ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2018, Vol. 303, pp. 178–192.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2018, Vol. 303, pp. 193–208.

On Irregular Sampling and Interpolation in Bernstein Spaces

Alexander Olevskii [a] **and Alexander Ulanovskii** [b]

Received April 30, 2018

Abstract—Sharp estimates of the sampling and interpolation constants in spaces of polynomials are obtained. These estimates are used to deduce asymptotically sharp estimates of the sampling and interpolation constants for Bernstein spaces as the density of a sampling set approaches the critical value.

DOI: 10.1134/S0081543818080151

1. INTRODUCTION

The main goal of this paper is to show that the classical Bernstein estimates for equidistant sampling and interpolation of bounded entire functions of exponential type remain optimal for non-equidistant sampling.

1.1. Bernstein space. The Bernstein space B_σ is the space of entire functions of exponential type $\sigma > 0$ bounded on the real axes, equipped with the uniform norm

$$\|f\| := \sup_{x\in\mathbb{R}}|f(x)|.$$

It is well known (see, e.g., [7, p. 17]) that every function $f \in B_\sigma$ satisfies the estimate

$$|f(x+iy)| \le \|f\| e^{\sigma|y|}, \qquad x, y \in \mathbb{R}. \tag{1.1}$$

Sampling at equidistant nodes in Bernstein (and other) spaces is a classical object of investigation. Consider, for simplicity, the set of integers $\mathbb{Z}$. Set

$$\|f|_{\mathbb{Z}}\| := \sup_{n\in\mathbb{Z}}|f(n)|.$$

Then for every $\sigma < \pi$, there is a constant K such that the following implication holds:

$$f \in B_\sigma \qquad \Rightarrow \qquad \|f\| \le K\|f|_{\mathbb{Z}}\|.$$

Denote by $K_{\mathrm{s}}(\mathbb{Z}, B_\sigma)$ the smallest possible value of K. Clearly,

$$K_{\mathrm{s}}(\mathbb{Z}, B_\sigma) := \sup_{f\in B_\sigma,\, f\ne 0} \frac{\|f\|}{\|f|_{\mathbb{Z}}\|}.$$

Observe that the example $f(x) = \sin(\pi x)$ shows that the implication above does not hold for $\sigma \ge \pi$.

Bernstein obtained a sharp asymptotic estimate of $K_{\mathrm{s}}(\mathbb{Z}, B_\sigma)$ as σ approaches the critical value π.

[a] School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel.
[b] University of Stavanger, N-4036 Stavanger, Norway.
E-mail addresses: olevskii@post.tau.ac.il (A. Olevskii), Alexander.Ulanovskii@uis.no (A. Ulanovskii).

Theorem A [1]. *Let $0 < \sigma < \pi$. Then*

$$K_{\mathrm{s}}(\mathbb{Z}, B_\sigma) = \frac{2}{\pi} \log \frac{\pi}{\pi - \sigma}(1 + o(1)), \qquad \sigma \uparrow \pi.$$

See [5] and the references therein for some other estimates of the constant $K_{\mathrm{s}}(\mathbb{Z}, B_\sigma)$.

Bernstein also obtained a sharp asymptotic estimate for the problem of interpolation of bounded sequences on $\mathbb{Z}$ by functions $f \in B_\sigma$, $\sigma > \pi$. Given a bounded sequence $\{c(n),\, n \in \mathbb{Z}\}$, one can construct a function $f \in B_\sigma$ satisfying

$$f(n) = c(n) \quad (n \in \mathbb{Z}), \qquad \|f\| \le K\|f|_{\mathbb{Z}}\|.$$

Here the constant K depends only on σ. Denote by $K_{\mathrm{i}}(\mathbb{Z}, B_\sigma)$ the minimal K which can be used.

Theorem B [1]. *Let $\sigma > \pi$. Then*

$$K_{\mathrm{i}}(\mathbb{Z}, B_\sigma) = \frac{2}{\pi} \log \frac{\sigma}{\sigma - \pi}(1 + o(1)), \qquad \sigma \downarrow \pi.$$

In this paper we are interested in estimates of this type in the general case of irregular sampling.

1.2. Sampling and interpolation sets. Given a set $\Lambda \subset \mathbb{R}$, let

$$\|f|_\Lambda\| := \sup_{\lambda \in \Lambda} |f(\lambda)|.$$

Definition 1.1. (i) A set $\Lambda \subset \mathbb{R}$ is called a *sampling set* for B_σ if there is a constant K such that

$$\|f\| \le K\|f|_\Lambda\| \qquad \text{for every function } f \in B_\sigma. \tag{1.2}$$

(ii) A discrete set $\Lambda \subset \mathbb{R}$ is called an *interpolation set* for B_σ if for every datum $\{c(\lambda)\} \in l^\infty(\Lambda)$ there exists $f \in B_\sigma$ such that

$$f(\lambda) = c(\lambda), \qquad \lambda \in \Lambda. \tag{1.3}$$

A set Λ is said to be *uniformly discrete* (u.d.) if

$$\inf_{\lambda, \lambda' \in \Lambda,\ \lambda \ne \lambda'} |\lambda - \lambda'| > 0.$$

The classical Beurling theorem characterizes u.d. sampling and interpolation sets for B_σ in terms of the lower and upper uniform densities of Λ:

$$D^-(\Lambda) := \lim_{l \to \infty} \min_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a + l))}{l}, \qquad D^+(\Lambda) := \lim_{l \to \infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a + l))}{l}.$$

Theorem C [2, 3]. *Let $\Lambda \subset \mathbb{R}$ be a u.d. set.*

(i) Λ *is a sampling set for B_σ if and only if $D^-(\Lambda) > \sigma/\pi$.*

(ii) Λ *is an interpolation set for B_σ if and only if $D^+(\Lambda) < \sigma/\pi$.*

1.3. Sampling and interpolation constants. Assume Λ is a sampling set for B_σ. We will call the minimal constant K for which (1.2) holds the *sampling constant* and denote it by $K_{\mathrm{s}}(\Lambda, B_\sigma)$. In other words,

$$K_{\mathrm{s}}(\Lambda, B_\sigma) := \sup_{f \in B_\sigma,\ f \ne 0} \frac{\|f\|}{\|f|_\Lambda\|}.$$

Now, let Λ be an interpolation set for B_σ. Banach theory implies that there is a constant K such that for every datum $\{c(\lambda)\} \in l^\infty(\Lambda)$, a function $f \in B_\sigma$ satisfying (1.3) can be chosen with the estimate

$$\|f\| \le K \sup_{\lambda\in\Lambda} |c(\lambda)|.$$

The minimal constant K for which this holds is called the *interpolation constant* and denoted by $K_{\mathrm{i}}(\Lambda, B_\sigma)$.

