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Turán–Erőd Type Converse Markov Inequalities on General Convex Domains of the Plane in the Boundary L^q Norm

Polina Yu. Glazyrina [a,b] **and Szilárd Gy. Révész** [c]

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Dedicated to Sergei V. Konyagin
on the occasion of his 60th birthday

Abstract—In 1939 P. Turán started to derive lower estimations on the norm of the derivatives of polynomials of (maximum) norm 1 on $\mathbb{I} := [-1, 1]$ (interval) and $\mathbb{D} := \{z \in \mathbb{C} \colon |z| \le 1\}$ (disk) under the normalization condition that the zeroes of the polynomial in question all lie in $\mathbb{I}$ or $\mathbb{D}$, respectively. For the maximum norm he found that with $n := \deg p$ tending to infinity, the precise growth order of the minimal possible derivative norm is $\sqrt{n}$ for $\mathbb{I}$ and n for $\mathbb{D}$. J. Erőd continued the work of Turán considering other domains. Finally, about a decade ago the growth of the minimal possible ∞-norm of the derivative was proved to be of order n for all compact convex domains. Although Turán himself gave comments about the above oscillation question in L^q norms, till recently results were known only for $\mathbb{D}$ and $\mathbb{I}$. Recently, we have found order n lower estimations for several general classes of compact convex domains, and conjectured that even for arbitrary convex domains the growth order of this quantity should be n. Now we prove that in L^q norm the oscillation order is at least $n/\log n$ for all compact convex domains.

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1. INTRODUCTION

Denote by $K \Subset \mathbb{C}$ a compact subset of the complex plane, with the most notable particular cases being the unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| \le 1\}$ and the unit interval $\mathbb{I} := [-1, 1]$.

As a kind of converse to the classical inequalities of Bernstein [5, 6, 27] and Markov [19] on the upper estimation of the norm of the derivative of polynomials, in 1939 Paul Turán [29] started to study inequalities of the form $\|p'\|_K \ge c_K n^A \|p\|_K$. Clearly such a converse can only hold if

[a] Institute of Natural Sciences and Mathematics, Ural Federal University named after the First President of Russia B. N. Yeltsin, ul. Kuibysheva 48, Yekaterinburg, 620026 Russia.

[b] N. N. Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620990 Russia.

[c] Alfred Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13–15, Budapest, 1053 Hungary.

E-mail addresses: polina.glazyrina@urfu.ru (P.Yu. Glazyrina), revesz.szilard@renyi.mta.hu (Sz.Gy. Révész).

further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n by $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n(K)$.

Denote by Γ the boundary of K. The (normalized) quantity under our study in the present paper is the "inverse Markov factor" or "oscillation factor"

$$M_{n,q} := M_{n,q}(K) := \inf_{p \in \mathcal{P}_n(K)} M_q(p) \qquad \text{with} \quad M_q(p) := \frac{\|p'\|_{L^q(\Gamma)}}{\|p\|_{L^q(\Gamma)}}, \tag{1.1}$$

where, as usual,

$$\begin{gathered} \|p\|_q := \|p\|_{L^q(\Gamma)} := \left(\int\limits_{\Gamma} |p(z)|^q \, |dz| \right)^{1/q}, \qquad 0 < q < \infty, \\ \|p\|_K := \|p\|_\infty := \|p\|_{L^\infty(\Gamma)} = \|p\|_{L^\infty(K)} = \sup_{z \in \Gamma} |p(z)| = \sup_{z \in K} |p(z)|. \end{gathered} \tag{1.2}$$

Note that for $0 < q < \infty$ the $L^q(\Gamma)$ norm remains finite if Γ is a rectifiable curve.

Theorem A (Turán)**.** *If $p \in \mathcal{P}_n(\mathbb{D})$, then we have*

$$\|p'\|_{\mathbb{D}} \ge \frac{n}{2} \|p\|_{\mathbb{D}}. \tag{1.3}$$

If $p \in \mathcal{P}_n(\mathbb{I})$, then we have

$$\|p'\|_{\mathbb{I}} \ge \frac{\sqrt{n}}{6} \|p\|_{\mathbb{I}}. \tag{1.4}$$

Inequality (1.3) of Theorem A is best possible. Regarding (1.4), Turán pointed out by example of $(1 - x^2)^n$ that the $\sqrt{n}$ order cannot be improved upon, even if the constant is not sharp (see also [4, 18]). The precise value of the constants and the extremal polynomials were computed for all fixed n by Erőd in [13].

We are discussing Turán-type inequalities (1.1) for general convex sets, so some geometric parameters of the compact convex domain K are involved naturally. We write $d := d_K := \operatorname{diam}(K)$ for the *diameter* of K, and $w := w_K := \operatorname{width}(K)$ for the *minimal width* of K. That is,

$$d := d_K := \max_{z', z'' \in K} |z' - z''|, \qquad w := w_K := \min_{\gamma \in [-\pi, \pi]} \Big(\max_{z \in K} \operatorname{Re}(ze^{i\gamma}) - \min_{z \in K} \operatorname{Re}(ze^{i\gamma}) \Big). \tag{1.5}$$

Note that a compact convex *domain* is a closed bounded convex set $K \subset \mathbb{C}$ *with nonempty interior*; hence $0 < w_K \le d_K < \infty$.

The key to (1.3) is the following straightforward observation.

Lemma B (Turán)**.** *Assume that $z \in \partial K$ and there exists a disk $D_R = \{\zeta \in \mathbb{C} \colon |\zeta - z_0| \le R\}$ of radius R such that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$|p'(z)| \ge \frac{n}{2R} |p(z)|. \tag{1.6}$$

For the easy and direct proof see any of the references [29, 18, 25, 26, 14]. Levenberg and Poletsky [18] found it worthwhile to formally define the crucial property of convex sets used here.

Definition 1 (Levenberg–Poletsky)**.** A set $K \Subset \mathbb{C}$ is called *R-circular* if for any $z \in \partial K$ there exists a disk D_R of radius R such that $z \in \partial D_R$ and $D_R \supset K$.

Thus for any R-circular K and $p \in \mathcal{P}_n(K)$, at the boundary point $z \in \partial K$ we can draw the disk D_R and get (1.6) to hold for $p \in \mathcal{P}_n(K)$ and $z \in \partial K$.

Erőd continued the work of Turán already in the same year, investigating the inverse Markov factors of domains with some favorable geometric properties. The most general domains with $M_{n,\infty}(K) \gg n$ found by Erőd were described in [13, Theorem IV].

Theorem C (Erőd)**.** *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles less than π, with all the arcs being C^2-smooth and being either straight line segments of length less than $\Delta(K)$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from zero by a fixed constant $\kappa > 0$. Then there is a constant $c(K)$ such that $M_{n,\infty}(K) \geq c(K)n$ for all $n \in \mathbb{N}$.*

As discussed in [14], this result covers the case of regular k-gons for $k \geq 7$, but not the square, which was also proved to have order n oscillation but only much later, by Erdélyi [12].

A lower estimate of the inverse Markov factor for all compact convex sets (of the same order $\sqrt{n}$ as was known for the interval) was obtained in full generality by Levenberg and Poletsky (see [18, Theorem 3.2]).

Since $\sqrt{n}$ was already known to be the right order of growth for the inverse Markov factor of $\mathbb{I}$, it remained to clarify the right order of oscillation for compact convex *domains* with nonempty interior. This was solved about a decade ago in [24].

Theorem D (Halász–Révész)**.** *Let $K \subset \mathbb{C}$ be any compact convex domain. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\|_K \geq 3 \cdot 10^{-4} \frac{w_K}{d_K^2} n \|p\|_K. \tag{1.7}$$

For the fact that it is indeed the precise order—moreover, $M_{n,\infty}(K)$ can only be within an absolute constant multiple of the above lower estimation—see [25, 14, 26].

There are many papers dealing with the L^q-versions of Turán's inequality for the disk $\mathbb{D}$, the interval $\mathbb{I}$, or the period (one-dimensional torus or circle) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (here with considering only real trigonometric polynomials). A nice review of the results obtained before 1994 is given in [20, Subsects. 6.2.6, 6.3.1].

Already Turán himself mentioned in [29] that on the perimeter of the disk $\mathbb{D}$ actually the pointwise inequality (1.6) holds *at all points* of $\partial\mathbb{D}$. As a corollary, for any $q > 0$,

$$\left(\int\limits_{|z|=1} |p'(z)|^q \, |dz| \right)^{1/q} \geq \frac{n}{2} \left(\int\limits_{|z|=1} |p(z)|^q \, |dz| \right)^{1/q}.$$

Consequently, Turán's result (1.3) extends to all weighted L^q norms on the perimeter, including all $L^q(\partial\mathbb{D})$ norms.

The estimation of the boundary L^q norm, or any weighted L^q norm on the boundary, goes the same way if we have a pointwise estimation for all (or for linearly almost all) boundary points. This observation was explicitly utilized first in [18].

In case we discuss maximum norms, one can assume that $|p(z)|$ is maximal, and it suffices to obtain a lower estimation of $|p'(z)|$ only at such a special point; for general norms, however, this is not sufficient. The above results work only if we have a pointwise inequality of the same strength *everywhere*, or almost everywhere. The situation becomes considerably more difficult when such a statement cannot be proved. For example, if the domain in question is not strictly convex, i.e., if there is a line segment on the boundary, then the zeroes of the polynomial can be arranged so that even some zeroes of the derivative lie on the boundary, and at such points $p'(z)$—and even $p'(z)/p(z)$—can vanish. As a result, at such points no fixed lower estimation can be guaranteed, and lacking a uniformly valid pointwise comparison of p' and p, one cannot draw a direct conclusion either.

This explains why already the case of the interval $\mathbb{I}$ proved to be much more complicated for L^q norms. In a series of papers [37–41] Zhou proved the inequality

$$\left(\int_{-1}^{1}\left|p^{(k)}(x)\right|^{p}dx\right)^{1/p} \geq C_{p,q}^{(k)}(n)\left(\int_{-1}^{1}|p(x)|^{q}\,dx\right)^{1/q}$$

for $k=1$ and $0<p\leq q\leq\infty$, $1-1/p+1/q\geq 0$, with $C_{p,q}^{(1)}(n)=c_{p,q}(\sqrt{n}\,)^{1-1/p+1/q}$. The best constants $C_{p,q}^{(k)}(n)$ were found by Babenko and Pichugov [3] for $p=q=\infty$ and $k=2$, by Bojanov [7] for $1\leq p\leq\infty$, $q=\infty$, and $1\leq k\leq n$, and by Varma [34] for $p=q=2$ and $k=1$.

For more discussions on these results, as well as related results on the interval, period, and circle, see the detailed survey in [14] and the introduction of [15], as well as the original works of Babenko and Pichugov [4], Bojanov [8], and Tyrygin [30, 31] (see also [32, 33, 35, 17]).

The classical inequalities of Bernstein and Markov are generalized for various differential operators (see [2]). In this context, Turán-type converses have also been already investigated, for example, by Akopyan [1] and Dewan et al. [11].

Involving the Blaschke rolling ball theorem, and even recent extensions of it, we proved that certain classes of domains admit order n oscillation factors in L^q (see [15, Theorem 2]). More importantly, however, combining these R-circular classes and the most general classes considered by Erőd in Theorem C (for $\|\cdot\|_\infty$), we could obtain the next result (see [15, Theorem 1]).

Theorem E (Glazyrina–Révész). *Let $K\Subset\mathbb{C}$ be an $E(d,\Delta,\kappa,\xi,\delta)$-domain. Then for any $q\geq 1$ there exists a constant $c=c_K$ (depending explicitly on the parameters q, d, Δ, κ, ξ, and δ) such that for all $n\in\mathbb{N}$ and $p\in\mathcal{P}_n(K)$ we have $\|p'\|_q\geq c_K n\|p\|_q$.*

Here the definition of a "generalized Erőd-type domain" $E(d,\Delta,\kappa,\xi,\delta)$ is basically the one used in Theorem C, but with skipping the assumption of C^2 smoothness and relaxing the $\ddot{\gamma}\geq\kappa$ everywhere assumptions on the curved pieces of the boundary: here $\ddot{\gamma}\geq\kappa$ is assumed only (linearly) almost everywhere.

Further discussion of this definition would lead us aside from our main line of progress, so we refer the reader for more details and explanations (as well as for the proof) to the original paper [15].

Recently, we obtained some order n oscillation results for certain further convex domains without any condition on the curvature. To formulate this, let us first recall another geometric notion, namely, the *depth* of a convex domain K:

$$h_K:=\sup\bigl\{h\geq 0\colon\ \forall\zeta\in\partial K\ \exists\text{ a normal line }\ell\text{ at }\zeta\text{ to }K\text{ with }|\ell\cap K|\geq h\bigr\}. \tag{1.8}$$

We say that a convex domain K has *fixed depth* or *positive depth* if $h_K>0$. The class of convex domains having positive depth contains all smooth compact convex domains, and also all polygonal domains with no vertex with an acute angle. However, observe that the regular triangle has $h_K=0$, as well as any polygon having some acute angle. For more about this class see [14], where the following was also proved.

Theorem F (Glazyrina–Révész). *Assume that $K\Subset\mathbb{C}$ is a compact convex domain having positive depth $h_K>0$. Then for any $q\geq 1$, $n\in\mathbb{N}$, and $p\in\mathcal{P}_n(K)$ it holds that*

$$\|p'\|_q\geq c_K n\|p\|_q,\qquad c_K:=\frac{h_K^4}{3000\,d_K^5}. \tag{1.9}$$

From the other direction, we also proved that one cannot expect more than order n growth of $M_{n,q}(K)$. In fact, in this direction our result was more general, but here we recall only a combination of Theorem 5 and Remark 6 of [14].

