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On Some Properties of Vector Measures

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Abstract—We study the properties of a parameterized sequence of countably additive vector measures with densities defined on a compact space T with a nonnegative nonatomic Radon measure μ and taking values in a separable Banach space. Each vector measure of this sequence depends continuously on a parameter belonging to a metric space. We assume that a countable locally finite open cover and a partition of unity inscribed into this cover are given in the metric space of parameters. We prove that, for each value of the parameter, there exists a sequence of μ-measurable subsets of the space T which is a partition of T. In addition, this sequence depends uniformly continuously on the parameter and, for each value of the parameter and each element of the initial parameterized sequence of vector measures, the relative value of the measure of the corresponding subset in the partition of T can be uniformly approximated by the corresponding value of the corresponding partition of unity function.

Keywords: Lyapunov theorem, countably additive vector measure, density of a vector measure, partition of unity, continuous mapping.

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INTRODUCTION

From the famous Lyapunov theorem [1, 2; 3, Ch. 8]) on vector measures defined on a compact topological space $T = (T, \mathcal{T}, \mu)$ with a σ-algebra $\mathcal{T}$ of measurable sets and a finite nonnegative nonatomic Radon measure μ, it follows that, for any countably additive vector measure $m : \mathcal{T} \to \mathbb{R}^n$ and any $\alpha \in [0, 1]$, there exists a set $A_\alpha \in \mathcal{T}$ such that $m(A_\alpha) = \alpha m(T)$.

Based on this corollary, we proved the following result (see [4, Lemma 3.1]): if $T = [t_0, t_1]$ is a segment of the real line with a Lebesgue measure μ and a family $\mathcal{T}$ of Lebesgue measurable sets, $m_s : \mathcal{T} \to \mathbb{R}^n$ is a family of countably additive measures continuously depending on a parameter s belonging to a compact metric space S, and $\{p_j(s)\}_{j=1}^J$ is a continuous partition of unity on S, then, for any $\delta > 0$, there exists a collection of disjoint sets $\{A_j(s)\}_{j=1}^J \subset \mathcal{T}$ continuously depending on $s \in S$ such that $\bigcup_{j=1}^J A_j(s) = T$ (i.e., for any $s \in S$, $\{A_j(s)\}_{j=1}^J$ is a measurable partition of T) and, for any $s \in S$ and any $j \in \overline{(1, J)}$, we have $\|m_s(A_j(s)) - p_j(s) m_s(T)\| < \delta$ and $\mu(A_j(s)) = p_j(s)\mu(T)$. Using this result, we proved the existence of a continuous mapping from a compact function space of parameters to a set of solutions of a differential inclusion. In turn, this allowed us to obtain necessary conditions for the optimality of the solution of an extremal Mayer problem where one of the constraints is a differential inclusion with Lipschitz right-hand side (see [5]).

In the case where a separable Banach space E and a compact topological space $T = (T, \mathcal{T}, \mu)$ are taken instead of $\mathbb{R}^n$ and $[t_0, t_1]$, respectively, the corresponding generalization of the Lyapunov

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theorem (see [6]) implies the weaker statement that, for any countably additive vector measure $m : \mathcal{T} \to E$ with density and any $\alpha \in [0,1]$ and $\delta > 0$, there exists a set $A_\alpha \in \mathcal{T}$ such that $\|m(A_\alpha) - \alpha m(T)\| < \delta$.

In optimal control problems for systems in a Banach space, there is a need to solve the following problem: for any sequence of vector measures $\{m_j\}_{j=1}^\infty$, $m_j : \mathcal{T} \to E$, with densities, any sequence of numbers $\{\alpha_j\}_{j=1}^\infty$ such that $\alpha_j \geq 0$ and $\sum_{j=1}^\infty \alpha_j = 1$, and any $\delta > 0$, find a partition of the space T into disjoint measurable subsets $\{A_j\}_{j=1}^\infty \subset \mathcal{T}$ such that $\|m_j(A_j) - \alpha_j m_j(T)\| < \delta$ for all $j \in \mathbb{N}$.

Of special interest is the case where each vector measure m_j depends continuously on a parameter s belonging to a metric space S (i.e., we have measures $m_{j,s} : \mathcal{T} \to E$) and the numbers $\{\alpha_j\}$ depend on the parameter and correspond to a partition of unity $\{p_j(s)\}$ of the space S. In this case, it is required to prove that, for any $\delta > 0$, there exists a family $\{A_j(s)\}_{j=1}^\infty$ of measurable partitions of the space T which is continuous in the parameter s and satisfies the estimates $\| \sum_{j=1}^\infty m_{j,s}(A_j(s)) - \sum_{j=1}^\infty p_j(s) m_{j,s}(T)\| < \delta$. This problem was largely solved in [7], which allowed the authors of that paper to prove a relaxation theorem for differential inclusions with Lipschitz right-hand side and values in a separable Banach space.

The present paper is devoted to the development of the results of the mentioned works. For a countable family of vector measures having densities and continuously depending on a parameter belonging to a metric space, we construct a countable family of measurable partitions of the compact support T of this family of measures. The constructed family of partitions is continuously parameterized by the same parameter and gives an approximate expansion of a convex combination of the values of the measures on T with more accurate uniform estimates of the form $\sum_{j=1}^\infty \|m_{j,s}(A_j(s)) - p_j(s)\, m_{j,s}(T)\| < \delta$.

1. MAIN NOTATION AND DEFINITIONS

In what follows, we assume that $(T, \mathcal{T}, \mu)$ is a compact topological space with a σ-algebra $\mathcal{T}$ of measurable subsets and a finite nonnegative nonatomic Radon measure μ on them. We also assume that E is a separable Banach space and S is a separable metric space of parameters. Define the interval $I := [0, 1]$.

Recall that a function $m : \mathcal{T} \to E$ is called a *finitely additive vector measure* if, for any disjoint sets $A_1, A_2 \in \mathcal{T}$, $m(A_1 \cup A_2) = m(A_1) + m(A_2)$. If, in addition, for any sequence $\{A_n\}$ of pairwise disjoint sets in $\mathcal{T}$, we have the equality

$$m\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m(A_n),$$

where the series on the right-hand side converges in the norm of the space E, then m is called a *countably additive vector measure*.