Assume that $D^-(\Lambda) = 1$. Then, by Theorem C(i), Λ is a sampling set for B_σ, $\sigma < \pi$, and it is not a sampling set for B_σ, $\sigma \ge \pi$. Using the compactness property of the Bernstein space (see [7, p. 19]), it is easy to see that the sampling constant $K_{\mathrm{s}}(\Lambda, B_\sigma)$ grows to infinity as σ approaches the critical value π from below.

Similarly, assume that $D^+(\Lambda) = 1$. Then, by Theorem C(ii), Λ is an interpolation set for B_σ, $\sigma > \pi$, and it is not if $\sigma \le \pi$. The compactness property shows that the constant $K_{\mathrm{i}}(\Lambda, B_\sigma)$ grows to infinity as σ approaches π from above.

When $\Lambda = \mathbb{Z}$, Theorems A and B show that the sampling and interpolation constants have precisely logarithmic growth. One may ask how fast these constants must grow in the general case of irregular sampling.

2. RESULTS

Our main result shows that the sampling and interpolation constants always have at least logarithmic growth as σ approaches the critical value.

In what follows we denote by C absolute positive constants.

Theorem 1. *Let $\Lambda \subset \mathbb{R}$ be a u.d. set.*

(i) *If $D^-(\Lambda) = 1$, then*

$$K_{\mathrm{s}}(\Lambda, B_\sigma) \ge C \log \frac{\pi}{\pi - \sigma}, \qquad 0 < \sigma < \pi. \tag{2.1}$$

(ii) *If $D^+(\Lambda) = 1$, then*

$$K_{\mathrm{i}}(\Lambda, B_\sigma) \ge C \log \frac{\sigma}{\sigma - \pi}, \qquad \pi < \sigma < 2\pi. \tag{2.2}$$

Statement (i) of this theorem was announced in [6], with a sketch of proof. Statement (ii) is new. Below we present complete proofs of both results. The main step of the proof is to get sharp estimates of the sampling and interpolation constants for complex polynomials.

Denote by P_n the space of all algebraic polynomials of degree $\le n$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$.

Let $\Lambda \subset \mathbb{T}$ be a finite set such that $\#\Lambda \ge n + 1$. Then every polynomial $P \in P_n$ is uniquely determined by its values on Λ, and one may introduce the corresponding sampling constant

$$K_{\mathrm{s}}(\Lambda, P_n) := \sup_{P\in P_n,\ P\ne 0} \frac{\max_{z\in\mathbb{T}} |P(z)|}{\max_{\lambda\in\Lambda} |P(\lambda)|}.$$

Assume $\Lambda \subset \mathbb{T}$ and $\#\Lambda \le n + 1$. Then for every datum $c(\lambda)$, $\lambda \in \Lambda$, there is a polynomial $P \in P_n$ satisfying

$$P(\lambda) = c(\lambda), \qquad \lambda \in \Lambda. \tag{2.3}$$

One may introduce the corresponding interpolation constant $K_{\mathrm{i}}(\Lambda, P_n)$ as the infimum over all K such that for every datum $c(\lambda)$, $\lambda \in \Lambda$, there exists $P \in P_n$ satisfying (2.3) and

$$\max_{z\in\mathbb{T}} |P(z)| \le K \max_{\lambda\in\Lambda} |c(\lambda)|.$$

The following version of Theorem 1 for complex polynomials holds.

Theorem 2. *Assume* $\Lambda \subset \mathbb{T}$.

(i) *If* $\#\Lambda \ge n+1$, *then*

$$K_{\mathrm{s}}(\Lambda, P_n) \ge C \log \frac{n}{\#\Lambda - n}. \tag{2.4}$$

(ii) *If* $\#\Lambda \le n+1$, *then*

$$K_{\mathrm{i}}(\Lambda, P_n) \ge C \log \frac{n}{n+2-\#\Lambda}. \tag{2.5}$$

3. SAMPLING

In this section we prove Theorems 1(i) and 2(i).

3.1. Sampling constant for polynomials. The following result essentially goes back to Faber (see [4, Ch. 7]):

Let U *be a projector from the space* $C(\mathbb{T})$ *onto the subspace* P_n. *Then* $\|U\| > C\log n$.

Faber's approach is based on averaging over translations. Different versions of the result have been obtained by this approach. We will use the following one due to Al. Privalov [8]:

For every projector U *above and every family of linear functionals* ψ_j $(1 \le j \le m)$ *in* $C^*(\mathbb{T})$, *there is a unit vector* f *in* $C(\mathbb{T})$ *such that* $\|Uf\| > C\log(n/m)$ *and the functionals vanish on* f.

Proof of Theorem 2(i). We may assume that $\#\Lambda > n+1$. Write $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_l\}$, where $l := n+m$ for some $m \in \mathbb{N}$. For every $f \in C(\mathbb{T})$ denote by $P(f)$ the polynomial of degree n satisfying

$$P(f)(\lambda_j) = f(\lambda_j), \qquad 0 \le j \le n.$$

Clearly, $P(f)$ is uniquely defined, and the operator $U\colon f \to P(f)$ is a projector from $C(\mathbb{T})$ onto P_n.

Set

$$\psi_j(f) := P(f)(\lambda_{n+j}), \qquad 1 \le j \le m.$$

By Privalov's theorem, there is a function $f \in C(\mathbb{T})$ satisfying

$$\|f\| = 1, \qquad P(f)(\lambda_j) = 0 \quad (n+1 \le j \le l), \qquad \|P(f)\| > C \log \frac{n}{m}.$$

Then (2.4) follows. $\square$

3.2. Sampling constant for trigonometric polynomials. Here we formulate an analogue of Theorem 2(i) for trigonometric polynomials.

Let $N > 0$ and $m \in \mathbb{N}$. Denote by $T_m(N)$ the space of all trigonometric polynomials of the form

$$T_m(N) := \left\{\varphi\colon\ \varphi(x) = \sum_{j=-m}^{m} a(j) e^{i\pi jx/N}\right\}. \tag{3.1}$$

Obviously, $T_m(N)$ consists of $2N$-periodic functions and $T_m(N) \subset B_{\pi m/N}$. Observe that every $2N$-periodic function $\varphi \in B_{\pi m/N}$ belongs to $T_m(N)$.