Theorem G (Glazyrina–Révész). *Let $K \Subset \mathbb{C}$ be any compact convex domain. Then for any $q \geq 1$ and any $n \in \mathbb{N}$ there exists a polynomial $p \in \mathcal{P}_n(K)$ satisfying $\|p'\|_q < 15 d_K^{-1} n \|p\|_q$.*

In [14] we formulated the following conjecture.

Conjecture 1. *For all compact convex domains $K \Subset \mathbb{C}$ there exist $c_K > 0$ such that for any $p \in \mathcal{P}_n(K)$ we have $\|p'\|_{L^q(\partial K)} \geq c_K n \|p\|_{L^q(\partial K)}$. That is, for any compact convex domain K the growth order of $M_{n,q}(K)$ is precisely n.*

Also we pointed out that in the positive (Turán–Erőd type oscillation) direction, apart from the above findings for various classes, no completely general result is known, not even with a lower estimation of any weaker order than conjectured. This situation was compared to the situation in the development of the ∞-norm case, where a general lower estimation result, valid for all compact convex domains, was first proved only in 2002.

The aim of the present work is to prove the validity of a general lower estimation.

Theorem 1. *Let $K \Subset \mathbb{C}$ be any compact convex domain and $q \geq 1$. Then there exist a constant c_K and $n_0(q,k) \in \mathbb{N}$ such that for $n \geq n_0(q,K)$ and all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\|_q \geq c_K \frac{n}{\log n} \|p\|_q. \tag{1.10}$$

In other words, for compact convex domains we always have $c_K n/\log n \leq M_{n,q} \leq C_K n$.

Note that this, although indeed falling short of Conjecture 1, clearly exceeds the order $\sqrt{n}$, known for the interval $\mathbb{I}$.

2. SOME BASIC GEOMETRIC NOTATION AND FACTS

We need to fix geometric notation. Let us start with a *convex compact domain* $K \Subset \mathbb{C}$. Then its interior $\operatorname{int} K$ is nonempty and $K = \overline{\operatorname{int} K}$, while its boundary $\Gamma := \partial K$ is a convex Jordan curve. More precisely, $\Gamma = \mathcal{R}(\gamma)$ is the *range* of a continuous convex closed Jordan curve γ on the complex plane $\mathbb{C}$.

If the parameter interval of the Jordan curve γ is $[0,L]$, then this means that $\gamma\colon [0,L] \to \mathbb{C}$ is continuous, convex, and one-to-one on $[0,L)$, while $\gamma(L) = \gamma(0)$. While this compact interval parametrization is the most used setup for curves, we need the essentially equivalent interpretations with this, too: one is the definition over the torus $\mathbb{T} := \mathbb{R}/L\mathbb{Z}$ and the other is the periodically extended interpretation with $\gamma(t) := \gamma(t - [t/L]L)$ defined periodically all over $\mathbb{R}$. If we need to distinguish, we will say that $\gamma\colon \mathbb{R} \to \mathbb{C}$ and $\gamma^*\colon \mathbb{T} := \mathbb{R}/L\mathbb{Z} \to \mathbb{C}$, or equivalently, $\gamma^*\colon [0,L] \to \mathbb{C}$ with $\gamma^*(L) = \gamma^*(0)$.

As the curves are convex, they always have finite arc length $L := |\gamma^*|$. Accordingly, we will restrict ourselves to *parametrization with respect to arc length.* The parametrization $\gamma\colon \mathbb{R} \to \partial K$ defines a unique ordering of points, which we assume to be positive in the counterclockwise direction, as usual. When considered locally, i.e., with parameters not extending over a set longer than the period, this can be interpreted as an ordering of the image (boundary) points themselves: we always implicitly assume that a proper cut of the torus $\mathbb{T}$ is applied at a point to which the consideration is not extended, and then for the part of boundary we consider, the parametrization is one-to-one and carries over the ordering of the cut interval to the boundary.

The arc length parametrization has an immediate consequence also regarding the derivative, which must then have $|\dot\gamma| = 1$, whenever it exists, i.e., (linearly) a.e. on $[0,L) \sim \mathbb{T}$. Since $\dot\gamma\colon \mathbb{R} \to \partial\mathbb{D}$, we can as well describe the value by its angle or argument: the derivative angle function will be denoted by $\alpha := \arg\dot\gamma\colon \mathbb{R} \to \mathbb{R}$. Since, however, the argument cannot be defined on the unit circle without a jump, we decide to fix one value and then define the extension continuously: this way α will not be periodic, but we will have rotational angles depending on the number of (positive or negative) revolutions, if started from the given point. With this interpretation, α is an a.e. defined nondecreasing real function with $\alpha(t) - (2\pi/L)t$ periodic (with period L) and bounded. By convexity, the angular values attained by $\alpha(t)$ are then ordered in the same way as the boundary points and parameters. In particular, for a subset not extending to a full revolution, the angular values are uniquely attached to the boundary points and parameter values, and they are ordered in the same way by considering a proper cut.

With the usual left and right limits, α_- and α_+ are the left- and right-continuous extensions of α, respectively. The geometric meaning is that if for a parameter value τ the corresponding boundary point is $\gamma(\tau) = \zeta$, then $[\alpha_-(\tau), \alpha_+(\tau)]$ is precisely the interval of values $\beta \in \mathbb{T}$ such that the straight lines $\{\zeta + e^{i\beta}s \colon s \in \mathbb{R}\}$ are supporting lines to K at $\zeta \in \partial K$. We will also talk about half-tangents: the left and right half-tangents are the half-lines emanating from ζ and progressing towards $-e^{i\alpha_-(\tau)}$ and $e^{i\alpha_+(\tau)}$, respectively. The union of the half-lines $\{\zeta + e^{i\beta}s \colon s \ge 0\}$ for all $\beta \in [\alpha_+(\tau), \pi - \alpha_-(\tau)]$ is precisely the smallest cone with vertex at ζ that contains K.

We will interpret α as a multi-valued function, assuming all the values in $[\alpha_-(\tau), \alpha_+(\tau)]$ at the point τ. Restricting to the periodic (finite interval) interpretation of $\gamma^* \colon [0, L) \to \mathbb{C}$, without loss of generality we may assume that $\alpha^* := \arg(\dot{\gamma}^*) \colon [0, L] \to [0, 2\pi]$. In this regard, we can say that $\alpha^* \colon \mathbb{R}/L\mathbb{Z} \to \mathbb{T}$ is of bounded variation, with total variation (i.e., total increase) 2π; the same holds for $\alpha \colon \mathbb{R} \to \mathbb{R}$ over one period.

The curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the unique tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$.

It is clear that interpreting α as a function on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function: to be fully precise, we would have to talk about $\widetilde{\gamma}$, $\widetilde{\gamma}^*$, $\widetilde{\alpha}$, and $\widetilde{\alpha}^*$. In line with the above, we consider $\widetilde{\alpha}$ and $\widetilde{\alpha}^*$ as *multivalued functions*, all admissible supporting line directions belonging to $[\alpha_-(\tau), \alpha_+(\tau)]$ at $\zeta = \gamma(\tau) \in \partial K$ being considered as $\widetilde{\alpha}$-function values at ζ. At points of discontinuity, $\alpha_\pm$ or $\alpha^*_\pm$ and similarly $\widetilde{\alpha}_\pm$ and $\widetilde{\alpha}^*_\pm$ are the left- and right-continuous extensions, respectively, of the same functions.

A convex domain K is called *smooth* if it has a unique supporting line at each of its boundary points. This occurs if and only if $\alpha_\pm := \alpha$ is continuously defined for all values of the parameter. For obvious geometric reasons we call the jump function $\Omega := \alpha_+ - \alpha_-$ the *supplementary angle* function. This is identically zero almost everywhere (and in fact except for a countable set) and has positive values such that the total sum of the (possibly infinite number of) jumps over a period does not exceed the total variation of α, i.e., 2π.

For a supporting line $\zeta + e^{i\beta}\mathbb{R}$ at the boundary point $\zeta \in \partial K$ that is oriented positively (so that K lies in the half-plane $\{z \in \mathbb{C} \colon \beta \le \arg(z - \zeta) \le \beta + \pi\}$) the corresponding (outer) normal vector is $\boldsymbol{\nu}(\zeta) := e^{i(\beta - \pi/2)}$.

The family of all the (outer) normal vectors consists precisely of the vectors satisfying the condition $\langle z - \zeta, \boldsymbol{\nu} \rangle \le 0$ (for all $z \in K$) with the usual $\mathbb{R}^2$ scalar product, or equivalently, $\mathrm{Re}((z - \zeta)\overline{\boldsymbol{\nu}}) \le 0$ (where $\overline{\boldsymbol{\nu}}$ is just the conjugate of the complex number $\boldsymbol{\nu}$).

Here we introduce some additional notation, too. First, we will write $\delta(\zeta, \varphi) := \delta_K(\zeta, \varphi) := |K \cap (\zeta + e^{i\varphi}\mathbb{R})|$. Further, to denote the "opposite endpoint" of the intersection line segment, we will use the notation

$$D := D(\zeta) := D(\zeta, \varphi) := D_K(\zeta, \varphi),$$

so that $K \cap (\zeta + e^{i\varphi}\mathbb{R}) = [\zeta, D(\zeta)]$; of course, in particular cases even $D(\zeta) = \zeta$ and $\delta(\zeta, \varphi) = 0$ is possible.

The following easy but useful observation will be used several times in various situations.

Proposition 1. *Let $\zeta \ne \zeta' \in \partial K$ and assume that $t = \zeta + e^{i\varphi}\mathbb{R}_+$ and $t' = \zeta' + e^{i\varphi'}\mathbb{R}_+$ are two half-lines emanating from ζ and ζ', respectively, and having the* (*subderivative or half-tangent*) *property that $t \cap \operatorname{int} K = \varnothing$ and also $t' \cap \operatorname{int} K = \varnothing$. Assume that these half-lines intersect at* *a point* $T := t \cap t'$. *Write* ℓ *for the straight line connecting* ζ *and* ζ', *and assume that neither* t *nor* t' *is included in (so is not parallel to)* ℓ, *so that* T *is in one of the open half-planes of* $\mathbb{C} \setminus \ell$; *denote this half-plane by* H. *Finally, let* $\triangle := \triangle_{\zeta,T,\zeta'} := \operatorname{con}(\zeta, T, \zeta')$ *be the triangle with vertices* ζ, T, *and* ζ'.

Then we have $H \cap K \subset \triangle$.

Proof. Assume, as we may, that $\zeta = -i$ and $\zeta' = i$, whence ℓ is the imaginary axis, and that $H = \{\operatorname{Re} z > 0\}$ is the right half-plane, say. This means that both half-lines t and t' are contained in H, cutting H into four convex components, all bounded by (parts of the) straight lines ℓ, t, and t': number them as $H_1, \dots, H_4$. One is essentially the triangle $\triangle$ but beware of the boundary: in precise terms, $H_1 = \triangle \setminus [\zeta, \zeta']$, as the side $[\zeta, \zeta']$ of $\triangle$ falls on ℓ, not contained in the open half-plane H. Also there are three other unbounded components H_2, H_3, and H_4.

The only component which has both points ζ and ζ' on its boundary ∂H_j is necessarily the one with $\zeta'' := (\zeta + \zeta')/2 = 0$ in its boundary: this is H_1. Note that there exists a small $r > 0$ with the property that $\{z = \rho e^{i\varphi} \colon -\pi/2 < \varphi < \pi/2,\ 0 < \rho < r\} \subset H_1$. Also, $0 \in K$ by the convexity of K.

If $\operatorname{int} K \cap H = \varnothing$, then also $K \cap H = \varnothing$ because H is an *open* half-plane and K is *fat* [36, Corollary 2.3.9]; i.e., all its (interior or boundary) points are limits of interior points. So in this case there remains nothing to prove.

So let us consider the case when $\operatorname{int} K \cap H \neq \varnothing$. As H is open, $(\operatorname{int} K) \cap H = \operatorname{int}(K \cap H)$. Now we want to prove that then $\operatorname{int} K \cap H \subset \triangle$. Once we prove this, it will suffice, as for $K \cap H$ being a convex domain with nonempty interior it is also *fat*, and thus $K \cap H \subset \operatorname{cl}(\operatorname{int} K \cap H) \subset \operatorname{cl}(\triangle) = \triangle$, as needed.

So take any point $Z \in \operatorname{int} K \cap H$ and assume for contradiction that $Z \notin \triangle$.

Let now $z := \rho e^{i \arg(Z)} = \rho Z/|Z|$ with some $\rho < r$; then $z \in \triangle \cap H$. As $0 \in K$, we will have $(0, Z] \subset \operatorname{int} K$ in view of the convexity of K; so in particular $[z, Z] \subset \operatorname{int} K$.

As $z \in \triangle$ and $Z \notin \triangle$, there exists a boundary point $B \in \partial\triangle$ on the segment $[z, Z]$: $B \in \partial\triangle \cap [z, Z]$. So, $B \in \partial\triangle \cap \operatorname{int} K = \partial H_1 \cap \operatorname{int} K$. But it is also within H, while the boundary line segments of any component of H can only consist of pieces of $t \cup t'$, free of $\operatorname{int} K$ by assumption, which is a contradiction. $\square$

There are obvious, yet important, consequences of the above, which we will use throughout our reasoning. First, if $\zeta, \zeta' \in \partial K$ are two boundary points with $s := |\zeta - \zeta'| < w$, then the tangent lines at these points cannot be distinct and parallel (as K is not contained in any strip of width less than w_K). So, if we assume that $t \neq \ell$ and $t' \neq \ell$, then appropriate half-lines of these tangents intersect at a point T. Therefore, when the plane, and hence K, is cut into two parts by the line ℓ of ζ and ζ', one part—the part of K in the same half-plane as T—will be contained in the triangle $\triangle_{\zeta,T,\zeta'}$.