Recall that a family of sets $\{A_\alpha\}_{\alpha\in I}$ is called *increasing* if $A_\alpha \subset A_\beta$ for any numbers $\alpha, \beta \in I$ such that $\alpha < \beta$.

The most important example of countably additive vector measures is the indefinite Bochner integral. To explain this, let $f : T \to E$ be a Bochner integrable function. Then the function $m_f : \mathcal{T} \to E$ defined by the formula

$$m_f(A) := \int_A f(t) d\mu(t), \quad A \in \mathcal{T},$$

is a countably additive and μ-continuous vector measure of bounded variation (see, for instance, [6]). In this case, the function $f(\cdot) \in L^1(T, E)$ is called the *density* (*or Radon–Nikodym derivative*) *of the vector measure* m_f.

Note that we consider only countably additive vector measures with densities. The set of all such measures is denoted by $\mathcal{M}(T, E)$.

For any vector measure $m \in \mathcal{M}(T, E)$ with density $f(\cdot) \in L^1(T, E)$, we define the *norm of this measure* as the norm of its density:

$$\|m\| = \|f(\cdot)\|_{L^1}.$$

Therefore, if a vector measure $m_s \in \mathcal{M}(T, E)$ with density $f_s(\cdot) \in L^1(T, E)$ is defined for any $s \in S$, then the mapping m_s from S to $\mathcal{M}(T, E)$ is continuous if and only if the mapping f_s from S to $L^1(T, E)$ is continuous.

In the case of a finite set of vector measures $m_1, \ldots, m_n \in \mathcal{M}(T, E)$, we define the *composite vector measure* $\widetilde{m}$ as follows:

$$\widetilde{m} := (m_1, \ldots, m_n) \in \mathcal{M}(T, E \times \ldots \times E).$$

The norm of this measure is defined by the formula

$$\|\widetilde{m}\| := \max\{\|m_1\|, \ldots, \|m_n\|\}.$$

Recall that the *characteristic function of a set* $B \subset T$ is the function χ_B such that $\chi_B(t) = 1$ if $t \in B$ and $\chi_B(t) = 0$ if $t \notin B$.

For any vector measure $m \in \mathcal{M}(T, E)$ and any set $B \in \mathcal{T}$, the measure $m_{|B}$ defined by the formula

$$m_{|B}(D) := m(D \cap B) \quad \forall D \in \mathcal{T}$$

is called the *restriction of the measure* m *to the set* B. It is easy to see that if $f(\cdot) \in L^1(T, E)$ is the density of a vector measure m and $B \in \mathcal{T}$, then the function $f(\cdot)\,\chi_B(\cdot)$ is the density of the vector measure $m_{|B}$.

For measurable subsets of the space T (more precisely, for equivalence classes of $\mathcal{T}$), we define a metric by the formula

$$\varrho(B, C) := \mu(B \triangle C) \quad \forall B, C \in \mathcal{T},$$

where $B \triangle C$ denotes the symmetric difference of sets B and C.

In what follows, the *continuity* of mappings from S to $\mathcal{T}$ is understood in the sense of this metric.

Proposition 1. *For any* $s \in S$, *let* $m(s) \in \mathcal{M}(T, E)$, $A(s) \in \mathcal{T}$, *and* $\hat{m}(s) := m(s)_{|A(s)}$. *Assume that the mappings* $m : S \to \mathcal{M}(T, E)$ *and* $A : S \to \mathcal{T}$ *are continuous. Then the mapping* $\hat{m} : S \to \mathcal{M}(T, E)$ *is continuous.*

Proof. Let $f(s)(\cdot) \in L^1(T, E)$ be the density of the vector measure $m(s)$ for any $s \in S$. We fix a point $s_0 \in S$ and an arbitrary $\varepsilon > 0$. By the absolute continuity of the Lebesgue integral, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any $A \in \mathcal{T}$ with $\mu(A) < \delta$, the following inequality holds: $\displaystyle\int_A \|f(s_0)(t)\|d\mu(t) < \varepsilon/3$. In turn, there exists a neighborhood $U(s_0) \subset S$ of the point s_0 such that, by the continuity of the measures $s \to m(s)$ and the sets $s \to A(s)$, the inequalities

$\|f(s)(\cdot) - f(s_0)(\cdot)\|_{L^1} < \varepsilon/3$ and $\mu(A(s)\triangle A(s_0)) < \delta$ hold for all $s \in U(s_0)$. As a result, we have

$$\|\hat{m}(s) - \hat{m}(s_0)\| = \int\limits_T \|f(s)(t)\,\chi_{A(s)}(t) - f(s_0)(t)\,\chi_{A(s_0)}(t)\|d\mu(t)$$

$$\leq 2\int\limits_T \|f(s)(t) - f(s_0)(t)\|d\mu(t) + \int\limits_{A(s)\triangle A(s_0)} \|f(s_0)(t)\|d\mu(t) < \varepsilon,$$

which completes the proof of the proposition. □

2. SOME COROLLARIES OF THE LYAPUNOV THEOREM

Recall the statement of the Lyapunov theorem on vector measures [1].

Theorem 1 (A. A. Lyapunov). *Let $(T, \mathcal{T}, \mu)$ be a compact topological space with a finite nonnegative nonatomic Radon measure μ, and let $f\colon\ T \to \mathbb{R}^n$ be a Lebesgue integrable vector function. We define a vector measure $m : \mathcal{T} \to \mathbb{R}^n$ as follows*:

$$m(A) := \int\limits_A f(t)d\mu(t), \quad A \in \mathcal{T}.$$

Then the set $m(\mathcal{T})$ defined by the formula

$$m(\mathcal{T}) := \{m(A) \mid A \in \mathcal{T}\}$$

is convex and compact in $\mathbb{R}^n$.

Note that, for any measurable set $D_0 \subset T$ of measure zero (for instance, for the empty set), we have $m(D_0) = 0 \in m(\mathcal{T})$. Similarly, $m(T) \in m(\mathcal{T})$. Consequently, by the Lyapunov theorem, the segment $[0, m(T)]$ belongs to the convex compact set $m(\mathcal{T})$. Therefore, for any $\alpha \in [0, 1]$, we have $\alpha m(T) \in m(\mathcal{T})$. Thus, we obtain the following result.

