x)$ such that for any $x \in \operatorname{bdry} C$ and $i \in I(x)$

$$\|V_i'(x)\| \le \beta(x). \tag{4.7}$$

(iv) At any point $x \in \operatorname{bdry} C$ for any convex coefficients $\{\lambda_i\}_{I(x)}$ (that is, such that $\lambda_i \ge 0$ and $\sum_{i \in I(x)} \lambda_i = 1$),

$$\sum_{i \in I(x)} \lambda_i V_i'(x) \ne 0. \tag{4.8}$$

It is obvious that (4.6) is satisfied if V_i are strictly convex C^2 functions with some uniform lower bound on the eigenvalues of the Hessians of V_i. Condition (4.8) is satisfied if V_i are convex and there exists x_* such that $V_i(x_*) < 0$ for all $i \in I$ (Slater condition).

The surjectivity condition (4.5) implies (see, e.g., [14]) that for any y_0 and x_0 near C such that

$$y_0 \in G(x_0)$$

and for any sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y_0 + \varepsilon B \subset G(x_0 + \delta B).$$

In particular, this implies the following two important facts:

(a) the set $S = G(C)$ is closed;

(b) if for some $x \in C$ and $k_x \in K$ the point $F(x) + k_x$ is a boundary point of S, then x is a boundary point of C.

Later we will give an example demonstrating that in the absence of the surjectivity condition (4.5) the set $G(C)$ can be nonconvex.

It should also be noted that Assumptions 4.1 and 4.2 have a global character: they require some uniform behavior of the mapping F and functions V_i inside the entire set C.

Theorem 4.1. *Let Assumptions* 4.1 *and* 4.2 *be satisfied and for any* $x \in \operatorname{bdry} C$

$$\frac{\alpha(x)\nu(x)}{\beta(x)} \geq \frac{1}{2}L(x). \tag{4.9}$$

Then the set S (4.3) *is convex.*

Proof. To prove the convexity of the image S, we apply the tangential convexity criteria from Theorem 1.1.

We use the following lemma, which characterizes vectors w that are not Bouligand tangent vectors of the set S (4.3) at the boundary point $F(x) + k_x$ with $x \in \operatorname{bdry} C$ and $k_x \in K$.

Lemma 4.1. *Let* $w \notin T_S^{\mathrm{B}}(F(x) + k_x)$. *Then there exist* $\varepsilon > 0$, $n(x) \in \mathbb{Y}^*$, $\|n(x)\| = 1$, *and nonnegative coefficients* $\{\lambda_i\}_{I(x)}$ *such that*

$$F'^*(x)n(x) = \sum_{i \in I(x)} \lambda_i V_i'(x), \qquad \langle n(x), k_x \rangle = 0, \qquad n(x) \in K^*, \tag{4.10}$$

$$\langle n(x), w \rangle \geq \varepsilon. \tag{4.11}$$

Proof. Note that it follows from the definition of the Bouligand tangent cone that there exists a constant $\varepsilon > 0$ and a sequence $\{t_j\}$, $t_j \to +0$ as $j \to \infty$, such that

$$d_S\big(F(x) + k_x + t_j w\big) \geq \varepsilon t_j.$$

This relation, the exact penalization result from [5], and the metric regularity of the system of inequalities in (4.2) (due to (4.8) in Assumption 4.2) imply that there exists a positive ρ such that for any $u \in \mathbb{X}$ near x and any $y \in K$

$$\|F(x) + k_x + t_j w - F(u) - y\| + \rho \max\Big\{0, \max_{i \in I} V_i(u)\Big\} \geq \varepsilon t_j.$$

Consider the case $k_x \neq 0$. Then by choosing $u = x + t_j e$ and $y = (1 - t_j\gamma)k_x + t_j k$ for arbitrary $e \in \mathbb{X}$, $k \in K$, and $\gamma \in \mathbb{R}$, we obtain

$$\frac{1}{t_j}\Big(\big\|F(x) + k_x + t_j w - F(x + t_j e) - (1 - t_j\gamma)k_x - t_j k\big\| + \rho \max\Big\{0, \max_{i \in I} V_i(x + t_j e)\Big\}\Big) \geq \varepsilon$$

for all j large enough. By taking the limit as $j \to \infty$, we arrive at

$$\inf_{e,k,\gamma} \max_{f \in B^*,\, \lambda} \left[\big\langle f, w - F'(x)e + \gamma k_x - k \big\rangle + \left\langle \sum_{i \in I(x)} \lambda_i V_i'(x), e \right\rangle \right] \geq \varepsilon,$$

where the maximum is taken over all $f \in B^*$ and $\lambda = \{\lambda_i\}_{I(x)}$, $0 \leq \lambda_i \leq \rho$, $i \in I(x)$.