We want to underline that *this part is smaller in a precise sense* than the other, left-over, part of K. For example, the maximal chord in the direction of $\zeta' - \zeta$ is $s = |\zeta' - \zeta| < w$ (for it cannot exceed the maximal chord of $\triangle_{\zeta,T,\zeta'}$ in the same direction). Note that we are talking about the direction of ℓ, whence the part of K in the other half-plane must have maximal chord in this direction *at least* w, as the maximal chord of K in any direction is at least w (cf. [36, Theorem 7.6.1]). Similarly, the part of K lying in $\triangle_{\zeta,T,\zeta'}$ has width in the direction orthogonal to t at most the height of the $\triangle_{\zeta,T,\zeta'}$, which does not exceed the chord $s < w$, while the minimal with of the totality of K is w, whence the left-over part also has points at least w-far from t, and at the same time ζ is also in the boundary of this part, so the width (in this direction) of this left-over part must be at least w. In this sense thus it is precise if we distinguish these two sides as the "smaller side/part of K" (in the same half-plane as T) and the "bigger/larger side/part of K."

Further, considering the positive orientation of the boundary curve, we may fix a branch of the arc length parametrization which is continuous over the small part; equivalently, we may apply a cut, or fix a starting point of parametrization, in the complementary part. In this sense the parametrization defines a unique ordering of points over the smaller part, even if the whole boundary ∂K cannot be ordered. In the following we will always say that two points—or their parameter values—are in precedence according to this choice of ordering, so that we compare only points in some unambiguously given "smaller part" and then $\zeta \prec \zeta'$ has the meaning of precedence in the positively ordered arc length parametrization, used continuously along this smaller part. We will also assume the tangent angle function α being defined according to the same continuity condition, so that $\zeta \prec \zeta'$ if and only if $\alpha(\zeta) < \alpha(\zeta')$ (or, more precisely, with $\zeta = \gamma(a)$ and $\zeta' = \gamma(b)$, we have $\alpha(a) < \alpha(b)$).

As for the precedence of boundary points of ∂K, we can equivalently say that whenever $\zeta, \zeta' \in \partial K$ and some positively oriented tangents to K at ζ and ζ' are t and t', then we say that $\zeta \prec \zeta'$ if and only if the positively oriented half-tangent of t intersects the negatively oriented half-tangent of t'. Of course, these tangents intersect only if they are not parallel; but distinct parallel tangents can exists only if they are at least at a distance of w from each other, so, for example, if the chord length $s := |\zeta - \zeta'|$ is less than w, then it is certainly not the case. In the case when ζ and ζ' lie in a straight line segment piece of ∂K (and when again either the intersection of the positively oriented half-tangent of t and the negatively oriented half-tangent of t' is empty or, conversely, the intersection of the negatively oriented half-tangent of t and the positively oriented half-tangent of t' is empty), this definition of precedence also works. Finally, if t and t' are distinct and not parallel, then there is a unique such point T, and the precedence is unambiguously defined. So, defining precedence only for point pairs $(\zeta, \zeta') \in \partial K \times \partial K$ this way, we create a partial relation in $\partial K \times \partial K$, which is asymmetric but is not transitive (so we cannot consider it an ordering); yet it is quite consistent with ordering of points if we apply a certain fixed cut of the boundary and consider the ordering of points of ∂K accordingly.

Proposition 2. *Let $\zeta, \zeta' \in \Gamma$, $0 < |\zeta - \zeta'| = s < w$, and let $t := \zeta + e^{i\alpha}\mathbb{R}$ and $t' := \zeta' + e^{i\alpha'}\mathbb{R}$ be two positively oriented tangent lines at these points. Assume that neither t nor t' is equal to the chord line $\ell := \overline{\zeta\zeta'}$. Then there exists a unique point of intersection $T := t \cap t'$; moreover, we have $T \notin \ell$.*

Furthermore, writing H for the open half-plane of $\mathbb{C} \setminus \ell$ with $T \in H$ and $\overline{H}$ for its closure, we also have

(i) $K \cap \overline{H} \subset \triangle := \triangle_{\zeta,T,\zeta'} := \operatorname{con}(\zeta, T, \zeta')$;

(ii) *if, in the triangle $\triangle$, $\beta := \angle(\zeta, T, \zeta') = |\arg((\zeta - T)/(\zeta' - T))|$, then* $\beta \ge \arcsin((w - s)/d)$;

(iii) $\operatorname{diam}(K \cap \overline{H}) \le sd/(w - s)$ *and, in particular,* $\operatorname{diam}(K \cap \overline{H}) \le 2sd/w$;

(iv) $|\Gamma \cap \overline{H}| \le 2sd/(w - s)$ *and, in particular,* $|\Gamma \cap \overline{H}| \le 4sd/w$.

Note that in this fully general case $\alpha' - \alpha$ and $\sin|\alpha' - \alpha|$ can be arbitrarily small (in case α' is not much different from α), but in the other direction we assert that their difference is bounded away from reaching π. In fact, even $\alpha' = \alpha$ would be possible (exactly if $[\zeta, \zeta']$ is a part of the boundary curve Γ and both tangents t and t' coincide with ℓ), but for easier formulation we assume in the claim that neither t nor t' is ℓ, which entails that $\alpha' \ne \alpha$. The degenerate cases when $[\zeta, \zeta'] \subset \partial K$ and some of t and t' equals ℓ are somewhat inconvenient, for then even the assertions may fail in cases when $\ell \cap \partial K$ exceeds $[\zeta, \zeta']$. Instead of describing these situations in an overcomplicated manner right here, we will also avoid dealing with them in the forthcoming applications of Propositions 1 and 2 either by assuming $\overline{\zeta\zeta'} \cap K = [\zeta, \zeta']$ or by discussing concretely the cases when $t' = \ell$ or $t = \ell$.

We also note that working with the maximal chord parallel to the chord $[\zeta, \zeta']$, one can get in a somewhat easier way the estimate[1] $\beta \ge \arctan((w - s)/d)$; as arcsin exceeds arctan, we opted for the presentation of this slightly sharper version.

[1] An observation kindly offered to us by Sándor Krenedits in personal communication.

Proof of Proposition 2. First, let us check that $t \neq \ell$ and $t' \neq \ell$ implies $\alpha \neq \alpha'$. For a *convex* domain and *positively oriented tangents*, $\alpha = \alpha'$ would be possible only if $t = t'$, while $\zeta \in t$ and $\zeta' \in t'$ entails that $t = t'$ could happen only if $t, t' = \ell$, which is excluded; so $t \neq t'$ and $\alpha \neq \alpha'$. Second, $t \parallel t'$ while $t \neq t'$ (i.e., with positive orientation, $\alpha' = \alpha + \pi \bmod 2\pi$) is also impossible, for then K would have two parallel tangents with a positive distance not exceeding $s < w$, which then would imply that $\operatorname{width}(K) < w$, a contradiction. So, t and t' are not parallel and indeed $T := t \cap t'$ exists uniquely; moreover, $T \notin \ell$ is clear (for in case $T \in \ell$ either $T \neq \zeta$ and so $t = \overline{T\zeta} = \ell$ or $T \neq \zeta'$ and $t' = \overline{\zeta' T} = \ell$, but these possibilities were both excluded by assumption). This proves the assertions about T itself.

As for (i), we have $K_0 := K \cap H \subset \triangle := \operatorname{con}(\zeta, T, \zeta')$ in view of Proposition 1, so it remains to see that the same also holds for the closure $\overline{H}$ in this case. In other words, we must show additionally that $\ell \cap K \subset \triangle$, or, equivalently, that $\ell \cap K \subset [\zeta, \zeta']$, i.e., $\ell \cap K = [\zeta, \zeta']$. Now the tangent line t, not matching to ℓ, must cut this chord line into proper half-lines starting from ζ, with only one of these half-lines containing points of K; so the said half-line must be the half-line emanating from ζ towards ζ'. Arguing in the same way for t' and ζ', we find that $K \cap \ell$ is covered by $[\zeta, \zeta']$, as stated. (Note that this latter property may easily fail if $t = \ell$ or $t' = \ell$ is allowed.)

For the following assume, as we may, that the precedence of points ζ and ζ' is chosen so that $\zeta \prec \zeta'$, or, equivalently, $\alpha < \alpha' < \alpha + \pi$. Note that this is equivalent to T being the intersection of the half-lines $t_+ := \zeta + e^{i\alpha}\mathbb{R}_+$ and $t'_- := \zeta' - e^{i\alpha'}\mathbb{R}_+$. Therefore, in the triangle $\triangle = \triangle_{\zeta, T, \zeta'}$, the angle at T is

$$\beta := \angle(\zeta, T, \zeta') = \arg(\zeta - T) - \arg(\zeta' - T) = \alpha + \pi - \alpha' = \pi - (\alpha' - \alpha) < \pi.$$

Further, the tangent angle function can be fixed so that it changes nondecreasingly between α and $\alpha + \pi$, with the cut (negative jump by -2π) occurring at some point with tangent direction, say, $\alpha + 3\pi/2 \pmod{2\pi}$.

So, let us prove (ii). Our task is to estimate the angle β from below: we want to show that $\beta \geq \arcsin((w - s)/d)$. Note that β can be close to π, even if it cannot reach it, but we claim that it cannot be too small.

For an arbitrary point $A \in \partial K$ with tangent direction $\alpha(A) = \alpha + \pi$ (so with a tangent parallel to t but oriented oppositely), we have $\alpha < \arg(\zeta' - \zeta) < \alpha' < \alpha + \pi = \alpha(A)$, and $\zeta \prec \zeta' \prec A$. In fact from the very definition of width it follows for the point A that $a := \operatorname{dist}(A, t) \geq w$, while for boundary points P with $\zeta \prec P \prec \zeta'$, i.e., for points of $\Gamma \cap H \subset K \cap H \subset \triangle$ we necessarily have $\operatorname{dist}(P, t) \leq \max_{z \in \triangle} \operatorname{dist}(z, t) = m := \operatorname{dist}(\zeta', t) \leq s < w$, so that $P = A$ is not possible.

As $A \notin \overline{H}$ (because that would entail $A \in K \cap \overline{H} \subset \triangle$), we also find that $A \in \mathbb{C} \setminus \overline{H}$, whence also $[\zeta', A] \subset \mathbb{C} \setminus H$. So let us draw the chord line $f := \overline{\zeta' A}$. By convexity, for the positively oriented direction φ of the chord f we have $\alpha' = \alpha(\zeta') \leq \varphi = \arg(A - \zeta') \leq \alpha(A) = \alpha + \pi$. Note that for points $z \in f_+$ on the positive half-line $f_+ := \zeta' + e^{i\varphi}\mathbb{R}_+$ we have $\operatorname{dist}(z, t) \geq \operatorname{dist}(\zeta', t) = m > 0$, whence $t \cap f_+ = \varnothing$. On the other hand, the intersection point $C := f \cap t$ exists uniquely, as f is not parallel to t (for $a := \operatorname{dist}(A, t) \neq \operatorname{dist}(\zeta', t) = m$). So, $C \in f_- \cap t$, i.e., $C = f_- \cap t_+$ (in accordance with $\zeta \prec A$). It follows that at C the angle

$$\theta := \angle(\zeta, C, \zeta') = \arg(\zeta - C) - \arg(\zeta' - C) = (\alpha + \pi) - \varphi \leq \alpha + \pi - \alpha' = \beta.$$

Consider the orthogonal projection of ζ' to t, and denote this point by M; then the height of $\triangle$ at ζ' is $m = |\zeta' - M|$, and $0 < m \leq s$. Further, take also the orthogonal projection of A to t and denote this point by B; then $a = \operatorname{dist}(A, t) = |A - B| \geq w$.

It remains to estimate $\sin\theta$ from below. Note that the triangles $\triangle_{A,B,C}$ and $\triangle_{\zeta', M, C}$ are similar triangles with right angles at B and M, respectively, whence for the angle $\angle(BCA) = \angle(MC\zeta')$

at the homothety center point C we have $\sin\angle(BCA) = |A-B|/|A-C| = |\zeta'-M|/|\zeta'-C|$ and so also $\sin\angle(MC\zeta') = (a-m)/|A-\zeta'|$. However, either $\angle(MC\zeta') = \theta$ or $\angle(MC\zeta') = \pi-\theta$, depending on whether $\overrightarrow{CM}$ is directed to the negative or positive direction of t (both cases are possible), i.e., $\arg(M-C) = \alpha+\pi$ or $\arg(M-C) = \alpha$. So finally $\sin\theta = \sin\angle(MC\zeta')$ in both cases, and we are led to $\sin\theta = (a-m)/|A-\zeta'|$. Therefore, since $A, \zeta' \in K$ entails $|A-\zeta'| \le d$, we find that $\sin\theta \ge (a-m)/d \ge (w-s)/d$, and so in particular $\beta \ge \theta \ge \arcsin((w-s)/d)$, proving assertion (ii).

Let us prove (iii). Using (i) we get $\operatorname{diam}(K\cap\overline{H}) \le \operatorname{diam}(K\cap\triangle_{\zeta,T,\zeta'}) = \max\{s, |\zeta-T|, |\zeta'-T|\}$.

As for $|\zeta'-T|$, with the above notation and using (ii) we easily obtain $|\zeta'-T| = m/\sin\beta \le s/\sin\theta \le sd/(w-s)$.