Corollary 1. *Let $f\colon T \to \mathbb{R}^n$ be a Lebesgue integrable vector function. Then, for any $\alpha \in [0, 1]$, there exists a measurable set $A_\alpha \subset T$ such that $m(A_\alpha) = \alpha m(T)$*, i.e.,

$$\alpha\int\limits_T f(t)d\mu(t) = \int\limits_{A_\alpha} f(t)d\mu(t).$$

Note that it is not possible to directly extend the Lyapunov theorem to arbitrary countably additive nonatomic vector measures without density and with values in an infinite-dimensional space.

The known generalization of the Lyapunov theorem for a vector measure with density and with values in the separable Banach space E (see [6]) implies that, in this case, only the closure of the set of all vectors $m(D)$ corresponding to all possible measurable subsets D of T is a convex compact set in E.

Therefore, similarly to Corollary 1, from the generalized Lyapunov theorem, we obtain only an inequality; i.e., the following result holds.

Corollary 2. *Let $f\colon T \to E$ be a Bochner integrable function. Then, for any $\varepsilon > 0$ and any $\alpha \in [0,1]$, there exists a measurable set $A_\alpha \subset T$ such that*

$$\left\| \alpha \int\limits_T f(t)d\mu(t) - \int\limits_{A_\alpha} f(t)d\mu(t) \right\| < \varepsilon.$$

Combining Corollaries 1 and 2, we obtain the following result.

Corollary 3. *Let $f\colon T \to E$ be a Bochner integrable function, and let $g\colon T \to \mathbb{R}^p$ be a Lebesgue integrable function. We define vector measures $m : \mathcal{T} \to E$ and $m_0 : \mathcal{T} \to \mathbb{R}^p$ as follows:*

$$m(A) := \int\limits_A f(t)d\mu(t), \quad m_0(A) := \int\limits_A g(t)d\mu(t) \quad \forall A \in \mathcal{T}. \tag{2.1}$$

Then, for any $\varepsilon > 0$ and any $\alpha \in [0,1]$, there exists a set $A_\alpha \in \mathcal{T}$ such that $\|m(A_\alpha) - \alpha m(T)\| < \varepsilon$ and $m_0(A_\alpha) = \alpha m_0(T)$, i.e.,

$$\left\| \alpha \int\limits_T f(t)d\mu(t) - \int\limits_{A_\alpha} f(t)d\mu(t) \right\| < \varepsilon, \tag{2.2}$$

$$\alpha \int\limits_T g(t)d\mu(t) = \int\limits_{A_\alpha} g(t)d\mu(t). \tag{2.3}$$

Note that Corollary 3 says nothing about the relationship between the measurable sets $\{A_\alpha\}_{\alpha \in I}$ besides the fact that they satisfy relations (2.2) and (2.3). We would like this family of sets to be increasing.

3. SEGMENTS AND ε-SEGMENTS

Definition 1 [8]. Let $f \in L^1(T, E)$, and let $m : \mathcal{T} \to E$ be the vector measure with density f. An increasing family of sets $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ such that $A_0 = \varnothing$ and $A_1 = T$ is called:

(1) a segment for the measure m if and only if, for any $\alpha \in I$, $m(A_\alpha) = \alpha m(T)$;

(2) an ε-segment for the measure m if and only if, for any $\alpha \in I$, $\|m(A_\alpha) - \alpha m(T)\| < \varepsilon$.

In what follows, we use the following theorem proved in [9].

Theorem 2 [9, Theorem 15]. *Let $f \in L^1(T, E)$ and $g \in L^1(T, \mathbb{R}^p)$, and let $m : \mathcal{T} \to E$ and $m_0 : \mathcal{T} \to \mathbb{R}^p$ be the corresponding vector measure (see formulas (2.1)). Then, for any $\varepsilon > 0$, there exists a family $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ which is an ε-segment for the vector measure m and is a segment for the vector measure m_0.*

In what follows, we need three lemmas which are either proved in [9] or are small modifications of the statements in [9].

Lemma 1 [8; 9, Proposition 18]. *Let S be a compact Hausdorff topological space, let $\{m_s\}_{s \in S} \subset \mathcal{M}(T, E)$ be a set of vector measures continuously depending on the parameter $s \in S$, and let $m_0 \in \mathcal{M}(T, \mathbb{R}^p)$. Then, for any $\varepsilon > 0$, there exists a family $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ which is a segment for the measure m_0 and is an ε-segment for any vector measure m_s, $s \in S$.*

Note that we can construct several different ε-segments in $\mathcal{T}$ for the same vector measure $m \in \mathcal{M}(T, E)$. In what follows, we need to find ways to pass continuously from one ε-segment of such a measure to another. A slight strengthening of Proposition 2 from [7] (or Proposition 19 from [9]) leads to the following result.

Lemma 2. *Let $m \in \mathcal{M}(T, E)$, and let $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\alpha\}_{\alpha\in I}$ be families in $\mathcal{T}$ each of which is an ε-segment for the measure m and is a segment for the measure μ. Then there exists a continuous mapping $D\colon I \times I \to \mathcal{T}$ with the following properties*:

(1) *$D(0, \alpha) = A_\alpha$ and $D(1, \alpha) = B_\alpha$ for any $\alpha \in I$;*

(2) *for any $z \in I$, the family $\{D(z, \alpha)\}_{\alpha\in I}$ is an ε-segment for both the measure m and the measure μ;*

(3) *$\mu(D(z_1, \alpha_1)\triangle D(z_2, \alpha_2)) \le (|z_1 - z_2| + 2|\alpha_1 - \alpha_2|)\mu(T)$, $\alpha_1, \alpha_2, z_1, z_2 \in I$.*

Proof. Since the families $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\alpha\}_{\alpha\in I}$ are segments for the measure μ, the mappings $\alpha \to \chi_{A_\alpha}(\cdot)$ and $\alpha \to \chi_{B_\alpha}(\cdot)$ from I to $L^1(T, \mathbb{R}^1)$ are continuous. Hence, the mappings $\alpha \to m(A_\alpha)$ and $\alpha \to m(B_\alpha)$ are continuous. Consequently, defining

$$a := \max\{\max_{\alpha\in I} \|m(A_\alpha) - \alpha\, m(T)\|,\, \max_{\alpha\in I} \|m(B_\alpha) - \alpha\, m(T)\|\},$$

we find that $a < \varepsilon$. Let η be a positive number such that $a + 2\eta < \varepsilon$.