Let $\Gamma \subset \mathbb{R}$ be a $2N$-periodic set, $\Gamma + 2N = \Gamma$. Set $\Gamma_0 := \Gamma \cap [-N, N)$.

Assume $\#\Gamma_0 \ge 2m+1$. Then every $\varphi \in T_m(N)$ is uniquely determined by the values on Γ, and we may introduce the sampling constant

$$K_{\mathrm{s}}(\Gamma, T_m(N)) := \sup_{\varphi \in T_m(N),\ \varphi \ne 0} \frac{\|\varphi\|}{\|\varphi|\Gamma\|}.$$

Proposition 3.1. *Assume* $N>0$, $m\in\mathbb{N}$ *and* $\Gamma_0\subset[-N,N)$, $\#\Gamma_0>2m$. *Then*

$$K_{\mathrm{s}}(\Gamma,T_m(N))\ge C\log\frac{2m}{\#\Gamma_0-2m},\qquad \Gamma:=\Gamma_0+2N\mathbb{Z}. \tag{3.2}$$

It easy to check that Proposition 3.1 is equivalent to Theorem 2(i) for even n. Indeed, let $\Lambda\subset\mathbb{T}$. Write

$$\Lambda=\bigl\{e^{i\lambda_1},\dots,e^{i\lambda_l}\bigr\},\qquad -\pi\le\lambda_1<\ldots<\lambda_l<\pi,$$

and set $\Gamma:=\Gamma_0+2N\mathbb{Z}$, where

$$\Gamma_0:=\left\{\frac{N\lambda_1}{\pi},\dots,\frac{N\lambda_l}{\pi}\right\}\subset[-N,N).$$

Clearly, Γ is $2N$-periodic and $\#\Gamma_0=\#\Lambda$.

Let n be an even number. Set $m:=n/2$. Every polynomial $P\in P_n$ can be written as

$$P(z)=z^m\sum_{j=-m}^{m}a(j)z^j.$$

By the change of variable $z=e^{i\pi t/N}$, we see that Theorem 2(i) is equivalent to Proposition 3.1 with $m=n/2$.

On the other hand, one may easily check that Theorem 2(i) for odd n follows from the result for even n.

3.3. Sampling in B_σ. Here we prove Theorem 1(i). The proof is based on Proposition 3.2 below.

Let $0<\sigma<\pi$, τ be a positive integer and $\Gamma_0\subset[-\tau,\tau]$, $\#\Gamma_0\le2\tau$. Consider the union of Γ_0 with the rays $x\ge\tau$ and $x\le-\tau$. It easily follows from Beurling's Theorem C(i) that the set $\Gamma_0\cup\{x\colon |x|\ge\tau\}$ is a sampling set for B_σ, for *every* $\sigma>0$. We show that if the number $\pi-\sigma$ is small and the number τ is large, then the corresponding sampling constant must be large.

Proposition 3.2. *Let* $\tau\in\mathbb{N}$ *and* $0<\sigma<\pi$. *For every set* $\Gamma_0\subset[-\tau,\tau)$, $\#\Gamma_0\le2\tau$, *we have*

$$K_{\mathrm{s}}\bigl(\Gamma_0\cup\{x\colon |x|\ge\tau\},B_\sigma\bigr)\ge C\log\frac{\pi}{\pi-\sigma+C\tau^{-1/3}}. \tag{3.3}$$

When τ is sufficiently large,

$$\tau\ge\frac{1}{\pi-\sigma}, \tag{3.4}$$

inequality (3.3) implies

Corollary 3.1. *Assume* $0<\sigma<\pi$ *and* $\tau\in\mathbb{N}$ *satisfy* (3.4). *Then for every set* $\Gamma_0\subset[-\tau,\tau)$, $\#\Gamma_0\le2\tau$, *we have*

$$K_{\mathrm{s}}\bigl(\Gamma_0\cup\{x\colon |x|\ge\tau\},B_\sigma\bigr)\ge C\log\frac{\pi}{\pi-\sigma}. \tag{3.5}$$

Proof of Proposition 3.2. Set $N:=\tau-\sqrt{\tau}$ and $\Gamma:=\Gamma_0+2N\mathbb{Z}$. To prove the proposition, it suffices to construct a function $g\in B_\sigma$ satisfying the conditions

$$\|g|_{\Gamma_0}\|\le1,\qquad |g(x)|\le1\quad(|x|\ge\tau),\qquad |g(x_0)|=C\log\frac{1}{\pi-\sigma+C\tau^{-1/3}}, \tag{3.6}$$

with some $x_0\in[-N,N)$.

We may assume that $\sigma > \pi/2$ and that the number τ is so large that

$$\frac{\tau}{N} < 1 + 2\tau^{-1/2}$$

and that we can find an integer m satisfying

$$1 \le \sigma - 2\tau^{-1/3} \le \frac{\pi m}{N} \le \sigma - \tau^{-1/3}. \tag{3.7}$$

Then the following inequality holds:

$$\frac{m}{\tau - m} = \frac{\pi m/N}{\pi\tau/N - \pi m/N} \ge \frac{1}{\pi - \sigma + C\tau^{-1/3}}. \tag{3.8}$$

Recall that $\#\Gamma_0 \le 2\tau$. Using Proposition 3.1 and (3.8), we get

$$K_{\mathrm{s}}(\Gamma, T_m(N)) \ge C \log \frac{2m}{2\tau - 2m} \ge C \log \frac{1}{\pi - \sigma + C\tau^{-1/3}}. \tag{3.9}$$

Choose now an exponential polynomial $H \in T_m(N)$ which is small on Γ and has a large norm, and let $x_0 \in [-N, N)$ be its maximum modulus point. By (3.9), we may assume that

$$\|H|_\Gamma\| \le 1, \qquad |H(x_0)| = \|H\| = C \log \frac{1}{\pi - \sigma + C\tau^{-1/3}}. \tag{3.10}$$

It follows that

$$\|H\| \le C \log \tau. \tag{3.11}$$

Set

$$h(x) := \frac{\sin x}{x} \qquad \text{and} \qquad g(x) := H(x) h\big(\tau^{-1/3}(x - x_0)\big). \tag{3.12}$$

By the right inequality in (3.7),

$$g \in B_{\pi m/N + \tau^{-1/3}} \subset B_\sigma.$$

From (3.10) we see that the first and last conditions in (3.6) are true.