By using a one-sided variant of the minimax theorem, we establish the existence of $n(x) \in B^*$ and nonnegative numbers $\{\lambda_i\}_{I(x)}$ such that

$$\langle n(x), w \rangle + \inf_{\gamma} \gamma \langle n(x), k_x \rangle - \sup_{e} \left\langle F'^*(x)n(x) - \sum_{i \in I(x)} \lambda_i V_i'(x), e \right\rangle - \sup_{k} \langle n(x), k \rangle \geq \varepsilon.$$

It is clear from this relation that $n(x) \neq 0$ and that $n(x)$ satisfies (4.10) and w satisfies (4.11). The case $k_x = 0$ is treated in a similar way.

The lemma is proved. $\square$

Now we return to the proof of the convexity of S (4.3) by assuming that the convexity criteria do not hold for some boundary point $F(x) + k_x$ of S. This means in accordance with Lemma 4.1 that there exist $u \in C$ and $k \in K$ such that for the vector $w = F(u) + k - F(x) - k_x$

$$\bigl\langle n(x), F(u) - F(x) + k - k_x \bigr\rangle \geq \varepsilon,$$

where $n(x)$, $0 < \|n(x)\| \leq 1$, satisfies (4.10) and (4.11).

Because of (4.10), from the previous inequality we have

$$\langle n(x), F(u) - F(x) \rangle \geq \varepsilon. \tag{4.12}$$

It follows from the first equality in (4.10) and the surjectivity condition (4.5) that

$$\sum_{i \in I(x)} \lambda_i \geq \frac{\nu(x)\|n(x)\|}{\max_{i \in I(x)} \|\nabla V_i(x)\|}.$$

Then we use this inequality and (4.7) to write the next estimate for $\{\lambda_i\}$:

$$\sum_{i \in I(x)} \lambda_i \geq \frac{\nu(x)\|n(x)\|}{\beta(x)}. \tag{4.13}$$

Since

$$\begin{aligned} F(u) - F(x) &= \int_0^1 F'(x + t(u - x))(u - x)\, dt \\ &= F'(x)(x - u) + \int_0^1 (F'(x + t(u - x)) - F'(x))(u - x)\, dt, \end{aligned}$$

we obtain from (4.4) the following estimate:

$$\langle n(x), F(u) - F(x) \rangle \leq \langle F'^*(x)n(x), u - x \rangle + \frac{1}{2} L(x)\|u - x\|^2 \|n(x)\|.$$

Due to the representation for $F'^*(x)n(x)$ in (4.10), we obtain

$$\langle n(x), F(u) - F(x) \rangle \leq \sum_{i \in I(x)} \lambda_i \langle V_i'(x), u - x \rangle + \frac{1}{2} L(x)\|u - x\|^2 \|n(x)\|. \tag{4.14}$$

Due to (4.6) and the fact that $u \in C$, we have

$$\langle V_i'(x), u - x \rangle \leq V_i(u) - V_i(x) - \alpha(x)\|u - x\|^2 \leq -\alpha(x)\|u - x\|^2$$

for $i \in I(x)$. We use this inequality and estimate (4.13) for λ_i to derive from (4.14) that

$$\langle n(x), F(u) - F(x) \rangle \leq -\left[\frac{\nu(x)\alpha(x)}{\beta(x)} - \frac{1}{2} L(x)\right] \|u - x\|^2 \|n(x)\| \leq 0.$$

The last inequality follows from (4.9) and implies that (4.12) is not satisfied. Thus, we have proved that inclusion (1.4) holds for any boundary point of the set $G(C)$ and this set is convex.

The theorem is proved. □

Let us show how one can use Theorem 4.1 to generalize the sufficient conditions due to Polyak [16] for the convexity of images of small balls in a Hilbert space.