For any $\alpha \in I$, we define a vector measure m_α as follows:

$$m_\alpha := (m_{|A_\alpha}, m_{|B_\alpha}, \mu_{|A_\alpha}, \mu_{|B_\alpha}) \in \mathcal{M}(T, E \times E \times \mathbb{R}^1 \times \mathbb{R}^1). \tag{3.1}$$

By Proposition 1, the mapping $\alpha \to m_\alpha$ is continuous. In view of Lemma 1, there exists a family $\{C_z\}_{z\in I}$ which is an η-segment for each measure of the family $\{m_\alpha\}_{\alpha\in I}$ and is a segment for the measure μ. We define

$$D(z, \alpha) := (B_\alpha \cap C_z) \cup (A_\alpha \cap (T \setminus C_z)), \quad z, \alpha \in I, \tag{3.2}$$

and show that $D(z, \alpha)$ are the desired sets. By construction, it follows from (3.2) that property (1) holds. By the definition of an η-segment, for the family $\{C_z\}_{z\in I}$, we have

$$\|m_\alpha(C_z) - z m_\alpha(T)\| < \eta. \tag{3.3}$$

Therefore,

$$\|m_\alpha(T \setminus C_z) - (1 - z)m_\alpha(T)\| < \eta. \tag{3.4}$$

Inequality (3.3) implies, in particular, that, for the second component in (3.1),

$$\|m(B_\alpha \cap C_z) - z m(B_\alpha)\| < \eta.$$

Since the family $\{B_\alpha\}_{\alpha\in I}$ is an ε-segment for the measure m, we have

$$\|m(B_\alpha \cap C_z) - z\alpha m(T)\| \le \|m(B_\alpha \cap C_z) - z m(B_\alpha)\| + \|z m(B_\alpha) - z\alpha m(T)\| < \eta + za. \tag{3.5}$$

Similarly, we deduce from (3.4) that, for the first component in (3.1),

$$\|m((T \setminus C_z) \cap A_\alpha) - (1 - z)\alpha m(T)\| < \eta + (1 - z)a. \tag{3.6}$$

Summing inequalities (3.5) and (3.6), we find that, for any $\alpha \in I$,

$$\|m(D(z, \alpha)) - \alpha m(T)\| \le 2\eta + a < \varepsilon.$$

Similarly, we deduce from inequalities (3.3) and (3.4) that, for the last two components in (3.1),

$$|\mu(D(z,\alpha)) - \alpha\mu(T)| < \varepsilon.$$

To prove the continuity of the mapping $D : I \times I \to \mathcal{T}$ (more precisely, property (3)), we note that

$$D(z,\alpha)\triangle D(y,\alpha) = (A_\alpha \triangle B_\alpha) \cap (C_z \triangle C_y).$$

Since the family $\{C_z\}_{z\in I}$ is a segment for the measure μ, for $z_1 > z_2$, we have $C_{z_1} \supset C_{z_2}$ and, therefore, $C_{z_1}\triangle C_{z_2} = C_{z_1} \setminus C_{z_2}$. As a result, we obtain

$$\mu(D(z_1,\alpha)\triangle D(z_2,\alpha)) \le \mu(C_{z_1}\triangle C_{z_2}) \le |z_1 - z_2|\mu(T).$$

On the other hand, it follows from (3.2) that

$$\mu(D(z,\alpha_1)\triangle D(z,\alpha_2)) = \int\limits_T \big|\chi_{D(z,\alpha_1)}(t) - \chi_{D(z,\alpha_2)}(t)\big| d\mu(t)$$

$$= \int\limits_T \big|\chi_{B_{\alpha_1}\cap C_z}(t) + \chi_{A_{\alpha_1}\cap(T\setminus C_z)}(t) - \chi_{B_{\alpha_2}\cap C_z}(t) - \chi_{A_{\alpha_2}\cap(T\setminus C_z)}(t)\big| d\mu(t)$$

$$\le \int\limits_T \big|\chi_{B_{\alpha_1}}(t) - \chi_{B_{\alpha_2}}(t)\big|\chi_{C_z}(t) d\mu(t) + \int\limits_T \big|\chi_{A_{\alpha_1}}(t) - \chi_{A_{\alpha_2}}(t)\big|\chi_{T\setminus C_z}(t) d\mu(t)$$

$$\le \mu(A_{\alpha_1}\triangle A_{\alpha_2}) + \mu(B_{\alpha_1}\triangle B_{\alpha_2}) \le 2|\alpha_1 - \alpha_2|\mu(T).$$

Summing these inequalities, we obtain property (3). □

Lemma 3 [9, Theorem 17]. *Let* $\{m_j\}_{j=1}^{\infty} \subset \mathcal{M}(T,E)$, *let* $m_0 = \mu$, *and let vector measures* $\widetilde{m}_j$ *be defined as follows*:

$$\widetilde{m}_j := (m_0, m_1, \ldots, m_j) : \mathcal{T} \to \mathbb{R}^1 \times E \times \ldots \times E \quad \forall j = 0, 1, 2, \ldots.$$

Then, for any $\varepsilon > 0$, *there exists a continuous mapping* $D : [0,\infty) \times I \to \mathcal{T}$ *with the following properties*:

(1) *for any* $z \in [0,\infty)$, *the family* $\{D(z,\alpha)\}_{\alpha\in I}$ *is an* ε*-segment for the measure* $\widetilde{m}_j$ with $j = [z]$;

(2) $\mu(D(z_1,\alpha_1)\triangle D(z_2,\alpha_2)) \le (|z_1 - z_2| + 2|\alpha_1 - \alpha_2|)\mu(T)\ \forall \alpha_1, \alpha_2 \in I,\ z_1, z_2 \in [0,\infty)$.