The distance from x_0 to the points $\pm\tau$ is at least $\sqrt{\tau}$, which gives

$$\big|h\big(\tau^{-1/3}(x - x_0)\big)\big| \le \frac{1}{\tau^{-1/3}\sqrt{\tau}} = \tau^{-1/6}, \qquad |x| \ge \tau.$$

So, by (3.11),

$$|g(x)| \le \|H\| \big|h\big(\tau^{-1/3}(x - x_0)\big)\big| \le C\tau^{-1/6} \log \tau < 1, \qquad |x| \ge \tau,$$

provided τ is sufficiently large. This completes the proof of (3.6). $\square$

We also need a simple

Lemma 3.1. *Assume that $\Lambda \subset \mathbb{R}$ and $\tau \in \mathbb{N}$.*

(i) *Let $D^-(\Lambda) = 1$. Then there is an interval $I = [a - \tau, a + \tau)$, $a \in \mathbb{R}$, which contains at most 2τ points of Λ.*

(ii) *Let $D^+(\Lambda) = 1$. Then there is an interval $I = [a - \tau, a + \tau)$, $a \in \mathbb{R}$, which contains at least 2τ points of Λ.*

Proof. Indeed, if statement (i) is not true, then every interval $[-\tau, \tau) + 2k\tau$, $k \in \mathbb{Z}$, contains at least $2\tau + 1$ points of Λ. It easily follows that $D^-(\Lambda) \ge (2\tau + 1)/(2\tau) > 1$. A contradiction.

The proof of statement (ii) is similar. $\square$

Statement (ii) of this lemma will be used later in the proof of Theorem 1(ii).

Proof of Theorem 1(i). Assume that $\sigma < \pi$ and $D^-(\Lambda) = 1$. Fix any integer τ satisfying (3.4). By Lemma 3.1(i), there is an interval I of length 2τ containing at most 2τ points of Λ. We may assume that $I = [-\tau, \tau)$. Set $\Gamma_0 := \Lambda \cap [-\tau, \tau)$. Clearly,

$$K_{\mathrm{s}}(\Lambda, B_\sigma) \geq K_{\mathrm{s}}\big(\Gamma_0 \cup \{x\colon |x| \geq \tau\}, B_\sigma\big).$$

The result now follows from (3.5). $\square$

4. INTERPOLATION

In this section we prove statements (ii) of Theorems 1 and 2. The proofs are more technical than those of statements (i).

Let $[a]$ denote the integer part of a positive number a. We say that a u.d. set Λ possesses a uniform density $D(\Lambda)$ if its upper and lower uniform densities coincide, and write $D(\Lambda) = D^-(\Lambda) = D^+(\Lambda)$.

Given $N > M > 1$, introduce two additional constants

$$\rho := \left(\frac{N}{M}\right)^{1/3}, \qquad r := \rho - \sqrt{\rho}. \tag{4.1}$$

We will assume that N and N/M are so large that $\rho > 64$ and

$$\frac{N}{\rho} > 30M\rho. \tag{4.2}$$

Clearly, the following estimates hold:

$$\rho < r + C\sqrt{r}, \qquad \log\frac{N}{M} > C\log r. \tag{4.3}$$

4.1. Auxiliary lemmas. Recall that we denote by C absolute constants.

Lemma 4.1. *Given a set $\Gamma_0' \subset [-\rho, \rho)$, $\#\Gamma_0' \leq 2\rho + 1$, there is a function $g \in B_\pi$ and a point $x_0 \in [-r, r)$ such that*

$$\|g|_{\Gamma_0'}\| \leq 1, \qquad |g(x_0)| \geq C\log r, \qquad |g(x)| \leq \frac{C}{(|x| - r)^3} \quad (|x| \geq \rho). \tag{4.4}$$

Proof. Set

$$\Gamma' := \Gamma_0' + 2r\mathbb{Z}.$$

Then

$$D(\Gamma') \leq \frac{\#\Gamma_0'}{2r} \leq \frac{2\rho + 1}{2r} \leq 1 + \frac{C}{\sqrt{r}}. \tag{4.5}$$

Set

$$w := \pi - 4r^{-1/9}.$$

From (4.5), we see that $\pi - w/D(\Gamma') < Cr^{-1/9}$.

Observe that the set $D(\Gamma')\Gamma' := \{D(\Gamma')\gamma\colon \gamma \in \Gamma'\}$ has density 1. Hence, using Theorem 1(i) with $\sigma = w/D(\Gamma')$, we obtain

$$K_{\mathrm{s}}(\Gamma', B_w) = K_{\mathrm{s}}\big(D(\Gamma')\Gamma', B_{w/D(\Gamma')}\big) \geq C\log\frac{\pi}{\pi - w/D(\Gamma')} \geq C\log r.$$

Using the last inequality, we may find a function $g_1 \in B_w$ and a point x_0 satisfying

$$\|g_1|_{\Gamma'}\| \le 1, \qquad \|g_1\| = C\log r, \qquad |g_1(x_0)| \ge \frac{\|g_1\|}{2}. \tag{4.6}$$

We may assume that $x_0 \in [-r, r)$. Indeed, if $x_0 \in [(2j-1)r, (2j+1)r)$, we consider the function $g_1(x + 2jr)$. Observe that $|g_1(x + 2jr)| \le 1$ for $x \in \Gamma'$, due to the $2r$-periodicity of Γ'.

Finally, set

$$g(x) := g_1(x)h^4\big(r^{-1/9}(x - x_0)\big),$$

where h is defined in (3.12). Clearly, g belongs to $B_{w+4r^{-1/9}} = B_\pi$. By (4.6), it satisfies the first two inequalities in (4.4).