Corollary 4.1. *Let $\mathbb{X}$ be a Hilbert space, $\mathbb{Y}$ a Banach space, $K \subset \mathbb{Y}$ a closed convex cone, and $F\colon \mathbb{X} \to \mathbb{Y}$ a nonlinear mapping such that for some $x_0 \in \mathbb{X}$ and positive L, ν, and r it satisfies the inequalities*

$$\|F'(x) - F'(u)\| \le L\|x-u\| \qquad \forall\, x, u \in x_0 + rB, \tag{4.15}$$

$$\|F'^*(x)e\| \ge \nu\|e\| \qquad \forall\, e \in K^*. \tag{4.16}$$

Then for any $0 < \Delta \le \min\{r, \nu/(2L)\}$ the set $G(x_0 + \Delta B)$ is convex.

Proof. Consider the function

$$V(x) := \frac{1}{2}\|x-a\|^2 - \frac{1}{2}\Delta^2,$$

where Δ satisfies the hypothesis of the corollary.

Note that in this case $C = x_0 + \Delta B$ and the function V satisfies Assumption 4.2 and, consequently, condition (4.6) with

$$\alpha(x) = \frac{1}{2}, \qquad \beta(x) = \Delta.$$

Note also that due to (4.16) and (4.15) the mapping F satisfies Assumption 4.1 with $L(x) = L$ and $\nu(x) = \nu$. Since $\|x - x_0\| = \Delta$ for any $x \in \operatorname{bdry} C$, we can easily check that inequality (4.9) is also satisfied. Then this corollary immediately follows from Theorem 4.1. $\square$

Consider another application of Theorem 4.1 which allows us to answer the question raised by Polyak in [17] about the convexity of images of small balls (1.1) in the finite-dimensional space $\mathbb{R}^n_p$ with the norm $\|\cdot\|_p$ (1.2). In [18] Reissig constructed an example of a nonlinear mapping (1.3) and demonstrated that for any $p > 2$ and any Δ small enough $F(C^p_\Delta)$ is nonconvex.

Here we will show that for a mapping F with surjective F' the image $F(C^p_\Delta)$ is convex for any $p \in (1,2]$ and all small Δ. This answers Polyak's question in the affirmative for the case of $p \in (1,2]$.

Proposition 4.1. *Let $p \in (1,2]$ and $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be a nonlinear mapping such that for some positive L, ν, and r conditions* (4.15) *and* (4.16) *are satisfied for some closed convex cone $K \subset \mathbb{R}^m$. Then for all positive Δ such that*

$$\Delta \le \min\left\{\frac{r}{\sqrt{n}}, \frac{(p-1)\nu}{\sqrt{n}L}\right\}, \tag{4.17}$$

the set $G(C^p_\Delta)$ is convex.

Proof. We only need to check that Assumptions 4.1, 4.2 and (4.9) are satisfied for F and the function $V(x) := \|x\|^p_p - \Delta^p$ under conditions (4.15), (4.16) and the bound (4.17) for $x_0 = 0$.

Indeed, if $x \in C^p_\Delta$, then its every component x_i satisfies the estimate $|x_i| \le \Delta$ and for the Euclidean norm we have $\|x\| \le \sqrt{n}\Delta$. Since $V'(x) = p\big[|x_1|^{p-1}\operatorname{sign} x_1, \dots, |x_n|^{p-1}\operatorname{sign} x_n\big]$, we obtain $\|V'(x)\| \le p\sqrt{n}\Delta^{p-1}$ for $x \in C^p_\Delta$. This implies that $\beta = p\sqrt{n}\Delta^{p-1}$.

To find $\alpha(x)$, we use the following representation:

$$V(u) - V(x) = \int\limits_0^1 \langle V'(y_t), u - x\rangle\, dt = \langle V'(x), u-x\rangle + \int\limits_0^1 \left\langle \int\limits_0^s V''(y_s)(u-x)\,ds, u-x \right\rangle dt,$$

where $y_t = x + t(u-x)$.