At this point, however, one may apply the symmetry of the situation; if the distance from one endpoint of the chord $[\zeta,\zeta']$ to $T = t\cap t'$ cannot exceed $sd/(w-s)$, then neither the distance from the other endpoint can do so: i.e., $|T-\zeta| \le sd/(w-s)$ holds, too. Consequently, $\operatorname{diam}(K\cap\overline{H}) \le sd/(w-s)$, as $s \le sd/(w-s)$ is immediate.

Finally, if $0 < s \le w/2$ then $\operatorname{diam}(K\cap\overline{H}) \le sd/(w-s) \le 2sd/w$ is obvious, while for $w/2 < s < w$ we trivially have $\operatorname{diam}(K\cap\overline{H}) \le d \le 2sd/w$.

Let us prove (iv). Since Γ is convex, the arc length of the part of Γ in $\triangle_{\zeta,T,\zeta'}$ joining ζ and ζ' cannot exceed the sum $|\zeta-T| + |\zeta'-T|$ (because it is well known for convex curves that the included one is not longer than the including one; see, e.g., [9, § 7, Sect. 31, property 5]). As discussed above, this can be estimated by $2sd/(w-s)$ and also by $4sd/w$, as claimed. □

In the following we will use the notation $S_z[\alpha,\beta] := \{z+\rho e^{i\varphi}\colon \varphi\in[\alpha,\beta]\}$ and $S_z(\alpha,\beta) := \{z+\rho e^{i\varphi}\colon \varphi\in(\alpha,\beta)\}$ for sectors with vertex at $z\in\mathbb{C}$ and angles between α and β.

Proposition 3. *Let $\zeta\in\partial K$ and $\boldsymbol{\nu} = -e^{i\sigma}$ be (one) outer normal vector to K at ζ, and let $t := \zeta + e^{i\alpha}\mathbb{R}$ be the corresponding positively oriented tangent line at ζ with $\alpha = \sigma-\pi/2$. Fix any angle $0<\varphi<\pi/2$. Define*

$$\ell_- := \zeta + e^{-\varphi i}\boldsymbol{\nu}\mathbb{R} = \zeta + e^{(\sigma-\varphi)i}\mathbb{R}, \qquad [\zeta, D_-] := \ell_-\cap K, \qquad \textit{and} \qquad \delta_- := |D_- - \zeta| = |\ell\cap K|$$

and similarly

$$\ell_+ := \zeta + e^{+\varphi i}\boldsymbol{\nu}\mathbb{R} = \zeta + e^{(\sigma+\varphi)i}\mathbb{R}, \qquad [\zeta, D_+] := \ell_+\cap K, \qquad \textit{and} \qquad \delta_+ := |D_+ - \zeta| = |\ell\cap K|.$$

If $0 < \delta_- \le \delta_+ < w$, then any tangent line t'_- drawn to K at D_- has negative slope with respect to t, i.e., t'_- is not parallel to t and the point of intersection $T = t\cap t'_-$ is on the half-line $\zeta + e^{i(\sigma-\pi/2)}\mathbb{R}_+$; equivalently, $\zeta \prec D_-$ in the sense discussed above, and from the parts of K arising from the cut of $\mathbb{C}$ (and thus of K) by the straight line ℓ_-, the one in the sector $S_\zeta[\sigma-\pi/2, \sigma-\varphi]$ is the "small part" of K.

Symmetrically, if $0 < \delta_+ \le \delta_- < w$, then any tangent line t'_+ drawn to K at D_+ has positive slope, $D_+ \prec \zeta$, $T \in \zeta - e^{i(\sigma-\pi/2)}\mathbb{R}_+$, and from the two parts of K determined by ℓ_+, the small part lies in the sector $S_\zeta[\sigma+\varphi, \sigma+\pi/2]$.

Note that we assumed here the condition $\max(\delta_-, \delta_+) < w$; but this is not necessary. However, the slightly weaker assumption that $\min(\delta_-, \delta_+) < w/\cos\varphi$ cannot be dropped: if both $\delta_\pm \ge w/\cos\varphi$, then the tangents can go in any direction (with both positive or negative slope) including the possibility of being parallel to t. We do not discuss these because in our later application in Lemma 4 we will be at ease if any of the chords is as large as w, and so we do not need further details. Similarly, it will also be easy to deal with the case when either δ_- or δ_+ vanishes, whence our other assumption on $\min(\delta_-, \delta_+) > 0$ is not too restrictive. Note that in the case $\min(\delta_-, \delta_+) = 0$, for example if $\delta_- = 0$, ℓ_- is also tangent to K (as it does not contain any interior points, only $\zeta\in\partial K$); thus K lies entirely in some of the sectors lying above t and determined by the line ℓ_-; however, we cannot always tell which side is the small/large side, as any of these two sectors $S_\zeta[\sigma-\pi/2,\sigma-\varphi]$ or $S_\zeta[\sigma-\varphi,\sigma+\pi/2]$ may contain K. Of course, if the other chord is nonzero, i.e., $\delta_+>0$, then clearly that side, i.e., the latter sector, will contain K. The situation is similar if we start with $\delta_+=0$.

Proof of Proposition 3. By symmetry, we may, and hence will, assume $0<\delta_-\le\delta_+<w$.

It is clear that the tangent t'_- cannot be parallel to t, for in this case we would have K contained between the *distinct* parallel supporting lines t and t'_- at a distance of $(0<)\,\delta_-\cos\varphi<\delta_-<w$, a contradiction. Now if t'_- had a positive slope, i.e., $T=t\cap t'_-$ fell on the half-line $\zeta-e^{i(\sigma-\pi/2)}\mathbb{R}_+$, then we would obviously have D_+ below this tangent, and $\delta_+<\delta_-$, contrary to the assumption.

So there remains the only possibility of t'_- having negative slope. That is, $T\in\zeta+e^{i(\sigma-\pi/2)}\mathbb{R}_+$, $\zeta\prec D_-$, and the above Proposition 1 applies. It means that the triangle $\triangle_{\zeta,T,D_-}$ covers the part of K in the respective sector $S_\zeta[\sigma-\pi/2,\sigma-\varphi]$, whence this can only be the "small part" of K. $\square$

3. TECHNICAL PREPARATIONS FOR THE INVESTIGATION OF $L^q(\partial K)$ NORMS

Lemma 1. *For any polynomial of degree at most n we have*

$$\|p\|_{L^q(\partial K)}\ge\left(\frac{d}{2(q+1)}\right)^{1/q}\|p\|_{L^\infty(\partial K)}\,n^{-2/q}. \tag{3.1}$$

For a proof of this Nikolskii-type estimate, see [14, Lemma 1].

Next, let us define the subset $\mathcal{H}:=\mathcal{H}^q_K(p)\subset\partial K$ in the following way:

$$\mathcal{H}:=\mathcal{H}^q_K(p):=\bigl\{\zeta\in\partial K\colon\ |p(\zeta)|>cn^{-2/q}\|p\|_\infty\bigr\},\qquad c:=\frac{1}{2}(8\pi(q+1))^{-1/q}. \tag{3.2}$$

Then in [14, Sect. 3.1] it was deduced from the above lemma that we have

Lemma 2. *Let $\mathcal{H}\subset\partial K$ be defined according to* (3.2)*. Then for all $p\in\mathcal{P}_n$ we have*[2]

$$\int\limits_{\mathcal{H}}|p|^q\ge\frac{1}{2}\|p\|^q_{L^q(\partial K)}. \tag{3.3}$$

Furthermore, for any point $\zeta\in\mathcal{H}$ and for any $p\in\mathcal{P}_n(K)$ we also have

$$\log\frac{\|p\|_\infty}{|p(\zeta)|}\le\log(16\pi)+2\log n\quad\forall\,n\in\mathbb{N},\qquad\log\frac{\|p\|_\infty}{|p(\zeta)|}\le\frac{107}{40}\log n\quad\forall\,n\ge73. \tag{3.4}$$

The other key and innovative feature of the original work of Erőd was invoking Chebyshev's lemma, which we recall here.

Lemma H (Chebyshev)**.** *Let $J=[u,v]$ be any interval on the complex plane with $u\ne v$. Then for all $k\in\mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k\in\mathbb{C}}\ \max_{z\in J}\left|\prod_{j=1}^k(z-w_j)\right|\ge2\left(\frac{|J|}{4}\right)^k. \tag{3.5}$$

Actually, we will also use this lemma in the next slightly more general form of an estimation using the transfinite diameter.

Lemma I (transfinite diameter lemma)**.** *Let $K\Subset\mathbb{C}$ be any compact set and $p\in\mathcal{P}_n(K)$ be a monic polynomial, i.e., assume that $p(z)=\prod_{j=1}^n(z-z_j)$ with all $z_j\in K$. Then we have $\|p\|_\infty\ge\Delta(K)^n$.*

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[2] Hereinafter all integrals are understood with respect to the arc length measure.

Lemma H is essentially the classical result of Chebyshev for a real interval [28] (cf. [21, Part 6, Problem 66; 10; 20]). The form with the transfinite diameter was first proved in various forms by Fekete, Faber, and Szegő. For details and references see [14, Lemma P] and its discussion there.

In the below proofs we will need the following straightforward calculation of the type usually considered in connection with transfinite diameter.

Lemma 3. *Let $K' \Subset K \Subset \mathbb{C}$ be two compact sets with diameters $d' := \operatorname{diam}(K')$ and $d := \operatorname{diam}(K)$, and assume $d' \le d/k$ with some parameter $k > 10$, say. If a polynomial $p \in \mathcal{P}_n(K)$ has $m \ge 3(\log 2/\log k)n$ zeroes in K', then $\|p\|_{K'} < 2^{-n}\|p\|_K$.*

Proof. Assume, as we may, that the leading coefficient of p is just 1, so $p(z) = \prod_{j=1}^n (z - z_j)$. It is well known (see, e.g., [15] or [22, Sect. 1.7.1][3]) that the capacity, or transfinite diameter, of a compact set is at least its diameter divided by 4 (and is, on the other hand, at most the diameter divided by 2). Using this or directly Chebyshev's lemma, we certainly have $\|p\|_K (\ge \Delta(K)^n) \ge (d/4)^n$.

Estimating from the other side, we have for any point $z' \in K'$ the estimate $|p(z')| \le d'^m d^{n-m}$, whence $\|p\|_{K'} \le d'^m d^{n-m}$, and after dividing these two estimates we get

$$\frac{\|p\|_{K'}}{\|p\|_K} \le \frac{d'^m d^{n-m}}{(d/4)^n} = 4^n \left(\frac{d'}{d}\right)^m \le 4^n k^{-m} = 2^{2n - m\log k/\log 2} \le 2^{2n-3n} = 2^{-n}. \quad \square$$

4. REFINED ESTIMATE BY TILTING THE NORMAL LINE

The method in our recent works [14, 15] was to consider an upper subinterval $J \subset [\zeta, D] := \boldsymbol{\nu} \cap K$, with $\boldsymbol{\nu}$ a normal line at $\zeta \in \partial K$, apply a suitable classification of zeroes, and, for the, say, k zeroes lying close to J, select a maximum point τ_0 of the corresponding product of the respective k terms $(z - z_j)$. This direct approach can be used to get some general infinity norm estimates (in fact, an order $n^{2/3}$ lower estimation [23]) even if the depth may tend to zero. Also, we succeeded to obtain the right order (i.e., order n) lower estimate for some *special* classes of domains in [14, 15]. However, this method incorporates some losses with respect to depth, and for fully general cases there seems to be no way to obtain the optimal or close-to-optimal order by this method.

Instead, here we pursue a significantly modified method based on an insightful idea of G. Halász and exploited, for the case of the maximum norm, in the proof of Theorem D in [24]. For more explanations and the heuristic reasons for the key idea of tilting the normal line in this approach, the interested reader may consult [24, 26].

In the main proof in [24] one could make use of the maximality of $|p(z)|$; as before in [14, 15], we now have to take a general boundary point and derive pointwise estimates in this more general case.

Here we work out the following version of the main proof from [24].

Lemma 4 (tilted normal estimate)**.** *Let $\zeta \in \partial K$ and $\boldsymbol{\nu} = -e^{i\sigma}$ be (some) outer normal vector to K at ζ. Fix the angles*

$$\psi := \arctan\frac{w}{d} \in \left(0, \frac{\pi}{4}\right] \qquad \textit{and} \qquad \theta := \frac{\psi}{20} \in \left(0, \frac{\pi}{80}\right]. \tag{4.1}$$

Define

$$\ell_\pm := \zeta + e^{\pm 2\theta i}\boldsymbol{\nu}\mathbb{R} = \zeta + e^{(\sigma \pm 2\theta)i}\mathbb{R}, \qquad [\zeta, D_\pm] := \ell_\pm \cap K, \qquad \textit{and} \qquad \delta_\pm := |D_\pm - \zeta| = |\ell \cap K|,$$

with the two alternatives with respect to $\pm$ understood separately. Then we have the following.

[3] However, note a disturbing misprint in this fundamental reference: in Section 1.7.2, the first two displayed formulas must be corrected to have the opposite direction of the inequality sign.