When $|x| \ge \rho$, we have

$$|x - x_0| \ge |x| - r \ge \rho - r \ge \sqrt{r},$$

and so $|x - x_0|^4 \ge \sqrt{r}(|x| - r)^3$. This yields

$$|g(x)| \le \|g_1\|\,\big|h\big(r^{-1/9}(x - x_0)\big)\big|^4 \le C\log r\left(\frac{r^{1/9}}{|x - x_0|}\right)^4 \le \frac{C}{(|x| - r)^3}, \qquad |x| \ge \rho,$$

which completes the proof. $\square$

Lemma 4.2. *Given a set $\Gamma_0 \subset [-N, N)$, $\#\Gamma_0 \le 2N + 1$, there are at least $k := [8M]$ disjoint intervals $I_j \subset (-N, N)$ such that the length of each interval is equal to 2ρ and $\#(\Gamma_0 \cap I_j) \le 2\rho + 1$ for every j.*

Proof. Clearly, the interval $[-N, N)$ contains at least $[N/\rho] - 1$ disjoint intervals of length 2ρ. Assume that at most $k - 1$ of these intervals contain $\le 2\rho + 1$ points of Γ_0. Then there are at least $[N/\rho] - k$ intervals containing $> 2\rho + 1$ points, so that

$$\#\Gamma_0 > (2\rho + 1)\left(\frac{N}{\rho} - 8M - 1\right) = 2N + \frac{N}{\rho} - (8M + 1)(2\rho + 1).$$

By (4.2), this inequality implies $\#\Gamma_0 > 2N + 1$. A contradiction. $\square$

Recall that the space $T_m(N)$ is defined in (3.1).

The next lemma plays a key role in the proof of Theorem 2(ii). We construct a real trigonometric polynomial φ which is small on a given set Γ_0, while φ is large and has prescribed signs on a sufficiently large set Q.

Lemma 4.3. *Assume $\Gamma_0 \subset [-N, N)$ and $\#\Gamma_0 \le 2N + 1$. Then there is a k-point set $Q = \{x_1, \dots, x_k\} \subset [-N, N)$, $k := [8M]$, such that for every function $\psi\colon Q \to \{-1, 1\}$ there is a real function $\varphi \in T_N(N)$ satisfying*

$$\|\varphi|_{\Gamma_0}\| \le 1, \qquad \psi(x_j)\varphi(x_j) > C\log r \quad (j = 1, \dots, k). \tag{4.7}$$

Proof. Let $I_j := [a_j - \rho, a_j + \rho)$, where $-N + \rho \le a_1 < \dots < a_k \le N - \rho$, be the intervals from Lemma 4.2. Set $J_j := [a_j - r, a_j + r)$, $j = 1, \dots, k$.

For every j, $1 \le j \le k$, set $\Gamma_j := \Gamma_0 \cap I_j$. By Lemma 4.1, there is a function $g_j \in B_\pi$ and a point $x_j \in J_j$ such that

$$|g_j(x_j)| \ge C\log r \tag{4.8}$$

and

$$\|g_j|_{\Gamma_j}\| \le 1, \qquad |g_j(x)| \le \frac{C}{(|x - a_j| - r)^3} \quad (|x - a_j| \ge \rho). \tag{4.9}$$

It is easy to check that we may assume that g_j is real on $\mathbb{R}$.

Fix any function $\psi\colon \{x_1,\dots,x_k\}\to\{-1,1\}$. Since both g_j and $-g_j$ satisfy the conditions above, we may assume that $\psi(x_j)g_j(x_j)>0$, $j=1,\dots,k$.

Set

$$g(x):=\sum_{j=1}^{k} g_j(x).$$

Then g is real and $g\in B_\pi$.

Recall that $\rho-r>\sqrt{r}$. In what follows we will use the simple inequality

$$|x-a_j|-r>\sqrt{r}|l-j| \qquad \text{for every } \ x\in J_l, \quad l\neq j.$$

Using this and (4.9), one may easily get the estimate

$$\|g|_{\Gamma_0}\|\le C. \tag{4.10}$$

Further, from (4.8) and (4.9), we obtain

$$\psi(x_l)g(x_l)\ge |g_l(x_l)|-\sum_{j\neq l}|g_j(x_l)|>C\log r-\sum_{j\neq l}\frac{C}{r^{3/2}|j-l|^3}>C\log r, \qquad l=1,\dots,k. \tag{4.11}$$

Now, assume $x\le -N$. Then, $|x-a_j|-r\ge |x+N|+j\sqrt{r}$, and by (4.9),

$$|g(x)|\le\sum_{j=1}^{k}\frac{C}{(|x+N|+j\sqrt{r})^3}\le\frac{C}{(|x+N|+\sqrt{r})^2}.$$

A similar estimate hods for $x\ge N$, which gives

$$|g(x)|\le\frac{C}{(|x|+\sqrt{r}-N)^2}, \qquad |x|\ge N. \tag{4.12}$$

Set

$$f(x):=\sum_{j\in\mathbb{Z}} g(x+2Nj).$$

From (4.12) it follows that this series converges uniformly on $\mathbb{R}$. By (1.1), the convergence is also uniform on compact subsets of the complex plane. It follows that $f\in B_\pi$. Since f is $2N$-periodic, we conclude that $f\in T_N(N)$.

It follows from (4.10) and (4.12) that $\|f|_{\Gamma_0}\|<C$, while from (4.11) and (4.12) one can deduce that $\psi(x_j)f(x_j)>C\log r$, $j=1,\dots,k$. Finally, let $\varphi(x):=f(x)/C$, where $C:=\|f|_{\Gamma_0}\|$. Then the conclusions of the lemma hold. $\square$

4.2. Interpolation in $T_m(N)$ and P_n. Assume $\Gamma_0\subset[-N,N)$ and $\#\Gamma_0\le 2m+1$. For every $2N$-periodic datum

$$\bigl\{c(\gamma)\colon\ \gamma\in\Gamma=\Gamma_0+2N\mathbb{Z}\bigr\},$$

there is a trigonometric polynomial $H\in T_m(N)$ satisfying

$$H(\gamma)=c(\gamma), \quad \gamma\in\Gamma, \qquad \text{and} \qquad \|H\|\le K\|H|_\Gamma\|,$$

where K depends only on Γ_0. Denote by $K_{\mathrm{i}}(\Gamma,T_m(N))$ the corresponding interpolation constant, i.e., the smallest possible value of K.