Note that the Hessian $V''(x)$ is a diagonal matrix $\operatorname{diag}\{p(p-1)|x_1|^{p-2}, \dots, p(p-1)|x_n|^{p-2}\}$ at a point $x \in C^p_\Delta$ where it exists. This implies that (4.6) is valid for any $x, u \in C^p_\Delta$ with $\alpha = p(p-1)\Delta^{p-2}/2$.

Finally we observe that if $\sqrt{n}\Delta \le r$, then $C^p_\Delta \subset B(0,r)$, and for such Δ condition (4.9) holds if

$$\frac{p(p-1)\Delta^{p-2}\nu}{2p\sqrt{n}\Delta^{p-1}} \ge \frac{1}{2}L.$$

However, this inequality is valid if Δ satisfies (4.17). The proposition is proved. □

Case $p > 2$. We show the nonconvexity of $F(C^p_\Delta)$ by using the proximal normal convexity criteria. Consider a parametrization of the boundary points of the set C^p_Δ:

$$x^\tau = [x^\tau_1, x^\tau_2] := \bigl[\Delta - \tau, (\Delta^p - (\Delta-\tau)^p)^{1/p}\bigr], \qquad \tau \in [0,\Delta). \tag{4.18}$$

Then $F(x^\tau) = [\Delta - \tau + \varepsilon(x^\tau_2)^2, x^\tau_2]$, and it is easy to check that for any $\varepsilon \in (0,1/4]$ and $\Delta < 1$ the vector $h = [1,0]$ is a proximal normal vector to $F(C^p_\Delta)$ at the point $F(x^0) = [\Delta, 0]$.

Now we fix positive ε and Δ satisfying the bounds mentioned above and find that for all $\tau > 0$ small enough

$$\langle h, F(x^\tau) - F(x^0)\rangle = -\tau + \varepsilon(\Delta^p - (\Delta-\tau)^p)^{2/p} = -\tau + \varepsilon(p\Delta^{p-1}\tau + o(\tau))^{2/p}.$$

Since $2/p < 1$, it is clear that for all small $\tau > 0$ the previous expression takes positive values. This implies that the proximal normal criterion (1.7) fails for the point $F(x^0)$ and the set $F(C^p_\Delta)$ is not convex.

Remark 4.1. The assumption of the surjectivity of F' on the set C is an important one. Consider the set $C_\Delta := \{x = [x_1, x_2] \in \mathbb{R}^2 \colon \|x\|^2/2 \le \Delta\}$ and the mapping $F(x)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(x) := [x_1, x_1^2]$. It is clear that F' is not surjective for any x and the set $F(C_\Delta)$ is not convex for arbitrary small $\Delta > 0$.

4.2. Localized conditions for the convexity of images $G(C)$. The conditions for the convexity of images of the convex set C (4.2) in the form (4.9) have a global character since the functions $\alpha(x)$, $\beta(x)$, and $L(x)$ are defined in terms of points from the entire set C.

In this subsection we present an approach to the derivation of convexity conditions for $G(C)$ which are stated in terms of the local behavior of the mapping F and functions $V_i(x)$ near the boundary points of C. The only global requirement is the surjectivity of the derivative F' at any point $x \in C$ (see (1.12)). However, we should note that in view of Remark 4.1 this assumption is essential for the convexity of the image.

Our approach is based on the known result of Klee [12] that a closed connected subset S of a linear topological space is convex if each point of S has a convex neighborhood in the relative topology. In the context of convexity of images of sets, it was first suggested in [2].

Now we state assumptions on the properties of the space $\mathbb{X}$, mapping F, and functions V_i. The first of them states some "smoothness" properties of $\mathbb{X}$ in terms of the existence of a smooth convex function.

Assumption 4.3. There exist a differentiable convex function $g\colon \mathbb{X} \to [0,+\infty)$ and positive constants c_1, c_2, and c_3 such that $g(0) = 0$, $g(x) > c_1$ for all $x \notin B$, $\|g'(x)\| \le c_2$ for all $x \in B$, and

$$g(u) - g(x) \ge \langle g'(x), u - x\rangle + c_3\|u - x\|^2$$

for all $x, u \in B$.

Note that in the case of a Hilbert space $\mathbb{X}$ the function $g(x) = \|x\|^2/2$ satisfies Assumption 4.3 with $c_1 = 1/2$, $c_2 = 1$, and $c_3 = 1/2$.