Proof. We fix $\varepsilon > 0$. By Theorem 2, for any $j = 0, 1, 2, \ldots$, there exists a family $\{D(j,\alpha)\}_{\alpha\in I}$ which is an ε-segment for the vector measure $\widetilde{m}_j$ and is a segment for the measure μ. To extend this family to values $\{D(z,\alpha)\}_{\alpha\in I}$ defined for all $z \in [0,\infty)$, for any $j = 0, 1, 2, \ldots$, we apply Lemma 2 to the families $\{D(j,\alpha)\}_{\alpha\in I}$ and $\{D(j+1,\alpha)\}_{\alpha\in I}$ which are ε-segments for the vector measure $\widetilde{m}_j$ and are segments for the measure μ. By this lemma, there exists a continuous mapping $C_j : I \times I \to \mathcal{T}$ connecting these families. Then we define the mappings $D(z,\alpha) := C_j(z - [z], \alpha)$, $z \in [j, j+1)$, which obviously have properties (1) and (2). □

4. COVERS AND PARTITIONS

In what follows, we assume that $S = (S,d)$ is a separable metric space with metric d. An *open cover of the metric space* (S,d) is a family $\{V_\lambda\}_{\lambda\in\Lambda} \subset S$ of nonempty open sets such that $\bigcup_{\lambda\in\Lambda} V_\lambda = S$ and $V_\lambda \neq S$ for any $\lambda \in \Lambda$.

An open cover $\{V_n\}_{n=1}^{\infty}$ of the separable metric space (S,d) is called *locally finite* if, for any point $s_0 \in S$, there exists a neighborhood $U(s_0)$ such that $V_n \cap U(s_0) \neq \varnothing$ only for a finite number of indices $n \in \mathbb{N}$.

We need the following property of open covers.

Lemma 4. *Let $\{V_\lambda\}_{\lambda\in\Lambda} \subset S$ be a locally finite cover of the metric space (S,d). Then there exists a locally finite cover $\{W_\lambda\}_{\lambda\in\Lambda}$ of the space (S,d) such that $\overline{W}_\lambda \subset V_\lambda$ for any $\lambda \in \Lambda$, where $\overline{W}_\lambda$ is the closure of the set W_λ.*

Proof. For any $s \in S$, we define $\Lambda_s := \{\lambda \in \Lambda \mid s \in V_\lambda\}$. Since the cover $\{V_\lambda\}_{\lambda\in\Lambda}$ is locally finite, any set Λ_s is finite and nonempty. For any $\lambda \in \Lambda$, we define

$$S_\lambda := \{s \in V_\lambda \mid d(s, S \setminus V_\lambda) = \max_{\nu\in\Lambda_s} d(s, S \setminus V_\nu)\}.$$

Thus, we associate any point $s \in S$ with sets V_λ containing this point together with the largest ball. As a result, we have: (i) for any $s \in S$, there exists $\lambda \in \Lambda$ such that $s \in S_\lambda$; (ii) $S_\lambda \subset V_\lambda$ for any $\lambda \in \Lambda$.

We fix an arbitrary $\lambda_0 \in \Lambda$ and consider an arbitrary point $s_1 \in S \setminus V_{\lambda_0}$. By the definition of a cover, there exists $\lambda_1 \in \Lambda$ such that $s_1 \in V_{\lambda_1}$. Therefore, there exists $\varepsilon > 0$ such that $B_{3\varepsilon}(s_1) \subset V_{\lambda_1}$. We show that $B_\varepsilon(s_1) \cap S_{\lambda_0} = \varnothing$. If this is not the case, then there exists a point $s_2 \in B_\varepsilon(s_1) \cap S_{\lambda_0}$. Therefore, $B_{2\varepsilon}(s_2) \subset B_{3\varepsilon}(s_1) \subset V_{\lambda_1}$. Consequently, $\lambda_1 \in \Lambda_{s_2}$ and

$$d(s_2, S \setminus V_{\lambda_0}) \leq d(s_2, s_1) \leq \varepsilon < 2\varepsilon \leq d(s_2, S \setminus V_{\lambda_1}),$$

which contradicts the inclusion $s_2 \in S_{\lambda_0}$. Thus, we have shown that any point $s_1 \in S \setminus V_{\lambda_0}$ has a neighborhood not intersecting S_{λ_0}. Hence, $\overline{S_{\lambda_0}} \subset V_{\lambda_0}$. Consider the closed sets $\overline{S_{\lambda_0}}$ and $\overline{U}_{\lambda_0} := S \setminus V_{\lambda_0}$. As we have just shown, they are disjoint. Since any metric space is a normal space, there exist disjoint neighborhoods $U(\overline{S_{\lambda_0}})$ and $U(\overline{U}_{\lambda_0})$ of the sets $\overline{S_{\lambda_0}}$ and $\overline{U}_{\lambda_0}$, respectively. Define $W_{\lambda_0} := U(\overline{S_{\lambda_0}})$. Then, by construction, $W_{\lambda_0} \subset S \setminus U(\overline{U}_{\lambda_0}) \subset V_{\lambda_0}$. In addition, the set $S \setminus U(\overline{U}_{\lambda_0})$ is closed. Consequently, W_{λ_0} is an open subset of V_{λ_0} contained in V_{λ_0} together with its closure.

Since, for any $s \in S$, there exists $\lambda \in \Lambda$ such that $s \in S_\lambda \subset W_\lambda$, the required cover $\{W_\lambda\}_{\lambda\in\Lambda}$ of the space S is constructed. □

A sequence of continuous functions $p_n : S \to I$, where $I = [0,1]$ and $n \in \mathbb{N}$, is called a *partition of unity subordinate to a locally finite open cover* $\{V_n\}_{n=1}^{\infty}$ of the space S if $\operatorname{supp} p_n \subset V_n$ for any $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} p_n(s) = 1$ for any $s \in S$. Note that this sum contains only a finite number of nonzero terms, which follows from the local finiteness of the cover $\{V_n\}_{n=1}^{\infty}$. We also recall that $\operatorname{supp} f$ is the closure of the set $\{s \in S \mid f(s) > 0\}$.

A *measurable partition of the space* T is a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{T}$ such that $A_i \cap A_j = \varnothing$ for $\forall i \neq j$ and $\bigcup_{n=1}^{\infty} A_n = T$.

If $\{A_n(s)\}_{n=1}^{\infty}$ is a measurable partition of the space T depending on the parameter $s \in S$ and if the mappings $A_n : S \to \mathcal{T}$ are continuous for any $n \in \mathbb{N}$, we say that $\{A_n(s)\}_{n=1}^{\infty}$ is a *continuous family of measurable partitions of the space* T.