Proposition 4.1. *Let $m \in \mathbb{N}$ and let $\Gamma_0 \subset [-N, N)$ satisfy $\#\Gamma_0 < 2m$. Then*

$$K_{\mathrm{i}}(\Gamma, T_m(N)) \geq C \log \frac{2m}{2m - \#\Gamma_0}, \qquad \Gamma = \Gamma_0 + 2N\mathbb{Z}. \tag{4.13}$$

Proof. Before proceeding with the proof, let us remark that we may make several assumptions. It is easy to see that we may assume that $\#\Gamma_0$ is an even number. After rescaling, we may assume that $N = \#\Gamma_0/2$, so that the assumptions of Lemma 4.3 are fulfilled. Set $M := m - N$. Then

$$\frac{2m}{2m - \#\Gamma_0} = \frac{m}{m - N} = \frac{m}{M} = 1 + \frac{N}{M}.$$

Since estimate (4.13) makes sense when the ratio $2m/(2m - \#\Gamma_0)$ is a large number, we may also assume that both N and m/M are so large that estimates (4.2) and (4.3) hold.

By (4.1) and (4.3), it is easy to check that $\log(N/M)$ lies between two constants times $\log r$. Hence, to prove Proposition 4.1, it suffices to prove the inequality

$$K_{\mathrm{i}}(\Gamma, T_m(N)) > C \log r. \tag{4.14}$$

Let $-N < x_1 < \ldots < x_k < N$ be the points from Lemma 4.3. Choose a function $\psi\colon \{x_1, \ldots, x_k\} \to \{-1, 1\}$ so that $\psi(x_j)\psi(x_{j+1}) = 1$ when the number $\#\Gamma \cap (x_j, x_{j+1})$ is odd and $\psi(x_j)\psi(x_{j+1}) = -1$ when $\#\Gamma \cap (x_j, x_{j+1})$ is even. Let $\varphi \in T_N(N)$ be the function from Lemma 4.3. Then $\|\varphi|_\Gamma\| \leq 1$.

Choose a datum $c(\gamma)$ as follows:

$$c(\gamma) := \varphi(\gamma), \qquad \gamma \in \Gamma.$$

To prove (4.14), we show that every trigonometric polynomial $H \in T_m(N)$ satisfying

$$H(\gamma) = c(\gamma), \qquad \gamma \in \Gamma,$$

must be large, $\|H\| \geq C \log r$. By (4.7), it suffices to show that

$$|H(x_j)| \geq |\varphi(x_j)| \qquad \text{for some } j, \quad 1 \leq j \leq k. \tag{4.15}$$

Since $H - \varphi$ vanishes on Γ, the following equality holds with some entire function G:

$$H(x) = \varphi(x) + G(x)\Phi(x), \qquad \Phi(x) := \prod_{\gamma \in \Gamma \cap [-N, N)} \sin\Bigl(\pi \frac{x - \gamma}{2N}\Bigr).$$

It is easy to check that $\Phi \in T_N(N)$, and so $G \in T_{m-N}(N) = T_M(N)$. Then G has at most $2M$ zeros on $[-N, N)$. We may assume that G is real (otherwise, we consider $\operatorname{Re} G$).

Assume (4.15) is not true, i.e., $|H(x_j)| < |\varphi(x_j)|$ for every j. Then, by the construction of function ψ, for every j, $1 \leq j < k$,

$$\operatorname{sign}(G(x_j)G(x_{j+1})) = \operatorname{sign}\biggl(\frac{\varphi(x_j)\varphi(x_{j+1})}{\Phi(x_j)\Phi(x_{j+1})}\biggr) = \frac{\psi(x_j)\psi(x_{j+1})}{\operatorname{sign}(\Phi(x_j)\Phi(x_{j+1}))} = -1,$$

where $\operatorname{sign}(G(x))$ denotes the sign of $G(x)$. Hence, G has at least $k - 1$ zeros on $[-N, N)$. This is a contradiction, since $k = [8M]$. $\square$

We have seen that Proposition 3.1 is equivalent to Theorem 2(i). In a similar fashion, one can check that Proposition 4.1 is equivalent to Theorem 2(ii). Therefore, Theorem 2(ii) is proved.

4.3. Interpolation in B_σ. Here we prove Theorem 1(ii). We will deduce Theorem 1(ii) from Proposition 4.2 below, which is an analogue of Proposition 3.2 for sampling.

It is clear that every finite set $\Gamma_0 \subset \mathbb{R}$ is an interpolation set for every space B_σ, $\sigma > 0$. Assume $\Gamma_0 \subset [-\tau, \tau)$ has $\geq 2\tau$ points. We show that if τ is large and $\sigma - \pi$ is small, then the interpolation constant $K_{\mathrm{i}}(\Gamma_0, B_\sigma)$ must be large.

Proposition 4.2. *Let $\tau \in \mathbb{N}$. For every set $\Gamma_0 \subset [-\tau, \tau)$, $\#\Gamma_0 \geq 2\tau$, we have*

$$K_{\mathrm{i}}(\Gamma_0, B_\sigma) \geq C \log \frac{\sigma}{\sigma - \pi + C\tau^{-1/3}}, \qquad \sigma > \pi.$$

This proposition implies

Corollary 4.1. *Assume $\sigma > \pi$ and $\tau \in \mathbb{N}$ satisfy*

$$\tau \geq \frac{1}{\sigma - \pi}.$$

For every $\Gamma_0 \subset [-\tau, \tau)$, $\#\Gamma_0 \geq 2\tau$, we have

$$K_{\mathrm{i}}(\Gamma_0, B_\sigma) \geq C \log \frac{\sigma}{\sigma - \pi}.$$

We will need two auxiliary results.

Lemma 4.4. *Assume $\Lambda \subset \mathbb{T}$ and $\#\Lambda \leq n + 1$. Assume a datum $c(\lambda)$, $\max_{\lambda \in \Lambda} |c(\lambda)| = 1$, is such that*

$$\max_{z \in \mathbb{T}} |P(z)| > \frac{K_{\mathrm{i}}(\Lambda, P_n)}{2} \tag{4.16}$$

for every polynomial $P \in P_n$ satisfying (2.3). *If a polynomial $Q \in P_n$ satisfies*

$$\max_{\lambda \in \Lambda} |Q(\lambda) - c(\lambda)| < \frac{1}{8},$$

then

$$\max_{z \in \mathbb{T}} |Q(z)| > \frac{K_{\mathrm{i}}(\Lambda, P_n)}{4}.$$

Proof. Indeed, there is a polynomial $H \in P_n$ such that

$$H(\lambda) = c(\lambda) - Q(\lambda), \qquad \lambda \in \Lambda,$$

and

$$\max_{z \in \mathbb{T}} |H(z)| < 2K_{\mathrm{i}}(\Lambda, P_n) \max_{\lambda \in \Lambda} |c(\lambda) - Q(\lambda)| < \frac{K_{\mathrm{i}}(\Lambda, P_n)}{4}.$$