Let us fix some $x_0 \in \mathbb{X}$ and $r > 0$ and define a function

$$V_0(x) := g\Bigl(\frac{x - x_0}{r}\Bigr) - c_1. \tag{4.19}$$

It is easy to see that for this function the nonempty convex closed set

$$D_r(x_0) := \{x \in \mathbb{X}\colon\ V_0(x) \le 0\} \tag{4.20}$$

is contained in the ball $x_0 + rB$ and for all $x, u \in D_r(x_0)$

$$\|V_0'(x)\| \le \frac{c_2}{r}, \qquad V_0(u) - V_0(x) \ge \langle V_0'(x), u - x\rangle + \frac{c_3}{r^2}\|u - x\|^2. \tag{4.21}$$

Note that in the next assumption about F and V_i, the functions $\widehat{\alpha}(x)$, $\widehat{\beta}(x)$, $\widehat{\nu}(x)$, and $\widehat{L}(x)$ are defined at boundary points of C and their values reflect the local behavior of the mapping F and functions V_i near these points.

Assumption 4.4. Let the mapping $F\colon \mathbb{X} \to \mathbb{Y}$ be continuously differentiable and satisfy (1.12) on C, and let the functions $V_i\colon \mathbb{X} \to \mathbb{R}$ be convex and differentiable at any $x \in \operatorname{bdry} C$. For any $x_0 \in \operatorname{bdry} C$ there exists positive $\widehat{r}(x_0)$, $\widehat{\nu}(x_0)$, $\widehat{L}(x_0)$, $\widehat{\alpha}(x_0)$, and $\widehat{\beta}(x_0)$ such that for any $u, x \in C \cap (x_0 + \widehat{r}(x_0)B)$

$$\|F'(x) - F'(u)\| \le \widehat{L}(x_0)\|x - u\|, \tag{4.22}$$

$$\|F'^*(x)e\| \ge \widehat{\nu}(x_0)\|e\| \qquad \forall\, e \in K^*, \tag{4.23}$$

$$\langle V_i'(x), u - x\rangle + \widehat{\alpha}(x_0)\|u - x\|^2 \le V_i(u) - V_i(x) \qquad \forall\, i \in I(x), \tag{4.24}$$

$$\|V_i'(x)\| \le \widehat{\beta}(x_0) \qquad \forall\, i \in I(x). \tag{4.25}$$

Finally, we assume that the functions V_i and V_0 (see (4.19)) satisfy an analog of the Slater condition from mathematical programming.

Assumption 4.5. For any $x_0 \in \operatorname{bdry} C$ and $r \in (0, r(x_0))$ there exists $x_* \in x_0 + rB$ such that

$$V_i(x_*) < 0, \qquad i = 0, 1, \dots, m. \tag{4.26}$$

Theorem 4.2. *Let Assumptions* 4.3–4.5 *be satisfied and for any* $x_0 \in \operatorname{bdry} C$

$$\frac{\widehat{\alpha}(x_0)\widehat{\nu}(x_0)}{\widehat{\beta}(x_0)} \ge \frac{1}{2}\widehat{L}(x_0). \tag{4.27}$$

Then the set $G(C)$ *is convex.*

Proof. Since C and the cone K are convex, the set $S = G(C)$ is connected. For any $x_0 \in C$ consider the set

$$S_r(x_0) := G(C \cap D_r(x_0)) \tag{4.28}$$

with a sufficiently small $r > 0$. Note that if x_0 is an interior point of C, then, because of (1.12), $S_r(x_0)$ contains a ball of sufficiently small radius, which is of course a convex set.

Thus we need to consider only boundary points $x_0 \in \operatorname{bdry} C$. We will show that the set $S_r(x_0)$ is convex for sufficiently small r. Then in accordance with Klee's theorem the set S is convex.

For $x_0 \in \operatorname{bdry} C$ we fix some positive $r < \min\{\widehat{r}(x_0), c_2/(c_3\widehat{\beta}(x_0))\}$.