A family $\{A_n(s)\}_{n=1}^{\infty}$ of measurable partitions of the space T is called *finite* if, for any $s_0 \in S$, the measurable partition $\{A_n(s_0)\}_{n=1}^{\infty}$ is finite, i.e., $\mu(A_n(s_0)) > 0$ for a finite number of indices $n \in \mathbb{N}$ and $A_n(s_0) = \varnothing$ for the remaining indices n.

The following remarkable theorem was proved in [9].

Theorem 3 [9, Theorem 18]. *Suppose that $\{V_n\}_{n=1}^{\infty}$ is a locally finite open cover of the metric space S, $\{p_n(\cdot)\}_{n=1}^{\infty}$ is a partition of unity subordinate to this cover, a sequence $\{m_{n,s}\}_{n=1}^{\infty}$ of vector measures with densities (i.e., $\{m_{n,s}\}_{n=1}^{\infty} \subset \mathcal{M}(T,E)$) is given for any $s \in S$, and the mapping $m_{n,\cdot} : S \to E$ is continuous for any $n \in \mathbb{N}$. Then, for any $\delta > 0$, there exists a finite continuous family $\{A_n(s)\}_{n=1}^{\infty} \subset \mathcal{T}$ of measurable partitions of the space T such that, for any $s \in S$,*

$$\Big\| \sum_{n=1}^{\infty} m_{n,s}(A_n(s)) - \sum_{n=1}^{\infty} p_n(s) m_{n,s}(T) \Big\| < \delta,$$

$$|\mu(A_n(s)) - p_n(s)\mu(T)| < \delta \quad \forall n \in \mathbb{N}.$$

Our goal is to give a generalization of this theorem important for applications.

5. MAIN RESULT

Theorem 4. *Suppose that $\{V_n\}_{n=1}^{\infty}$ is a locally finite open cover of the metric space S, $\{p_n(\cdot)\}_{n=1}^{\infty}$ is a partition of unity subordinate to this cover, a sequence $\{m_{n,s}\}_{n=1}^{\infty}$ of vector measures with densities (i.e., $\{m_{n,s}\}_{n=1}^{\infty} \subset \mathcal{M}(T,E)$) is given for any $s \in S$, and the mapping $m_{n,\cdot} : S \to E$ is continuous for any $n \in \mathbb{N}$. Then, for any $\delta > 0$, there exists a finite continuous family $\{A_n(s)\}_{n=1}^{\infty} \subset \mathcal{T}$ of measurable partitions of the space T such that:*

(1) *for $s \in S$ and $n \in \mathbb{N}$, if $\mu(A_n(s)) > 0$, then $p_n(s) > 0$;*

(2) *for any $s \in S$,*

$$\sum_{n=1}^{\infty} \|m_{n,s}(A_n(s)) - p_n(s) m_{n,s}(T)\| < \delta, \tag{5.1}$$

$$\sum_{n=1}^{\infty} |\mu(A_n(s)) - p_n(s)\mu(T)| < \delta, \tag{5.2}$$

$$\lim_{s' \to s} \sum_{n=1}^{\infty} \mu(A_n(s') \triangle A_n(s)) = 0. \tag{5.3}$$

Proof. Let $f_n(s)(\cdot) \in L^1(T,E)$ be the corresponding densities of the vector measures $m_{n,s} : \mathcal{T} \to E$. By the assumption of the theorem, for any $n \in \mathbb{N}$, the mapping $f_n : S \to L^1(T,E)$ is continuous. For any $s \in S$, we define

$$N_s := \{n \in \mathbb{N} \mid p_n(s) > 0\}.$$

For any $n \in \mathbb{N}$, let $h_n : S \to I$ be a continuous function such that $h_n(s) = 1$ for $s \in \operatorname{supp} p_n$ and $\operatorname{supp} h_n \subset V_n$. Define $r(s) := \sum_{n=1}^{\infty} h_n(s)$. In view of the local finiteness of the cover $\{V_n\}_{n=1}^{\infty}$, for any $s \in S$, the cardinality of the set N_s (denoted by $\operatorname{card}\{N_s\}$) is finite. More precisely, $\operatorname{card}\{N_s\} \le r(s) < \infty$.

We define functions $\widetilde{f_n} : S \to L^1(T, \mathbb{R}^1 \times E)$ as follows: $\widetilde{f_n}(s)(\cdot) := (1, f_n(s)(\cdot))$.

For any $n \in \mathbb{N}$ and any $s \in S$, we define a function $k_n(s) : T \to L^1(T, \mathbb{R}^1 \times E)$ as follows:

$$k_n(s)(t) := r(s) h_n(s) \widetilde{f_n}(s)(t), \quad t \in T. \tag{5.4}$$

Obviously, $r(s) \geq 1$ for any $s \in S$ and the mappings $r : S \to \mathbb{R}^1$ and $k_n : S \to L^1(T, \mathbb{R}^1 \times E)$ are continuous. For any $s_0 \in S$, we define

$$U_{s_0} := \bigcap_{n \in N_{s_0}} \Big\{ s \in S \big| p_n(s) > 0, \|f_n(s) - f_n(s_0)\|_{L^1} < \frac{\delta}{16 r(s_0)}, 3r(s) < 4r(s_0) \Big\}. \tag{5.5}$$

Clearly, the family $\{U_s\}_{s \in S}$ is an open cover of the metric space S. By virtue of the paracompactness of the space S and Lemma 4, there exists a sequence of continuous functions $r_i : S \to I$ such that the sequence $\{\mathrm{int}\,\mathrm{supp}\, r_i\}_{i=1}^{\infty}$ is a locally finite subcover of the cover $\{U_s\}_{s \in S}$ and the sets $\overline{W}_i := \{s \in S \mid r_i(s) = 1\}$ cover the space S. Therefore, there exist points $s_i \in S$ such that $\overline{W}_i \subset U_{s_i}$ for any $i \in \mathbb{N}$. We define the functions $u_j \in L^1(T, \mathbb{R}^1 \times E)$, $j \in \mathbb{N}$, by the formula

$$u_j(t) := \begin{cases} k_n(s_i)(t) & \text{if } j = 2^i 3^n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Using these functions as densities, we define vector measures $m_j(A)$ by the formula

$$m_j(A) := \int_A u_j(t) d\mu(t), \quad j \in \mathbb{N},$$

and let $m_0(A) = \mu(A)$ for $A \in \mathcal{T}$. We also define vector measures $\widetilde{m}_j$ as follows:

$$\widetilde{m}_j := (m_0, m_1, \dots, m_j), \quad j = 0, 1, 2, \dots .$$

By Lemma 3, for the sequence of vector measures $\{m_j\}_{j=0}^{\infty}$, there exists a continuous mapping $D : [0, \infty) \times I \to \mathcal{T}$ with the following properties:

(1) for any $z \in [0, \infty)$, the family $\{D(z, \alpha)\}_{\alpha \in I}$ is a $\dfrac{\delta}{4}$-segment for the measure $\widetilde{m}_j$ with $j = [z]$;

(2) $\mu(D(z_1, \alpha_1) \triangle D(z_2, \alpha_2)) \leq (|z_1 - z_2| + 2|\alpha_1 - \alpha_2|)\mu(T)$ if $\alpha_1, \alpha_2 \in I$ and $z_1, z_2 \in [0, \infty)$.

We define the function $\tau : S \to \mathbb{R}^1_+$ by the formula

$$\tau(s) := \sum_{n, i=1}^{\infty} r_i(s)\, h_n(s)\, 2^i\, 3^n. \tag{5.7}$$

In view of the local finiteness of the covers $\{\mathrm{supp}\, r_i\}_{i=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ of the space S, the sum in formula (5.7) is finite for any $s \in S$ and the function $\tau : S \to \mathbb{R}^1_+$ is continuous.

Define the following family of measurable sets in T:

$$A(s, \alpha) := D(\tau(s), \alpha), \quad s \in S, \ \alpha \in I. \tag{5.8}$$

From the above and formula (5.8), we deduce that the mapping $A : S \times I \to \mathcal{T}$ is continuous and, for any $s \in S$, the family $\{A(s, \alpha)\}_{\alpha \in I}$ is a $\dfrac{\delta}{4}$-segment for the vector measure $\widetilde{m}_j$ with $j = [\tau(s)]$. In addition, for any $s_1, s_2 \in S$ and any $\alpha_1, \alpha_2 \in I$, we have

$$\mu(A(s_1, \alpha_1) \triangle A(s_2, \alpha_2)) \leq (|\tau(s_1) - \tau(s_2)| + 2|\alpha_1 - \alpha_2|)\mu(T).$$

We fix a point $s \in S$. To estimate $\tau(s)$ and find j such that $j \leq [\tau(s)]$, we choose an index $n \in N_s$ (i.e., an index n such that $h_n(s) = 1$) and an index $m \in \mathbb{N}$ such that $s \in \overline{W}_m \subset U_{s_m}$. Hence, $r_m(s) = 1$ and, by the definition of the set U_{s_m} (see (5.5)), we have $n \in N_{s_m}$, i.e., $p_n(s_m) > 0$. Consequently, $h_n(s_m) = 1$. This and (5.7) yield $\tau(s) \geq r_m(s)h_n(s)2^m3^n = 2^m3^n$. Thus, choosing $j = 2^m3^n$, we obtain the inequality $j \leq [\tau(s)]$, which implies that the family $\{A(s,\alpha)\}_{\alpha\in I}$ is a $\dfrac{\delta}{4}$-segment for the vector measure $\widetilde{m}_j$. In particular, this family is a $\dfrac{\delta}{4}$-segment for the vector measure m_j with $j = 2^m3^n$. In view of (5.4) and (5.6), this means that the function $u_j(\cdot) = k_n(s_m)(\cdot) = r(s_m)\widetilde{f_n}(s_m)(\cdot)$ is the density of the measure m_j. Therefore, dividing the corresponding inequality by $r(s_m) > 0$, for any $\alpha \in I$, we obtain the inequalities

$$|\mu(A(s,\alpha)) - \alpha\mu(T)| < \frac{\delta}{4r(s_m)}, \tag{5.9}$$

$$\left\| \int\limits_{A(s,\alpha)} f_n(s_m)(t)d\mu(t) - \alpha \int\limits_T f_n(s_m)(t)d\mu(t) \right\| < \frac{\delta}{4r(s_m)}. \tag{5.10}$$

In turn, for the same point $s \in S$, we have

$$\left\| \int\limits_{A(s,\alpha)} f_n(s)(t)d\mu(t) - \alpha \int\limits_T f_n(s)(t)d\mu(t) \right\|$$

$$\leq \left\| \int\limits_{A(s,\alpha)} f_n(s_m)(t)d\mu(t) - \alpha \int\limits_T f_n(s_m)(t)d\mu(t) \right\| + 2\|f_n(s)(\cdot) - f_n(s_m)(\cdot)\|_{L^1}.$$

Since $s \in U_{s_m}$, formula (5.5) for U_{s_m} and inequality (5.10) imply that the right-hand side of the latter inequality is less than

$$\frac{\delta}{4r(s_m)} + 2\,\frac{\delta}{16r(s_m)} = \frac{3\delta}{8r(s_m)} < \frac{\delta}{2r(s)}. \tag{5.11}$$

Hence,

$$\left\| \int\limits_{A(s,\alpha)} f_n(s)(t)d\mu(t) - \alpha \int\limits_T f_n(s)(t)d\mu(t) \right\| < \frac{\delta}{2r(s)} \tag{5.12}$$

for any $s \in S$, any $n \in N_s$, and any $\alpha \in I$. Similarly, it follows from (5.9) and (5.11) that

$$|\mu(A(s,\alpha)) - \alpha\mu(T)| < \frac{\delta}{2r(s)} \tag{5.13}$$

for any $s \in S$ and any $\alpha \in I$. We define $z_0(s) := 0$, $z_n(s) := p_1(s) + \ldots + p_n(s)$, and