Since the polynomial $P := H + Q$ satisfies (2.3), it also satisfies (4.16). The statement of the lemma easily follows. □

Lemma 4.5. *For every $\tau \in \mathbb{N}$ there is a function $R \in B_{\tau^{-1/6}}$ and a constant $c > 1$ such that*

$$\|R\| < c, \qquad |R(x)| > \frac{1}{c}, \quad |x| \leq \tau, \tag{4.17}$$

$$|R(x + iy)| \leq C \frac{e^{\tau^{-1/3}|y|}}{(|x| - \tau)^{3/2}}, \qquad |x| \geq N, \quad y \in \mathbb{R}, \tag{4.18}$$

and

$$\sum_{j \in \mathbb{Z},\, j \neq 0} |R(x + 2Nj)| \leq C\tau^{-2/3}, \qquad |x| \leq N, \tag{4.19}$$

where $N := \tau + \tau^{2/3}$.

Proof. Set $\alpha := \tau^{-1/3}/4$ and denote by R the convolution between the characteristic function of $(-\tau,\tau)$ and the function $\alpha h^4(\alpha x)$, where h is defined in (3.12); i.e., for every complex number $z = x + iy$,

$$R(z) := \alpha \int_{-\tau}^{\tau} \left(\frac{\sin(\alpha(z-t))}{\alpha(z-t)}\right)^4 dt = \int_{\alpha(x-\tau)}^{\alpha(x+\tau)} \left(\frac{\sin(u+i\alpha y)}{u+i\alpha y}\right)^4 du. \tag{4.20}$$

Clearly, $R \in B_{4\alpha} = B_{\tau^{-1/3}}$. It is straightforward to check that this function satisfies (4.17).

Assume $x \le -\tau - \tau^{2/3}$. Since $\alpha(x+\tau) = -\alpha(|x|-\tau)$, we have

$$|R(x+iy)| < \int_{-\infty}^{-\alpha(|x|-\tau)} \frac{|\sin(i\alpha y)|^4}{u^4}\, du < \frac{e^{4\alpha|y|}}{(\alpha(|x|-\tau))^3} < C\frac{e^{\tau^{-1/3}|y|}}{(|x|-\tau)^{3/2}}.$$

A similar estimate holds for $x \ge \tau + \tau^{2/3}$, which proves (4.18).

Finally, observe that for every $|x| \le N$ and $j \in \mathbb{Z}$, $j \ne 0$, we have

$$|x + 2Nj| - \tau \ge (2|j|-1)N - \tau \ge (|j|-1)N + \tau^{2/3}.$$

This estimate and (4.18) imply (4.19). $\square$

Proof of Proposition 4.2. Set $N := \tau + \tau^{2/3}$ and $\Gamma := \Gamma_0 + 2N\mathbb{Z}$.

We may assume that τ is so large that the conclusions of Lemma 4.5 hold, and that there is an integer m satisfying

$$\sigma + \tau^{-1/3} \le \frac{\pi m}{N} \le \sigma + 2\tau^{-1/3}. \tag{4.21}$$

Then

$$\frac{m}{m-\tau} = \frac{\pi m/N}{\pi m/N - \pi\tau/N} \ge \frac{1}{\sigma - \pi + C\tau^{-1/3}}.$$

This estimate and (4.13) imply

$$K_{\mathrm{i}}(\Gamma, T_m(N)) \ge C \log\frac{2m}{2m-2\tau} \ge C\log\frac{\sigma}{\sigma-\pi+C\tau^{-1/3}}. \tag{4.22}$$

Fix a $2N$-periodic datum $c(\gamma)$ such that $\max_{\gamma\in\Gamma_0}|c(\gamma)| = 1$ and

$$\|H\| > \frac{K_{\mathrm{i}}(\Gamma, T_m(N))}{2}$$

for every $H \in T_m(N)$ satisfying $H(\gamma) = c(\gamma)$ for $\gamma \in \Gamma_0$.

Take any function $f \in B_\sigma$ satisfying

$$f(\gamma) = \frac{c(\gamma)}{R(\gamma)}, \qquad \gamma \in \Gamma_0,$$

where R is defined in (4.20). By the second inequality in (4.17),

$$\max_{\gamma\in\Gamma_0}\left|\frac{c(\gamma)}{R(\gamma)}\right| \le C.$$

Hence, to prove the proposition, it suffices to show that

$$\|f\| \ge C\log\frac{\sigma}{\sigma-\pi+C\tau^{-1/3}}. \tag{4.23}$$

Set

$$f_R(x) := \sum_{j\in\mathbb{Z}} (fR)(x + 2Nj).$$

Using (1.1) and (4.18), we see that the series above converges uniformly on compact subsets of the complex plane, so that $f_R \in B_{\sigma+\tau^{-1/3}}$. Since f_R is $2N$-periodic, (4.21) shows that $f_R \in T_m(N)$. One may also deduce from (4.18) that

$$\|f\| \ge C\|f_R\|. \tag{4.24}$$

Let $\gamma \in \Gamma_0$. Since $\Gamma_0 \subset [-\tau, \tau)$, from (4.19) we get the estimate

$$|f_R(\gamma) - c(\gamma)| = \left|\sum_{j\in\mathbb{Z},\ j\ne 0} (fR)(\gamma + 2Nj)\right| \le C\|f\|\tau^{-2/3}. \tag{4.25}$$

Assume f satisfies $\|f\| \ge (1/4C)\tau^{2/3}$, where C is the constant in (4.25). Then, clearly, it also satisfies (4.23).

If $\|f\| < (1/4C)\tau^{2/3}$, then (4.25) implies

$$\max_{\gamma\in\Gamma_0} |f_R(\gamma) - c(\gamma)| \le \frac{1}{4}.$$

By Lemma 4.4,

$$\|f_R\| \ge \frac{K_{\mathrm{i}}(\Gamma, T_m(N))}{4}.$$

This, (4.22) and (4.24) imply (4.23). □

Proof of Theorem 1(ii). Set

$$\tau := 1 + \left[\frac{1}{\sigma - \pi}\right].$$

By Lemma 3.1(ii), we may assume that $[-\tau, \tau)$ contains at least 2τ points of Λ. Then estimate (2.2) is an immediate consequence of Corollary 4.1. □

5. REMARKS

5.1. Good sampling and interpolation. The size of the sampling constant is important in the theory of reconstruction of signals from their sampled values. Loosely speaking, the sampling provided by a set Λ is "good" if the corresponding sampling constant is not "large." The following question arises: Given a constant $K > 0$, how dense should a u.d. set Λ be to satisfy the inequality $K_{\mathrm{s}}(\Lambda, B_\sigma) \le K$?