(i) *If* $\ell_+ \cap \operatorname{int} K = \varnothing$ *or* $\ell_- \cap \operatorname{int} K = \varnothing$*—in particular, if either* $\delta_- = 0$*, i.e.,* $D_- = \zeta$ *and* $\ell_- \cap K = \{\zeta\}$*, or* $\delta_+ = 0$*, i.e.,* $D_+ = \zeta$ *and* $\ell_+ \cap K = \{\zeta\}$*—then*

$$\left|\frac{p'}{p}(\zeta)\right| \geq \frac{1}{2d}n.$$

(ii) *If both intersections* $\ell_\pm \cap \operatorname{int} K$ *are nonempty—entailing that* $\delta_\pm > 0$*—and* $0 < \delta_\pm < w$*, then*

$$\left|\frac{p'}{p}(\zeta)\right| > 10^{-3}\frac{w}{d^2}n - \frac{2}{39\delta_\pm}\log\frac{\max_{K\cap\ell_\pm}|p|}{|p(\zeta)|} \geq 10^{-3}\frac{w}{d^2}n - \frac{2}{39\delta_\pm}\log\frac{\|p\|_\infty}{|p(\zeta)|}, \tag{4.2}$$

where the choice of the sign has to be such that $\delta_\pm = \min(\delta_-, \delta_+)$*. In particular, if* $\zeta \in \mathcal{H}$ *(with* $\mathcal{H}$ *defined in* (3.2)*) and* $n \geq 73$*, then according to the last estimate of* (3.4)

$$\left|\frac{p'}{p}(\zeta)\right| > 10^{-3}\frac{w}{d^2}n - \frac{3}{20\delta_\pm}\log n. \tag{4.3}$$

(iii) *Finally, if* $\max(\delta_-, \delta_+) \geq w/2$*, then the above estimates* (4.2) *and* (4.3) *hold for both choices of the sign, so also with the one providing* $\max(\delta_-, \delta_+)$*, irrespective of the size of the various parts of* K *as cut by the chord lines or of whether* $\operatorname{int} K \cap \ell_\pm = \varnothing$ *or not.*

Proof. Assume, as we may, $\zeta = 0$ and $\boldsymbol{\nu} = \boldsymbol{\nu}(\zeta) = \boldsymbol{\nu}(0) = -i$; i.e., the selected supporting line is the real line $\mathbb{R}$ (oriented positively) and $\sigma = \pi/2$. So, K lies in the upper half-plane: $K \subset \{z\colon \operatorname{Im} z \geq 0\}$.

Consider now the situation in (i); for example, let us consider the case when $\operatorname{int} K \cap \ell_- = \varnothing$, the other case being symmetrical. The ray (straight half-line) $e^{i(\pi/2-2\theta)}\mathbb{R}_+ = \ell \cap \{z\colon \operatorname{Im} z \geq 0\}$, emanating from $\zeta = 0$ in the direction of $e^{i(\pi/2-2\theta)}$ intersects K along the segment $[0, D]$, and if $\ell_- \cap \operatorname{int} K = \varnothing$, then we necessarily have $[0, D] \subset \partial K$. So, ℓ_- is a supporting line of K, and either $K \subset S[0, \pi/2 - 2\theta]$ or $K \subset S[\pi/2 - 2\theta, \pi]$. In either case a standard argument using, for example, Turán's Lemma B yields directly $|p'(\zeta)/|p(\zeta)| \geq n/(2d)$. Hence assertion (i) is proved.

It remains to discuss the cases when $\operatorname{int} K \cap \ell_\pm \neq \varnothing$, entailing that both $\delta_\pm$ are positive.

Again we choose to deal with one of the two entirely symmetrical cases and suppose that $0 < \delta_- \leq \delta_+ < w$ if $\min(\delta_-, \delta_+) < w/2$ and $0 < \delta_+ \leq \delta_-$ otherwise. Therefore, we can take δ_- in both cases (ii) and (iii). To further ease the notation, we will drop the minus sign from the index and will simply write δ, D, etc., for the previously given δ_-, D_-, etc., in the rest of the argument.

The small geometric claim whose proof ramifies here is the statement that we necessarily have

$$|z| \leq \frac{2\delta d}{w} \qquad \text{for} \quad z \in K \cap S[0, \theta]. \tag{4.4}$$

This is clearly true if $\delta \geq w/2$, because $|z| = |z - \zeta| \leq d$. However, if $\delta < w/2$, then Proposition 3 applies with $\varphi := 2\theta$, which in turn furnishes $\operatorname{diam}(K \cap S[0, \pi/2 - 2\theta]) \leq 2\delta d/w$, according to Proposition 2(iii). As $S[0, \theta] \subset S[0, \pi/2 - 2\theta]$, it is all the more true that $\operatorname{diam}(K \cap S[0, \theta]) \leq 2\delta d/w$; so again $|z| = |z - \zeta| \leq \operatorname{diam}(K \cap S[0, \theta]) \leq 2\delta d/w$, as wanted. This small statement will be soon used in the calculations with points of the forthcoming set $\mathcal{Z}_1$.

Denote by $\mathcal{Z} := \{z_j = r_j e^{i\varphi_j}\colon j = 1, \dots, n\}$ the n-element set of zeroes (listed according to multiplicities) of the fixed polynomial $p \in \mathcal{P}_n(K)$. Note that $0 \leq \varphi_j \leq \pi$ for $j = 1, \dots, n$.

Observe that for any subset $\mathcal{W} \subset \mathcal{Z}$ and $M := |p'(\zeta)/p(\zeta)|$ we have

$$M = \left|\frac{p'}{p}(0)\right| \geq -\operatorname{Im}\frac{p'}{p}(0) = \sum_{j=1}^n \operatorname{Im}\left(-\frac{1}{z_j}\right) \geq \sum_{z_j \in \mathcal{W}} \operatorname{Im}\left(-\frac{1}{z_j}\right) = \sum_{z_j \in \mathcal{W}} \frac{\sin\varphi_j}{r_j}, \tag{4.5}$$

because all terms in the full sum are nonnegative.

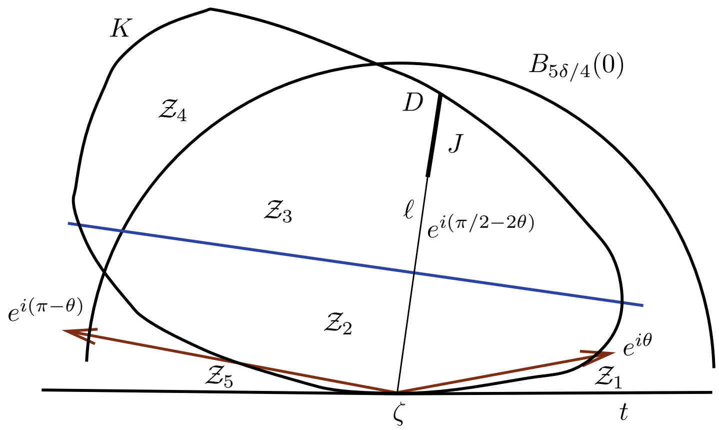

Fig. 1. The classification of zeroes according to location.

The segment J is defined to be

$$J := \left[\frac{\zeta+3D}{4}, D\right] = \left\{\tau := te^{i(\pi/2-2\theta)}\delta\colon\ \frac{3}{4} \le t \le 1\right\}. \tag{4.6}$$

Clearly, by convexity we have $J \subset K$.

Setting $B_r(0) := \{z\colon |z| \le r\}$ and writing $\mathcal{Z}[\alpha,\beta] := \mathcal{Z} \cap S[\alpha,\beta]$ and $\mathcal{Z}(\alpha,\beta) := \mathcal{Z} \cap S(\alpha,\beta)$, we split the set $\mathcal{Z}$ into the following parts (Fig. 1):

$$\begin{aligned}
&\mathcal{Z}_1 := \mathcal{Z}[0,\theta], && \mu := \#\mathcal{Z}_1,\\
&\mathcal{Z}_2 := \mathcal{Z}(\theta,\pi-\theta) \cap \left\{\mathrm{Im}(e^{i2\theta}z) < \frac{3}{8}\delta\right\}, && \nu := \#\mathcal{Z}_2,\\
&\mathcal{Z}_3 := \mathcal{Z}(\theta,\pi-\theta) \cap \left\{\mathrm{Im}(e^{i2\theta}z) \ge \frac{3}{8}\delta\right\} \cap B_{5\delta/4}(0), && \kappa := \#\mathcal{Z}_3,\\
&\mathcal{Z}_4 := \mathcal{Z}(\theta,\pi-\theta) \cap \left\{\mathrm{Im}(e^{i2\theta}z) \ge \frac{3}{8}\delta\right\} \setminus B_{5\delta/4}(0) = \mathcal{Z}(\theta,\pi-\theta) \setminus (\mathcal{Z}_2 \cup \mathcal{Z}_3), && k := \#\mathcal{Z}_4,\\
&\mathcal{Z}_5 := \mathcal{Z}[\pi-\theta,\pi], && m := \#\mathcal{Z}_5.
\end{aligned} \tag{4.7}$$

In the following we estimate $|p(\tau)/p(\zeta)|$ from below.

First we estimate the distance of any $z_j \in \mathcal{Z}_1$ from J. In view of the above discussed small claim (4.4), for any $z = re^{i\varphi} \in K \cap S[0,\theta]$ we have $|z| \le 2\delta d/w$, whence from the convexity of the tangent function

$$r\sin\theta \le \frac{2\delta d}{w}\sin\theta \le 2\delta\frac{d}{w}\tan\theta = 2\delta\frac{\tan\theta}{\tan(20\theta)} < \frac{\delta}{10}. \tag{4.8}$$

Now $\mathrm{dist}(z,J) = \min_{3/4\le t\le 1}|z-\tau|$ (where $\tau := te^{i(\pi/2-2\theta)}\delta$), and by the cosine theorem $|z-\tau|^2 = r^2 + t^2\delta^2 - 2rt\delta\cos(\pi/2-\varphi-2\theta)$. Because of $\cos(\pi/2-\varphi-2\theta) = \sin(\varphi+2\theta) \le \sin(3\theta) \le 3\sin\theta$, (4.8) implies

$$|z-\tau|^2 = r^2 + 10t^2\delta r\sin\theta - 6t\delta r\sin\theta = r^2 + (10t^2-6t)\delta r\sin\theta,$$

and thus $\min_{3/4\le t\le 1}|z-\tau|^2 = |z-\tau|^2\big|_{t=3/4} = r^2 + (9/8)\delta r\sin\theta$. It follows that we have

$$\frac{|z-\tau|^2}{|z|^2} \ge 1 + \frac{9}{8}\frac{\delta\sin\theta}{r} > 1 + \frac{9}{8}\frac{\delta\sin\theta}{d}, \qquad \tau \in J.$$

Now $\delta/d \le 1$ and $\sin\theta < \pi/80 < 1/10$; hence we can apply $\log(1+x) \ge x - x^2/2 \ge 9x/10$ for $0 < x < 1/10$ to get

$$\frac{|z-\tau|^2}{|z|^2} \ge \exp\left(\frac{9}{10}\,\frac{9\delta\sin\theta}{8d}\right) > \exp\left(\frac{\delta\sin\theta}{d}\right), \qquad \tau \in J.$$

Applying this estimate for all the μ zeroes $z_j \in \mathcal{Z}_1$, we finally find

$$\prod_{z_j\in\mathcal{Z}_1}\left|\frac{z_j-\tau}{z_j}\right| \ge \exp\left(\frac{1}{2}\,\frac{\delta\mu\sin\theta}{d}\right), \qquad \tau = t\delta e^{i(\pi/2-2\theta)} \in J. \tag{4.9}$$

The estimate of the contribution of zeroes from $\mathcal{Z}_5$ is somewhat easier, as now the angle between z_j and τ exceeds $\pi/2$. By the cosine theorem again, we obtain for any $z = re^{i\varphi} \in S[\pi-\theta,\pi]\cap K$ the estimate

$$\begin{aligned} |z-\tau|^2 &= r^2 + t^2\delta^2 - 2rt\delta\cos\Bigl(\varphi - \Bigl(\frac{\pi}{2} - 2\theta\Bigr)\Bigr) \\ &\ge r^2 + t^2\delta^2 + 2rt\delta\sin\theta > r^2\left(1 + \frac{3\delta\sin\theta}{2d}\right), \qquad \tau \in J, \end{aligned} \tag{4.10}$$

as $t \ge 3/4$ and $r \le d$. Hence, using again $\delta/d \le 1$ and $(3/2)\sin\theta < (3/2)\pi/80 < 1/10$, we can again apply $\log(1+x) \ge 9x/10$ for $0 < x < 1/10$ to get

$$\frac{|z-\tau|}{|z|} \ge \exp\left(\frac{1}{2}\,\frac{9}{10}\,\frac{3\delta\sin\theta}{2d}\right) > \exp\left(\frac{\delta\sin\theta}{2d}\right), \qquad \tau \in J,$$

which then yields

$$\prod_{z_j\in\mathcal{Z}_5}\left|\frac{z_j-\tau}{z_j}\right| \ge \exp\left(\frac{\delta m\sin\theta}{2d}\right), \qquad \tau = t\delta e^{i(\pi/2-2\theta)} \in J. \tag{4.11}$$

Observe that the zeroes belonging to $\mathcal{Z}_2$ have the property that they fall to the opposite side of the line $\operatorname{Im}(e^{i2\theta}z) = 3\delta/8$ than J; hence they are closer to 0 than to any point of J. It follows that

$$\prod_{z_j\in\mathcal{Z}_2}\left|\frac{z_j-\tau}{z_j}\right| \ge 1, \qquad \tau = t\delta e^{i(\pi/2-2\theta)} \in J. \tag{4.12}$$

Next we use Chebyshev's Lemma H to estimate the contribution of zero factors belonging to $\mathcal{Z}_3$. We find

$$\max_{\tau\in J}\prod_{z_j\in\mathcal{Z}_3}\left|\frac{z_j-\tau}{z_j}\right| \ge 2\left(\frac{|J|}{4}\right)^{\kappa}\prod_{z_j\in\mathcal{Z}_3}\frac{1}{r_j} \ge \left(\frac{1}{20}\right)^{\kappa} > \exp(-3\kappa) \tag{4.13}$$

since $|J| = \delta/4$, $r_j \le 5\delta/4$, and $\log 20 = 2.9957\ldots < 3$.