Note that the set $S_r(x_0)$ is the image $G(C_r(x_0))$ of the set

$$C_r(x_0) := \big\{x \in \mathbb{X}\colon\ V_0(x) \le 0,\ V_i(x) \le 0,\ i \in I\big\}.$$

Let us assume that the set $S_r(x_0)$ is not convex, which in accordance with Theorem 1.1 implies that at some boundary point $F(x) + k_x$ of $S_r(x_0)$ the vector $w = F(u) - F(x) + k_u - k_x$ is not a Bouligand tangent vector to this set for some $u \in C_r(x_0)$ and $k_u \in K$.

Note that the boundary point $F(x) + k_x$ of $S_r(x_0)$ should satisfy the condition $x \in \operatorname{bdry} C_r(x_0)$ because of (1.12).

In view of Assumption 4.5 we can use Lemma 4.1 to conclude that there exist $\varepsilon > 0$, nonnegative numbers λ_0 and λ_i, $i \in I(x)$, and $n(x) \in \mathbb{Y}^*$ such that $\|n(x)\| = 1$ and

$$F'^*(x)n(x) = \lambda_0 V_0'(x) + \sum_{i\in I(x)} \lambda_i V_i'(x), \qquad \langle n(x), k_x\rangle = 0, \qquad n(x) \in K^*, \tag{4.29}$$

$$\langle n(x), w\rangle \ge \varepsilon. \tag{4.30}$$

Then it follows from (4.29) that (4.12) is valid.

From (4.29), (4.21), (4.25), and the surjectivity assumption (4.23) we obtain

$$\lambda_0 \frac{c_2}{r} + \widehat{\beta}(x_0) \sum_{i\in I(x)} \lambda_i \ge \widehat{\nu}(x_0). \tag{4.31}$$

The next estimate follows from (4.22) and the integral representation for $F(u) - F(x)$:

$$\langle n(x), w\rangle \le \langle n(x), F(u) - F(x)\rangle \le \langle F'^*(x)n(x), u - x\rangle + \frac{1}{2}\widehat{L}(x_0)\|u - x\|^2.$$

By using the first equality in (4.29) we obtain

$$\langle n(x), F(u) - F(x)\rangle \le \lambda_0\langle \nabla V_0(x), u - x\rangle + \sum_{i\in I(x)} \lambda_i\langle \nabla V(x), u - x\rangle + \frac{1}{2}\widehat{L}(x_0)\|u - x\|^2. \tag{4.32}$$

Due to (4.24), (4.21), and the fact that $x, u \in C_r(x_0)$, we have

$$\langle \nabla V_i(x), u - x\rangle \le V_i(u) - V_i(x) - \widehat{\alpha}(x_0)\|u - x\|^2 \le -\widehat{\alpha}(x_0)\|u - x\|^2$$

for $i \in I(x)$ and

$$\langle \nabla V_0(x), u - x\rangle \le V_0(u) - V_0(x) - \frac{c_3}{r^2}\|u - x\|^2 \le -\frac{c_3}{r^2}\|u - x\|^2.$$

Using these inequalities and estimate (4.31) for λ_0 and λ_i, $i \in I(x)$, we find from (4.32) that

$$\begin{aligned}\langle n(x), F(u) - F(x)\rangle &\le \left[-\frac{c_3}{r^2}\lambda_0 - \sum_{i\in I_\gamma(x)} \lambda_i\widehat{\alpha}(x_0) + \frac{1}{2}\widehat{L}(x_0)\right]\|u - x\|^2 \\ &\le \left[\lambda_0\left(\frac{c_2}{\widehat{\beta}(x_0)r} - \frac{c_3}{r^2}\right) - \frac{\widehat{\alpha}(x_0)\widehat{\nu}(_0)}{\widehat{\beta}(x_0)} + \frac{1}{2}\widehat{L}(x_0)\right]\|u - x\|^2.\end{aligned}$$

Due to (4.27) for x_0 and the choice of r, we arrive at a contradiction, and this implies that the tangential criterion (1.4) for the convexity of the set S_r is satisfied. Thus, the set $S_r(x_0)$ is convex. In accordance with Klee's theorem this implies that $G(S)$ is convex. The theorem is proved. □

ACKNOWLEDGMENTS

The author is grateful to Sergey Aseev and Francis Clarke for useful discussions.

FUNDING

This work was partially supported by the National Science Foundation grant DMS-0708019.

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This article was submitted by the author simultaneously in Russian and English