Similarly, one may ask how sparse a set Λ should be so that $K_{\mathrm{i}}(\Lambda, B_\sigma) \le K$.

The following result gives necessary density conditions for "good" sampling and interpolation.

Corollary 5.1. *Suppose given constants $\sigma > 0$, $K > 1$ and a u.d. set $\Lambda \subset \mathbb{R}$.*

(i) *If $K_{\mathrm{s}}(\Lambda, B_\sigma) \le K$, then*

$$D^-(\Lambda) \ge \frac{\sigma}{\pi}(1 + e^{-CK}).$$

(ii) *If $K_{\mathrm{i}}(\Lambda, B_\sigma) \le K$, then*

$$D^+(\Lambda) \le \frac{\sigma}{\pi}(1 - e^{-CK}).$$

Proof. This result follows easily from Theorem 1: Set $a = D^-(\Lambda)$. Then $D^-(a\Lambda) = 1$. Since $K_s(\Lambda, B_\sigma) = K_s(a\Lambda, B_{\sigma/a})$, statement (i) easily follows from (2.1). Statement (ii) follows from (2.2) in a similar fashion. □

Observe that Beurling's Theorem C is an easy consequence of Corollary 5.1.

On the other hand, no sufficient conditions for "good" sampling and interpolation may be given in terms of the uniform densities of the sampling set Λ (see Theorem 3 below).

5.2. Growth of sampling and interpolation constants. As Theorems A and B prove, the asymptotic estimates (2.1) and (2.2) in Theorem 1 are sharp.

In fact, the sampling and interpolation constants may have arbitrarily fast growth.

Theorem 3. *Suppose given a function $\omega(\sigma) \uparrow \infty$, $\sigma \downarrow 0$.*

(i) *There exists a u.d. set Λ, $D^-(\Lambda) = 1$, such that*

$$K_s(\Lambda, B_\sigma) \ge \omega(\pi - \sigma), \qquad \sigma < \pi.$$

(ii) *There exists a u.d. set Λ, $D^+(\Lambda) = 1$, such that*

$$K_i(\Lambda, B_\sigma) \ge \omega(\sigma - \pi), \qquad \pi < \sigma < 2\pi.$$

Proof. Let us prove statement (ii). Fix a decreasing sequence σ_j such that $\sigma_1 = 2\pi$ and σ_j approaches π so fast that

$$\log \frac{1}{\sigma_{j+1} - \pi} > C\omega(\sigma_j - \pi), \qquad j \in \mathbb{N}, \tag{5.1}$$

where C is a large constant to be chosen later.

Set $a_j := \sigma_{j+2}/\sigma_j$. Clearly, $a_j \to 1$ as $j \to \infty$.

Let $k(j) \in \mathbb{N}$ be so large that

$$k(j) \ge \frac{1}{\sigma_{j+2} - \pi}.$$

Set

$$Z_k := \{-k, -k+1, \dots, k-1, k\}, \qquad \Lambda_j := \frac{1}{a_j} Z_{k(j)}.$$

Since $\sigma_j a_j = \sigma_{j+2}$, using Corollary 4.1 we obtain

$$K_i(\Lambda_j, B_{\sigma_j}) = K_i(Z_{k(j)}, B_{\sigma_{j+2}}) \ge C \log \frac{\sigma_{j+2}}{\sigma_{j+2} - \pi}.$$

We may assume that C in (5.1) is so large that we get

$$K_i(\Lambda_j, B_{\sigma_j}) \ge \omega(\sigma_{j+1} - \pi).$$

Now, set

$$\Lambda := \bigcup_{j=1}^{\infty} (\Lambda_j + N_j),$$

where $N_j \uparrow \infty$ grow so fast that $D^+(\Lambda) = 1$.

Take any $\sigma \in (\pi, 2\pi)$, and find j such that $\sigma_j \ge \sigma \ge \sigma_{j+1}$. Then we get

$$K_i(\Lambda, B_\sigma) \ge K_i(\Lambda, B_{\sigma_j}) \ge K_i(\Lambda_j + N_j, B_{\sigma_j}) = K_i(\Lambda_j, B_{\sigma_j}) \ge \omega(\sigma_{j+1} - \pi) \ge \omega(\sigma - \pi).$$

This proves statement (ii) of Theorem 3.

Statement (i) of Theorem 3 can be deduced from Corollary 3.1 in a similar fashion. □

ACKNOWLEDGMENTS

The research of the first author is supported in part by the Israel Science Foundation.

REFERENCES

1. S. N. Bernstein, “The extension of properties of trigonometric polynomials to entire functions of finite degree,” Izv. Akad. Nauk SSSR, Ser. Mat. **12** (5), 421–444 (1948).
2. A. Beurling, “Balayage of Fourier–Stieltjes transforms,” in *The Collected Works of Arne Beurling*, Vol. 2: *Harmonic Analysis* (Birkhäuser, Boston, 1989), Ch. IV, pp. 341–350.
3. A. Beurling, “Interpolation for an interval in $\mathbb{R}$,” in *The Collected Works of Arne Beurling*, Vol. 2: *Harmonic Analysis* (Birkhäuser, Boston, 1989), Ch. V, pp. 351–365.
4. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* (Nauka, Moscow, 1977; Pergamon Press, Oxford, 1982).
5. H. C. Liu and A. J. Macintyre, “Cartwright’s theorem on functions bounded at the integers,” Proc. Am. Math. Soc. **12**, 460–462 (1961).
6. A. Olevskii and A. Ulanovskii, “On irregular sampling in Bernstein spaces,” C. R., Math., Acad. Sci. Paris **353** (1), 47–50 (2015).
7. A. M. Olevskii and A. Ulanovskii, *Functions with Disconnected Spectrum: Sampling, Interpolation, Translates* (Am. Math. Soc., Providence, RI, 2016), Univ. Lect. Ser. **65**.
8. Al. A. Privalov, “Growth of the degrees of polynomial bases and approximation of trigonometric projectors,” Math. Notes **42** (2), 619–623 (1987) [transl. from Mat. Zametki **42** (2), 207–214 (1987)].