Note that for any point $z = re^{i\varphi} \in B_{5\delta/4}(0) \cap \{\operatorname{Im}(e^{i2\theta}z) \ge 3\delta/8\}$ we must have

$$\frac{3\delta}{8} \le \operatorname{Im}(e^{i2\theta}re^{i\varphi}) = r\sin(\varphi+2\theta);$$

hence, by $r \le 5\delta/4$ also

$$\sin(\varphi+2\theta) \ge \frac{3\delta}{8r} \ge \frac{3}{10} \qquad \text{and} \qquad \sin\varphi \ge \sin(\varphi+2\theta) - 2\theta \ge \frac{3}{10} - \frac{\pi}{40} > \frac{1}{5}.$$

Applying this for all the zeroes $z_j \in \mathcal{Z}_3$, we are led to

$$1 \le \frac{5\delta/4}{r_j} \le \frac{25}{4}\delta\frac{\sin\varphi_j}{r_j}, \qquad z_j \in \mathcal{Z}_3. \tag{4.14}$$

On combining (4.13) with (4.14) and writing in $3\cdot 25/4<19$, we are led to

$$\max_{\tau\in J}\prod_{z_j\in\mathcal{Z}_3}\left|\frac{z_j-\tau}{z_j}\right|>\exp\biggl(-19\delta\sum_{z_j\in\mathcal{Z}_3}\frac{\sin\varphi_j}{r_j}\biggr). \tag{4.15}$$

Finally we consider the contribution of the zeroes from $\mathcal{Z}_4$, i.e., the "far" zeroes, for which we have $\operatorname{Im}(z_je^{2i\theta})\ge 3\delta/8$, $\varphi_j\in(\theta,\pi-\theta)$, and $|r_j|\ge 5\delta/4$. Put now $Z:=z_je^{2i\theta}=u+iv=re^{i(\varphi_j+2\theta)}$, and $s:=|\tau|=t\delta$, say. We then have

$$\begin{aligned}\left|\frac{z_j-\tau}{z_j}\right|^2&=\frac{|Z-t\delta i|^2}{r^2}=\frac{u^2+(v-s)^2}{r^2}=1-\frac{2vs}{r^2}+\frac{s^2}{r^2}>1-\frac{2vs}{r^2}+\frac{s^2}{r^2}\frac{v^2}{r^2}\\&=\Bigl(1-\frac{vs}{r^2}\Bigr)^2\ge\Bigl(1-\frac{|v|\delta}{r^2}\Bigr)^2=\Bigl(1-\frac{\delta|\sin(\varphi_j+2\theta)|}{r}\Bigr)^2.\end{aligned} \tag{4.16}$$

Recall that $\log(1-x)>-x-x^2/(2(1-x))\ge -3x$ whenever $0\le x\le 4/5$. We can apply this for $x:=\delta|\sin(\varphi_j+2\theta)|/r_j\le\delta/r_j\le 4/5$ using $r=r_j=|z_j|=|u+iv|\ge 5\delta/4$. As a result, (4.16) leads to

$$\left|\frac{z_j-\tau}{z_j}\right|\ge\exp\biggl(-3\delta\frac{|\sin(\varphi_j+2\theta)|}{r_j}\biggr), \tag{4.17}$$

and using $|\sin(\varphi_j+2\theta)|\le\sin\varphi_j+\sin 2\theta\le 3\sin\varphi_j$ (in view of $\varphi_j\in(\theta,\pi-\theta)$), we finally get

$$\prod_{z_j\in\mathcal{Z}_4}\left|\frac{z_j-\tau}{z_j}\right|\ge\exp\biggl(-9\delta\sum_{z_j\in\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}\biggr),\qquad \tau=t\delta e^{i(\pi/2-2\theta)}\in J. \tag{4.18}$$

If we collect estimates (4.9), (4.11), (4.12), (4.15), and (4.18), we find for a certain point of maximum $\tau_0\in J$ in (4.15) the inequality

$$\frac{|p(\tau_0)|}{|p(0)|}=\prod_{z_j\in\mathcal{Z}}\left|\frac{z_j-\tau_0}{z_j}\right|>\exp\biggl\{\frac{1}{2}\delta\frac{\mu+m}{d}\sin\theta-19\delta\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}\biggr\},$$

or, after taking logarithms and canceling by $\delta/2$,

$$\frac{2}{\delta}\log\left|\frac{p(\tau_0)}{p(0)}\right|\ge(\mu+m)\frac{\sin\theta}{d}-38\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}. \tag{4.19}$$

Observe that for the zeroes in $\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4$ we have $\sin\varphi_j>\sin\theta$, whence also

$$(\nu+\kappa+k)\frac{\sin\theta}{d}-\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}\le 0. \tag{4.20}$$

Adding (4.20) to the right hand-side of (4.19) and taking into account $\#\mathcal{Z}=\sum_{j=1}^5\#\mathcal{Z}_j$, we obtain

$$\frac{2}{\delta}\log\left|\frac{p(\tau_0)}{p(0)}\right|\ge\frac{\sin\theta}{d}n-39\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}. \tag{4.21}$$

Making use of (4.5) with the choice of $\mathcal{W}:=\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4$, we arrive at

$$\frac{2}{\delta}\log\left|\frac{p(\tau_0)}{p(0)}\right|\ge\frac{\sin\theta}{d}n-39\left|\frac{p'}{p}(0)\right|;$$

that is, writing in again the normalization $\zeta:=0$,

$$\left|\frac{p'}{p}(\zeta)\right|>\frac{1}{39}\frac{\sin\theta}{d}n-\frac{2}{39\delta}\log\left|\frac{p(\tau_0)}{p(\zeta)}\right|. \tag{4.22}$$

It remains to recall (4.1) and to estimate

$$\sin\theta = \sin\frac{\arctan(w/d)}{20}.$$

As $\theta \in (0, \pi/80]$, we have $\sin\theta > \theta(1-\theta^2/6) \geq \theta(1-\pi^2/38\,400) > (1-10^{-3})\theta$, and as $0 < w/d \leq 1$, we have $\arctan(w/d) \geq (w/d)(\pi/4)$, whence

$$\sin\theta \geq (1-10^{-3})\frac{\arctan(w/d)}{20} \geq \frac{(1-10^{-3})\pi}{80}\frac{w}{d} > 39\cdot 10^{-3}\frac{w}{d}.$$

If we substitute this last estimate into (4.22), we get

$$\left|\frac{p'}{p}(\zeta)\right| > 10^{-3}\frac{w}{d^2}n - \frac{2}{39\delta}\log\left|\frac{p(\tau_0)}{p(\zeta)}\right|,$$

and the lemma follows. $\square$

5. COMBINED ESTIMATE FOR VALUES OF THE LOGARITHMIC DERIVATIVE

Lemma 5. *Let $\zeta, \zeta' \in \partial K$, $\zeta \prec \zeta'$, and let (some) tangents (supporting lines to K) be given at these points as $t := \zeta + e^{i\alpha}\mathbb{R}$ and $t' := \zeta' + e^{i\alpha'}\mathbb{R}$, respectively, with the directional vectors $e^{i\alpha}$ and $e^{i\alpha'}$ oriented positively and $\alpha < \alpha' < \alpha + \pi$. Define the angle $\beta := \pi - (\alpha' - \alpha)$ and write $s := |\zeta' - \zeta|$. If $s \leq s_0 := s_0(\beta) := \min(1, 2\sin\beta)d/384$, then for any $p \in \mathcal{P}_n(K)$ we have the following alternative*:

(i) *either $|p(\zeta)|, |p(\zeta')| \leq 2^{-n}\|p\|_\infty$ (in particular, $\zeta, \zeta' \notin \mathcal{H}$ for $n \geq 15$),*

(ii) *or $|p'(\zeta)/p(\zeta)| + |p'(\zeta')/p(\zeta')| \geq 3\sin\beta\cdot n/(8d)$.*

Proof. Let $T := t \cap t'$. Then with the above notation we have $\angle(\zeta T \zeta') = \beta$. Moreover, since t and t' are tangents of K, we have $K \subset S$, where $S = S_T[\alpha', \pi - \alpha]$ is the sector with vertex T containing the chord $[\zeta, \zeta']$.

Let now $z \in K \subset S$ be arbitrary. We can describe the location of z with respect to each of the three points ζ, ζ', and T. Let us write

$$r := |z-\zeta|, \qquad r' := |z-\zeta'|, \qquad \rho := |z-T|,$$
$$\varphi := \angle(z\zeta T), \qquad \varphi' := \angle(z\zeta' T), \qquad \phi := \angle(zT\zeta), \qquad \phi' := \angle(zT\zeta').$$

Then, of course, $\phi + \phi' = \beta$. Now, if the distances (heights) of z from the tangents are $m := \operatorname{dist}(z,t)$ and $m' := \operatorname{dist}(z,t')$, then we have $m = r\sin\varphi = \rho\sin\phi$ and $m' = r\sin\varphi' = \rho\sin\phi'$. If we further take $\rho \geq R := 3\max(|T-\zeta|, |T-\zeta'|)$, then we also have $r, r' \leq d$ and $r \leq \rho + |\zeta - T| \leq (4/3)R\rho$, $r' \leq \rho + |\zeta' - T| \leq (4/3)\rho$, whence

$$\frac{\sin\varphi}{r} + \frac{\sin\varphi'}{r'} = \frac{r\sin\varphi}{r^2} + \frac{r'\sin\varphi'}{r'^2} = \frac{\rho\sin\phi}{r^2} + \frac{\rho\sin\phi'}{r'^2} \geq \frac{3}{4d}(\sin\phi + \sin\phi')$$
$$= \frac{3}{2d}\sin\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2} \geq \frac{3}{2d}\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \frac{3\sin\beta}{4d}.$$

Denote now the subset of zeroes of K which lie at least R far from T by $\mathcal{Z}(R)$, i.e., write $\mathcal{Z}(R) := \mathcal{Z} \setminus B_R(T)$. Then for any $z_j \in \mathcal{Z}(R)$ we have $|z_j - T| \geq R$ and $z_j \in K \subset S$, i.e., the above calculation is valid, and we obtain

$$\frac{\sin(\arg(z_j-\zeta)-\alpha)}{|z_j-\zeta|} + \frac{\sin(\arg(z_j-\zeta')-\alpha')}{|z_j-\zeta'|} \geq \frac{3\sin\beta}{4d}.$$

Observe that the terms here are the general terms for the expression $\operatorname{Im}(e^{-i\alpha}/(\zeta - z_j)) + \operatorname{Im}(e^{-i\alpha'}/(\zeta' - z_j))$, whence denoting $\nu := \#\mathcal{Z}(R)$ we get similarly to (4.5)

$$\begin{aligned}\left|\frac{p'}{p}(\zeta)\right| + \left|\frac{p'}{p}(\zeta')\right| &\geq \operatorname{Im}\left(\frac{p'}{p}(\zeta)e^{-i\alpha}\right) + \operatorname{Im}\left(\frac{p'}{p}(\zeta')e^{-i\alpha'}\right) \\ &= \sum_{j=1}^{n}\left\{\operatorname{Im}\frac{e^{-i\alpha}}{\zeta - z_j} + \operatorname{Im}\frac{e^{-i\alpha'}}{\zeta' - z_j}\right\} \geq \sum_{z\in\mathcal{Z}(R)} \frac{3\sin\beta}{4d} = \frac{3\sin\beta}{4d}\nu. \end{aligned} \tag{5.1}$$

Next we would like to estimate the number $\mu := n - \nu$ of zeroes of a fixed $p \in \mathcal{P}_n(K)$ in $\mathcal{Z} \setminus \mathcal{Z}(R)$, i.e., in $\mathcal{Z}^* := \mathcal{Z} \cap B_R(T)$. Recall that $K \subset S$, whence also $\mathcal{Z}^* \subset K^* := K \cap B_R(T) \subset S \cap B_R(T) =: S^*$. For the sectorial part S^* of $B_R(T)$ the diameter is either the radius or the chord, depending on the central angle (on whether it exceeds $\pi/3$); so we obtain $d^* := \operatorname{diam}(K^*) \leq \operatorname{diam}(S^*) = \max(R, 2R\sin(\beta/2))$. Recall the definition of R as three times the maximum of the two sides from T of the triangle $\triangle(T\zeta\zeta')$. Calculating from the sine theorem, we thus obtain $R \leq 3s/\sin\beta$, and, moreover, in the case $\beta > \pi/2$ we even get $R \leq 3s$ (as then the side $\overline{\zeta\zeta'}$, opposite to the largest angle $\beta > \pi/2$, is necessarily the longest side of the triangle). On combining these estimates, we finally get

$$d^* \leq \begin{cases} \dfrac{3}{\sin\beta}s & \text{if } \beta \leq \dfrac{\pi}{3}, \\ \dfrac{3}{\cos(\beta/2)}s \leq 3\sqrt{2}s & \text{if } \dfrac{\pi}{3} \leq \beta \leq \dfrac{\pi}{2}, \\ 3\max\left(1, 2\sin\dfrac{\beta}{2}\right)s \leq 6s & \text{if } \beta > \dfrac{\pi}{2}, \end{cases} \qquad d^* \leq \frac{6}{\min(1, 2\sin\beta)}s.$$

Now let us assume that $s \leq s_0$. This means that

$$d^* \leq \frac{6}{\min(1, 2\sin\beta)}\frac{\min(1, 2\sin\beta)}{384}d = \frac{d}{64}.$$

Consider now the case when the number μ of zeroes of p in K^* is at least $n/2$; we claim that then (i) of the stated alternative holds true.