$$A_n(s) := A(s, z_n(s)) \setminus A(s, z_{n-1}(s)). \tag{5.14}$$

Obviously, property (1) holds; i.e., if $p_n(s) = 0$, then $A_n(s) = \varnothing$. Since $\{p_n(s)\}_{n=1}^{\infty}$ is a locally finite partition of unity on S, for any $s \in S$, the sequence $\{z_n(s)\}_{n=1}^{\infty}$ is nondecreasing and there exists an index n_s such that $z_n(s) = 1$ for all $n \geq n_s$; i.e., the family $\{A_n(s)\}_{n=1}^{\infty}$ is a finite measurable partition of the space T such that $A_n(s) = \varnothing$ if $n \notin N_s$. Moreover, since, for any $n \in \mathbb{N}$, the function $z_n : S \to I$ is continuous, the mappings $s \to A(s, z_n(s))$ are continuous from S to $\mathcal{T}$. This along with (5.14) and the inequality

$$\mu\Big(\big(A(s_1, z_n(s_1)) \setminus A(s_1, z_{n-1}(s_1))\big)\triangle\big(A(s_2, z_n(s_2)) \setminus A(s_2, z_{n-1}(s_2))\big)\Big)$$
$$\le \mu\big(A(s_1, z_n(s_1))\triangle A(s_2, z_n(s_2))\big) + \mu\big(A(s_1, z_{n-1}(s_1))\triangle A(s_2, z_{n-1}(s_2))\big)$$

implies that the mappings $A_n : S \to \mathcal{T}$ are continuous.

We show that the sets $A_n(s)$ satisfy inequalities (5.1) and (5.2). From formula (5.14), the inclusion $A(s, z_{n-1}(s)) \subset A(s, z_n(s))$, and inequalities (5.12) and (5.13) with $\alpha = z_n(s)$ and $\alpha = z_{n-1}(s)$, we deduce that, for any $n \in \mathbb{N}$,

$$\Bigg\| \int\limits_{A_n(s)} f_n(s)(t)d\mu(t) - p_n(s) \int\limits_T f_n(s)(t)d\mu(t)\Bigg\| < \frac{\delta}{r(s)}, \qquad |\mu(A_n(s)) - p_n(s)\mu(T)| < \frac{\delta}{r(s)}.$$

The cardinality of the set of all indices n for which $A_n(s) \neq \varnothing$ can be estimated as follows: $\operatorname{card}\{n \mid A_n(s) \neq \varnothing\} = \operatorname{card}\{N_s\} \le r(s)$. Summing these inequalities over all $n \in \mathbb{N}$, we take into account only $n \in N_s$, since $p_n(s) = 0$ and $A_n(s) = \varnothing$ for $n \notin N_s$. As a result, we obtain inequalities (5.1) and (5.2).

Since the cover $\{V_n\}_{n=1}^{\infty}$ of the space S is locally finite, for any fixed point $s_0 \in S$, we choose a neighborhood $U(s_0)$ for which there exists a finite set $N(s_0) \subset \mathbb{N}$ such that $V_n \cap U(s_0) \neq \varnothing$ only for $n \in N(s_0)$. Denoting the cardinality of the set $N(s_0)$ by $\widetilde{r}(s_0)$, we have $r(s) \le \widetilde{r}(s_0) < \infty$ for any $s \in U(s_0)$. This means that if $s \in U(s_0)$ and $n \notin N(s_0)$, then $p_n(s) = 0$, i.e., $A_n(s) = \varnothing$. As a result, for any $s \in U(s_0)$, we have

$$\sum_{n=1}^{\infty} \mu(A_n(s)\triangle A_n(s_0)) = \sum_{n\in N(s_0)} \mu(A_n(s)\triangle A_n(s_0)).$$

Hence, taking into account that each mapping $A_n : S \to \mathcal{T}$ is continuous (i.e., each term in this sum tends to zero as $s \to s_0$) and the number of terms is finite for any $s \in U(s_0)$, we obtain equality (5.3). □

In conclusion, we show how the finite measurable partition $\{A_n(s)\}_{n=1}^{\infty}$ of the space T obtained in Theorem 4 can be used to construct continuous mappings.

Proposition 2. *Let S be a separable metric space, and let, for any $s \in S$ and any $n \in \mathbb{N}$, $A_n(s)$ be a measurable subset of the compact space T. We assume that the following relations hold:*

$$\begin{cases} A_{n_1}(s) \cap A_{n_2}(s) = \varnothing \quad \forall n_1 \neq n_2, \quad T = \bigcup\limits_{n=1}^{\infty} A_n(s), \\ \lim\limits_{s\to s_0} \sum\limits_{n=1}^{\infty} \mu(A_n(s)\triangle A_n(s_0)) = 0 \quad \forall s_0 \in S. \end{cases} \tag{5.15}$$

Let $v_n(\cdot) \in L^1(T, E)$ for any $n \in \mathbb{N}$, and assume that there exists a function $k(\cdot) \in L^1(T, \mathbb{R}^1_+)$ such that $\|v_n(t)\| \le k(t)$ for any $n \in \mathbb{N}$ and almost every $t \in T$. Let $g\colon S \to L^1(T, E)$ be the mapping defined by the formula

$$g(s)(t) := \sum_{n=1}^{\infty} \chi_{A_n(s)}(t)v_n(t), \quad t \in T. \tag{5.16}$$

Then the mapping g is continuous.

Proof. Using (5.15) and (5.16), for any $s, s_0 \in S$, we obtain

$$\int_T \|g(s)(t) - g(s_0)(t)\| d\mu(t) \le \sum_{n=1}^{\infty} \int_T \left| \chi_{A_n(s)}(t) - \chi_{A_n(s_0)}(t) \right| \|v_n(t)\| d\mu(t)$$

$$\le \sum_{n=1}^{\infty} \int_T \chi_{A_n(s) \triangle A_n(s_0)}(t) k(t) d\mu(t) \le \sum_{n=1}^{\infty} \int_{A_n(s) \triangle A_n(s_0)} k(t) d\mu(t).$$

It remains to note that, by (5.15), the right-hand side of this inequality tends to zero as $s \to s_0$. □

CONCLUSIONS

The obtained result makes it possible to construct a continuous mapping from a set of functions which approximate solutions of a differential inclusion with unbounded right-hand side and values in a Banach space to the set of solutions of this inclusion. Using such a mapping and other results, we can generalize the class of optimization problems for which necessary conditions for the optimality of a solution in Euler–Lagrange form can be proved.

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