The condition $\mu \geq n/2$ can be written with $k = 64$ as $\mu \geq 3\log 2/\log kn$. Therefore, an application of Lemma 3 with $k = 64$ provides that we necessarily have $|p(\zeta)|$ and $|p(\zeta')|$ rather small, smaller than $2^{-n}\|p\|_\infty$, proving the first part of the claim in (i). (In fact, in this case we also find that the same must hold throughout all of K^*, so in particular at all points of the arc $\widetilde{\zeta\zeta'}$ between ζ and ζ'.)

As for the second part of (i), we certainly have $\zeta, \zeta' \notin \mathcal{H}$ whenever $cn^{-2/q} \geq 2^{-n}$ with the constant c defined in (3.2); reformulating, it suffices to have

$$\frac{1}{2}\left(\frac{1}{n^2 8\pi(q+1)}\right)^{1/q} \geq 2^{-n}.$$

As $q \geq 1$ and the left-hand side is easily seen to increase as a function of $q \geq 1$, it suffices to show this for $q = 1$; and for $q = 1$ the inequality becomes $2^n/n^2 \geq 32\pi$, which holds for $n \geq 15$, as for $n \geq 15$ the left-hand side is an increasing function of n and its value at $n = 15$ is $2^{15}/15^2 > 2^{15}/16^2 = 32 \cdot 4 > 32\pi$. Thus (i) is satisfied, concluding the proof in this case.

In the other case (when $\mu < n/2$), however, we must have $\nu \geq n/2$. Therefore, in this case (5.1) furnishes (ii) of the stated alternative and the proof concludes in this case as well. $\square$

6. PROOF OF THEOREM 1

In this section we prove the main result of the paper, that is, Theorem 1. More precisely, we also get the following explicit estimate of $M_{n,q}$ for large values of n.

Theorem 1′. *Let $K \Subset \mathbb{C}$ be any compact convex domain. Then for any $q \geq 1$, $n \geq n_0(K) = \max(10^{21}, d^5/w^5)$ and all $p \in \mathcal{P}_n(K)$, we have*

$$\|p'\|_q \geq \frac{1}{24 \cdot 10^4} \frac{w^2}{d^3} \frac{n}{\log n} \|p\|_q, \qquad \text{i.e.,} \qquad M_{n,q} \geq \frac{1}{24 \cdot 10^4} \frac{w^2}{d^3} \frac{n}{\log n}.$$

Proof. The proof is divided into four parts. In Subsection 6.1 we introduce the set $\mathcal{G}$ of "good points," for which the tilted normal estimate (4.3) may be applied. In Subsection 6.2 we construct and describe a set $\mathcal{L}$ which covers $\Gamma \setminus \mathcal{G}$. The integral of $|p'|^q$ over $\mathcal{H}$ is estimated in Subsections 6.3 and 6.4. These subsections are computationally quite expansive.

6.1. The subset $\mathcal{G}$ of "good points." As before in Lemma 4, we fix here the angle θ as $\theta := \arcsin(w/d)/80$, so that the angle $\varphi := \pi/2 - 2\theta$ will satisfy $2\pi/5 < \varphi < \pi/2$. Further, we will fix a parameter $r > 0$. The value of this will be of the order $\log n/n$, so very small for n large. Later in Subsection 6.3 the "tilted normal estimate" of Lemma 4 will be applied with chord lengths at least r.

At the outset we will assume $r < w/108$; later we will need more restrictions on r, but $r < w/108$ will certainly be satisfied. In all, our condition on r will be expressed as $0 < r \leq r_0 = r_0(w,d)$, so that r_0 depends only on the parameters $d = d_K$ and $w = w_K$, but not on anything else, in particular not on the degree n. On the other hand, we will also assume that $n \geq n_0 := n_0(w,d)$, again depending only on w and d, and on nothing else. The bound r_0 will be specified later; n_0 will be equal to the $n_0(K)$, already set in the assertion of Theorem 1′.

At any point $\zeta \in \partial K = \Gamma$ one can consider all normals and the respective tangents; and the tilted normal lines with angles φ from the tangent *measured from half-lines of the tangent lines*, i.e., with $\pm 2\theta = \pm(\pi/2 - \varphi)$ from the (inner) normal directions.

If any of these tilted lines intersects K along a sufficiently long chord, i.e., along a chord at least as long as r, or if one of the tilted lines does not intersect the interior of K at all, then we will apply the "tilted normal estimate" (4.3) of Lemma 4. Later in Subsection 6.3 we will see how for these points an application of Lemma 4 suffices. So we will call these points good points, the full set of good points being $\mathcal{G} \subset \Gamma$. Our main concern will be to deal with points in $\Gamma \setminus \mathcal{G}$.

6.2. The subsets $\mathcal{F}$ and $\mathcal{L}$ of ∂K.

6.2.1. *The family $\mathcal{I}$ of "elementary small arcs."* Once $\zeta \notin \mathcal{G}$, it means that for any (outer) normal direction $\boldsymbol{\nu} = -e^{i\sigma}$ to K at ζ, at least one of the two chords $(\zeta + e^{i(\sigma \pm 2\theta)}\mathbb{R}) \cap K$ is short—shorter than r. As stated in Propositions 2 and 3, the "small part of Γ" and the respective small part of K, encircled by this part of the boundary, are always proportional to the chord length $\delta = \delta(\zeta, \sigma \pm 2\theta) = |(\zeta + e^{i(\sigma \pm 2\theta)}\mathbb{R}) \cap K|$ and hence are small, too.

Together with $\zeta \in \Gamma \setminus \mathcal{G}$, there is thus a chord line ℓ of direction $\pm 2\theta$ from the inner normal direction $e^{i\sigma}$ with $\ell \cap K = [\zeta, D]$ with $D := D(\zeta, \sigma \pm 2\theta) \in \Gamma$ and $|D - \zeta| = \delta(\zeta, \sigma \pm 2\theta) < r$. Consider the smaller arc of Γ bounded by these two points ζ and D (endpoints included). Such arcs will be called *elementary small arcs*; thus $\zeta \notin \mathcal{G}$ if and only if ζ defines such an elementary small arc I, for which it is one of the endpoints and so in particular $\zeta \in I$. The family of all such elementary small subarcs will be denoted by $\mathcal{I}$, and their union, by $\mathcal{E}$, so that $\mathcal{E} := \bigcup_{I \in \mathcal{I}} I$. We clearly have $\Gamma \setminus \mathcal{G} \subset \mathcal{E}$. Note that the name "elementary *small* arc" is well justified because any elementary small arc $I \subset \Gamma$ is of length not exceeding $4dr/w$ in view of Proposition 2(iv).

However small these arcs $I \in \mathcal{I}$ are, they exhibit certain largeness, too. Namely, along any elementary small subarc I the total variation $\mathrm{Var}[\alpha, I]$ of α is at least φ. To see this, assume that

$I = \widetilde{\zeta D} := \{P \in \Gamma\colon \zeta \prec P \prec D\}$, say, with the chord $[\zeta, D] = K \cap (\zeta + e^{i(\sigma-2\theta)})\mathbb{R}$ small. (The other case when $I = \widetilde{D\zeta}$, i.e., $D \prec \zeta$ and $[\zeta, D] = K \cap (\zeta + e^{i(\sigma+2\theta)})\mathbb{R}$ small, is entirely symmetrical again.) So the (positively oriented) directional angle of (some) tangent t at ζ is $\sigma - \pi/2$, the chord $\overrightarrow{\zeta D}$ is of direction $\sigma - 2\theta$, and by convexity of Γ any tangent t' at D must have direction $\alpha' \geq \sigma - 2\theta$. Thus we indeed find $\mathrm{Var}[\alpha, I] = \alpha_+(D) - \alpha_-(\zeta) \geq \alpha' - (\sigma - \pi/2) \geq (\sigma - 2\theta) - (\sigma - \pi/2) = \varphi$.

As an immediate result, there can be *at most four disjoint such subarcs* in Γ: for already along five disjoint elementary subarcs the total variation would be at least $5\varphi > 2\pi = \mathrm{Var}[\alpha, \Gamma]$, a contradiction. So let us choose, once and for all, a *maximal family* of such *disjoint* elementary small subarcs I_j, $j = 1, \dots, k$, $k \leq 4$. We of course have then only $\mathcal{F} := \bigcup_{j=1}^k I_j \subset \mathcal{E}$, and cannot state that even $\mathcal{F}$ covers $\Gamma \setminus \mathcal{G}$, but on the other hand, we know that any point $z \in \mathcal{E}$ belongs to some elementary arc $I \in \mathcal{I}$, which intersects some of these I_j. As we have $|I| \leq 4rd/w$, always, it means that a point $z \in \mathcal{E}$ cannot be farther (measured in arc length along Γ) from $\mathcal{F}$ than $4rd/w$.

6.2.2. *A covering $\mathcal{L}$ of the sets $\mathcal{F}$ and $\mathcal{E}$.* In view of the above, extending each I_j along Γ *in both directions* by $4dr/w$ in arc length, we obtain a subset

$$\mathcal{L} := \left\{ z \in \Gamma\colon \ \mathrm{dist}(z, \mathcal{F}) \leq \frac{4rd}{w} \right\} \subset \Gamma,$$

which will contain all points of $\mathcal{E}$, and thus also cover $\Gamma \setminus \mathcal{G}$ again. So we find $\mathcal{G} \cup \mathcal{L} = \Gamma$. Let us also record right here that the total arc length measure of the so constructed set $\mathcal{L}$ is $|\mathcal{L}| \leq 4 \cdot 12rd/w = 48rd/w$.

So there are points in $\Gamma \setminus \mathcal{L}$, and fixing one such point $C \in \Gamma \setminus \mathcal{L}$, we can start the parametrization of Γ from that point. It means that $\gamma\colon [0, L] \to \Gamma$ will define a unique ordering of points of $\Gamma \setminus \{C\}$, so in particular of points of $\mathcal{L}$. In this ordering let us write $I_j = \widetilde{P_j P_j'}$; so it can be the case that $P_j' = D(P_j)$, but also that $P_j = D(P_j')$, depending on the initial point of small chord length in the construction of the arc.

So, $\mathcal{L} := \bigcup_{j=1}^k \widetilde{Q_j Q_j'}$, where $Q_j \prec P_j \prec P_j' \prec Q_j'$, and the arc length measures are $|\widetilde{Q_j P_j}| = 4rd/w$, $|\widetilde{P_j P_j'}| \leq 4rd/w$, $|\widetilde{P_j' Q_j'}| = 4rd/w$, and altogether $|\widetilde{Q_j Q_j'}| \leq 12rd/w$.

There is only a slight technicality here: these vicinities $\widetilde{Q_j Q_j'}$ of the disjoint small elementary arcs $I_j = \widetilde{P_j P_j'}$ need not remain disjoint. However, *no three of them may chain together.* Indeed, assume this to happen: that would result in a subarc Γ' of Γ, altogether not longer than $3 \cdot 12rd/w = 36rd/w$, with a total variation of the tangent direction already exceeding $3\varphi > 6\pi/5$. This is, however, impossible. Indeed, then the tangents t and t' at the endpoints $\zeta \prec \zeta'$ of Γ' would intersect at a point T on the other side of Γ', and the triangle $\triangle = \mathrm{con}(T, \zeta, \zeta')$ would contain $\Gamma \setminus \Gamma'$; and then using

$$\mathrm{diam}(\Gamma \setminus \Gamma') \leq \mathrm{diam}(\triangle) \leq \frac{1}{\sin(\pi/5)} |\zeta' - \zeta| < 2|\zeta' - \zeta|,$$

we would get

$$\begin{aligned}\mathrm{diam}(K) = \mathrm{diam}(\Gamma) &\leq \mathrm{diam}(\Gamma \setminus \Gamma') + \mathrm{diam}(\Gamma') < 2|\zeta' - \zeta| + \mathrm{diam}(\Gamma') \\ &\leq 2|\Gamma'| + |\Gamma'| \leq 3\frac{36rd}{w} = \frac{108rd}{w} < d,\end{aligned}$$

a contradiction.

In view of the above, $\mathcal{L} = \bigcup_{m=1}^{k_0} \mathcal{A}_m$, where $\mathcal{A}_m$ are to denote the connected components of $\mathcal{L}$, their number is $k_0 \leq k \leq 4$, and each of the components has arc length exceeding $8rd/w$. More precisely, each of the connected components (arcs) of $\mathcal{L}$ consists of some (one or two) of the prefixed disjoint elementary arcs $I_j = \widetilde{P_j P_j'}$—among which we can now choose one arbitrarily, if there are

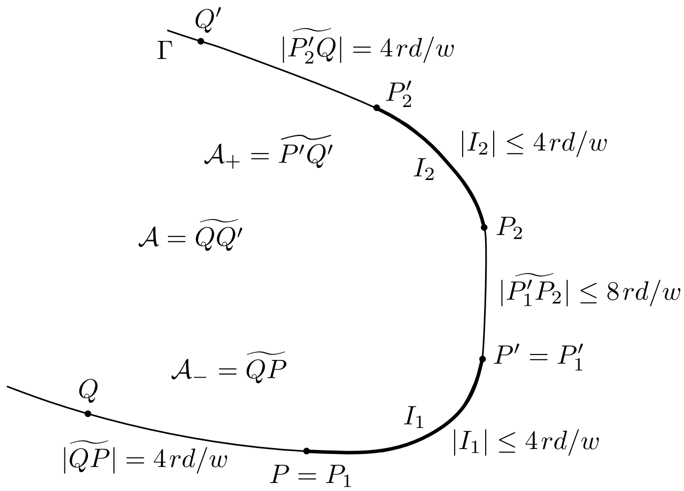

Fig. 2. One connected component $\mathcal{A}$ of the set $\mathcal{L}$.

two—and also some part preceding and some other part following this selected elementary arc. For one arbitrary connected component $\mathcal{A}_m$ of $\mathcal{L}$ we will thus write $\mathcal{A}_m = \widetilde{Q_mQ'_m}$ with $Q_m \prec P_{j(m)} \prec P'_{j(m)} \prec Q'_m$ with the parts $\mathcal{A}_{m-} := \widetilde{Q_mP_{j(m)}}$ and $\mathcal{A}_{m+} := \widetilde{P'_{j(m)}Q'_m}$ having arc length measure at least $4rd/w$ and at most $16rd/w$, and the intermediate ("central") part $I_{j(m)} = \widetilde{P_{j(m)}P'_{j(m)}}$ of arc length measure at most $4rd/w$; and in all,

$$\frac{8rd}{w} < |\mathcal{A}_m| \le \frac{24rd}{w}, \qquad m = 1, \dots, k_0. \tag{6.1}$$

Let us briefly summarize our construction of subsets of ∂K. We started with points $\zeta \notin \mathcal{G}$, considered the elementary short subarcs $I = \widetilde{\zeta D}$ or $\widetilde{D\zeta}$ generated by any such ζ, and took the union $\mathcal{E} := \bigcup_{I\in\mathcal{I}} I$ of these subarcs, obviously covering $\Gamma \setminus \mathcal{G}$. Next, we selected a maximal disjoint subset of elementary subarcs and their union $\mathcal{F} := \bigcup_{j=1}^{k} I_j$, which is only a subset of $\mathcal{E}$; but then took a proper neighborhood $\mathcal{L}$ of $\mathcal{F}$ to cover $\mathcal{E}$, and whence also $\Gamma \setminus \mathcal{G}$ again. The advantage of these steps back and forth is that the resulting set $\mathcal{L}$ not only covers $\mathcal{E} \supset \Gamma \setminus \mathcal{G}$ but also has a manageable structure: it consists of $k_0 \le k \le 4$ connected subarcs $\mathcal{A}_m$ of Γ, all of which have arc length measure between $8rw/d$ and $24rd/w$, and each of which is easily divided into three parts: one selected elementary small subarc $I_{j(m)}$ from the disjoint system $\{I_j\}_{j=1}^{k}$ as a "central part," and the preceding and following parts $\mathcal{A}_{m-}$ and $\mathcal{A}_{m+}$, both of size of order rd/w, too. It is important that the "central part" exhibits a change of the tangent angle function at least φ, and its arc length is bounded by that of the surrounding parts (i.e., of order rd/w). One such connected subarc $\mathcal{A} := \mathcal{A}_m$ is depicted in Fig. 2.

We have already seen in Lemma 2 that $\int_{\mathcal{H}}|p|^q \ge (1/2)\int_{\Gamma}|p|^q$. In the rest of the proof, we distinguish two cases:

$$\text{(I)} \quad \int\limits_{\mathcal{H}\cap\mathcal{L}} |p|^q \le \frac{1}{2}\int\limits_{\mathcal{H}} |p|^q; \qquad \text{(II)} \quad \int\limits_{\mathcal{H}\cap\mathcal{L}} |p|^q > \frac{1}{2}\int\limits_{\mathcal{H}} |p|^q.$$

6.3. Case (I). Note that in this case we have

$$\int\limits_{\mathcal{H}\setminus\mathcal{L}} |p|^q \ge \frac{1}{2}\int\limits_{\mathcal{H}} |p|^q \ge \frac{1}{4}\int\limits_{\Gamma} |p|^q.$$

So let then $\zeta \in \mathcal{H} \setminus \mathcal{L}$ be any point. As $\zeta \notin \mathcal{L}$, it follows that $\zeta \in \mathcal{G}$, that is, its "tilted normal" chord length is $\delta(\zeta, \sigma \pm 2\theta) \geq r$. So, from (4.3) of Lemma 4 we get the estimate

$$|p'(\zeta)| \geq \left(10^{-3}\frac{w}{d^2}n - \frac{3}{20r}\log n\right)|p(\zeta)|, \qquad n \geq 73. \tag{6.2}$$

Therefore, if

$$r := r(n) := 300\frac{d^2}{w}\frac{\log n}{n}, \tag{6.3}$$

then $|p'(\zeta)| \geq 5 \cdot 10^{-4} w d^{-2} n |p(\zeta)|$, and we get

$$\int\limits_{\mathcal{H}\setminus\mathcal{L}} |p'|^q \geq \left(5 \cdot 10^{-4}\frac{w}{d^2}n\right)^q \int\limits_{\mathcal{H}\setminus\mathcal{L}} |p|^q \geq \frac{1}{4}\left(5 \cdot 10^{-4}\frac{w}{d^2}n\right)^q \int\limits_{\Gamma} |p|^q.$$

It follows that in this case

$$\|p'\|_q \geq \left(\int\limits_{\mathcal{H}\setminus\mathcal{L}} |p'|^q\right)^{1/q} \geq \frac{5 \cdot 10^{-4}}{4^{1/q}}\frac{w}{d^2}n\left(\int\limits_{\Gamma}|p|^q\right)^{1/q} > 10^{-4}\frac{w}{d^2}n\|p\|_q, \tag{6.4}$$

which closes the argument for all n sufficiently large (so that $r(n) < r_0$ holds). Note that in this case we obtained a constant times n oscillation, not only of order $\log n/n$.

6.4. Case (II). In the remaining other case we have

$$\int\limits_{\mathcal{H}\cap\mathcal{L}} |p|^q > \frac{1}{2}\int\limits_{\mathcal{H}} |p|^q; \qquad \text{therefore,} \qquad \int\limits_{\mathcal{L}} |p|^q \geq \int\limits_{\mathcal{L}\cap\mathcal{H}} |p|^q \geq \frac{1}{2}\int\limits_{\mathcal{H}} |p|^q \geq \frac{1}{4}\int\limits_{\Gamma}|p|^q.$$

Recall that $\mathcal{L}$ consists of $k_0 \leq 4$ arcs, each of length between $8rd/w$ and $24rd/w$. Let us select one arc $\mathcal{A}_m = \widetilde{Q_m Q'_m}$, where $\int_{\mathcal{A}_m}|p|^q$ is *maximal* among these at most four arcs. To relax notation, from now on let us drop the indices m and $m(j)$ and write $\mathcal{A}$ for $\mathcal{A}_m$, P for $P_{j(m)}$, Q for Q_m, etc. So, as it was said before, we fix one elementary small subarc $I = I_{j(m)} = \widetilde{PP'} \subset \mathcal{A}$ as the "central part" of $\mathcal{A}$ and write $\mathcal{A}_- := \widetilde{QP}$ and $\mathcal{A}_+ := \widetilde{P'Q'}$ for the parts preceding and following it, respectively.

By construction, we necessarily have

$$\int\limits_{\mathcal{A}} |p|^q \geq \frac{1}{4}\int\limits_{\mathcal{L}}|p|^q \geq \frac{1}{16}\int\limits_{\Gamma}|p|^q. \tag{6.5}$$

We also put

$$u := \min_{\mathcal{A}}|p(z)| \qquad \text{and} \qquad v := \max_{\mathcal{A}}|p(z)| = \|p\|_{L^\infty(\mathcal{A})}.$$

With these quantities, we consider two subcases next:

$$(\mathrm{II}_1) \quad 2u < v; \qquad (\mathrm{II}_2) \quad 0 < u \leq v \leq 2u.$$

6.4.1. *Subcase* (II_1): $2u < v$. We estimate the integrals using the Hölder inequality and the trivial estimation of the variation of p on $\mathcal{A}$ as follows:

$$\begin{aligned} |\mathcal{A}|^{1-1/q}\left(\int\limits_{\mathcal{A}}|p'|^q\right)^{1/q} &\geq \int\limits_{\mathcal{A}}|p'| \geq |v-u| \geq \frac{v}{2} = \frac{1}{2}\|p\|_{L^\infty(\mathcal{A})} \\ &\geq \frac{1}{2}\left(\frac{1}{|\mathcal{A}|}\int\limits_{\mathcal{A}}|p|^q\right)^{1/q} = \frac{1}{2}|\mathcal{A}|^{-1/q}\left(\int\limits_{\mathcal{A}}|p|^q\right)^{1/q}; \end{aligned} \tag{6.6}$$

i.e., we obtain

$$2|\mathcal{A}|\left(\int_{\mathcal{A}}|p'|^q\right)^{1/q} \geq \left(\int_{\mathcal{A}}|p|^q\right)^{1/q} \geq \left(\frac{1}{16}\int_{\Gamma}|p|^q\right)^{1/q},$$

and so with $r = r(n)$

$$\|p'\|_q \geq \left(\int_{\mathcal{A}}|p'|^q\right)^{1/q} \geq \frac{1}{2|\mathcal{A}|}\left(\frac{1}{16}\int_{\Gamma}|p|^q\right)^{1/q} \geq \frac{\|p\|_q}{16^{1/q}\cdot 2\cdot 24rd/w}$$
$$> \frac{w}{800\,d\,r(n)}\|p\|_q = \frac{w}{800\,d\cdot 300\,d^2w^{-1}n^{-1}\log n}\|p\|_q = \frac{1}{24\cdot 10^4}\frac{w^2}{d^3}\frac{n}{\log n}\|p\|_q. \tag{6.7}$$

6.4.2. *Subcase* (II$_2$): $0 < u \leq v \leq 2u$. We will consider any two points $\zeta \in \mathcal{A}_-$ and $\zeta' \in \mathcal{A}_+$. Note that between these two points there lies an elementary small subarc I_j, whence if t and t' are tangent lines at ζ and ζ' to K, respectively, with directional angles α and α', respectively, then we necessarily have $\alpha' - \alpha \geq \varphi$.

This is the place where we need Lemma 5. The distance between the points is $s := |\zeta - \zeta'| \leq |\mathcal{A}| \leq 24rd/w$, which must not exceed s_0 from the condition of Lemma 5. For the angle $\beta := \pi - (\alpha' - \alpha)$ we already know by construction that $\beta \leq \pi - \varphi \leq 3\pi/5$, so $\sin\beta \geq 1/2$ unless $\beta < \pi/6$. Therefore, we need to care for small β only. However, according to Proposition 2 from Section 2, we also know that $\beta > \arcsin((w-s)/d)$, whence $\sin\beta \geq (w-s)/d$ whenever $0 < \beta < \pi/2$. So altogether we find that $\sin\beta \geq \min(1/2, (w-s)/d) \geq w/(2d)$ if we also assume $s \leq w/2$.

At this point we need to specify a sufficient condition in terms of r for the chord length $s := |\zeta' - \zeta|$ to stay below $\min(s_0, w/2)$: it suffices to have

$$\frac{24rd}{w} \leq \frac{w}{384} \quad \left(\leq \frac{\min(1, 2\sin\beta)}{384}d\right), \qquad \text{that is,} \qquad r \leq \frac{1}{24\cdot 384}\frac{w^2}{d},$$

so, for example, $r \leq r_1 := 10^{-4}w^2/d$ (which is much smaller than the initial bound $w/108$).

The alternative of the said Lemma 5 has (i) with rather small values of the polynomial p. However, the variation of the values all over $\mathcal{A}$ remains within a factor of 2 in our case. Thus we conclude that even for the maximum we must have $v := \|p\|_{L^\infty(\mathcal{A})} \leq 2^{1-n}\|p\|_{L^\infty(K)}$. So from the above

$$\|p\|^q_{L^q(\partial K)} = \int_{\Gamma}|p|^q \leq 16\int_{\mathcal{A}}|p|^q \leq 16|\mathcal{A}|v^q \leq 16\cdot 24\frac{d}{w}\cdot 300\frac{d^2}{w}\frac{\log n}{n}\cdot 2^{q-qn}\|p\|^q_{L^\infty(K)}.$$

Next we show that this is not possible. And indeed, according to the Nikolskii-type estimate of Lemma 1, we must have $\|p\|_{L^q(K)} \geq (d/(2(q+1)))^{1/q}\|p\|_{L^\infty(K)}n^{-2/q}$; so, combining it with the latter formula, we get

$$\|p\|^q_{L^q(\partial K)} \leq 16\cdot 24\frac{d}{w}\cdot 300\frac{d^2}{w}\frac{\log n}{n}\cdot 2^{q-qn}\frac{2(q+1)}{d}\|p\|^q_{L^q(\partial K)}n^2,$$

that is

$$\frac{2^{qn}}{n\log n} \leq 16\cdot 24\cdot 300\frac{d^2}{w^2}\cdot 2^{q+1}(q+1),$$

which clearly fails for n large enough.

To be more precise, consider any fixed $n \ge 73$ and the function $2^{(n-1)q}/(q+1)$; then this is clearly an increasing function of $q \in [1,\infty)$, so it suffices to establish a contradiction with $q = 1$; and so it suffices to demonstrate a contradiction with

$$\frac{2^n}{n\log n} \le 16\cdot 24\cdot 300\frac{d^2}{w^2}\cdot 8 = 900\cdot 2^{10}\frac{d^2}{w^2} = 921\,600\frac{d^2}{w^2}$$

for sufficiently large n. As for the left-hand side, we can write $2^x/(x\log x) > 2^{x/2}f(x)$ with $f(x) := 2^{x/2}/(x\log x)$, and the latter is an increasing function of the variable x for $x \ge 73$, so we have $f(x) \ge f(73) > 310\,290\,286$, $x \ge 73$. Therefore, the desired contradiction will arise if $2^{n/2}f(73) > 921\,600\,d^2/w^2$, in particular, if $2^{n/2} > d^2/w^2$. Actually, it suffices then to take

$$n \ge n_1 := \max\biggl(73, 6\log\frac{d}{w}\biggr).$$

It remains to consider alternative (ii) of Lemma 5. As is clarified above, we already know that the occurring angle β satisfies $\arcsin(w-s)/d \le \beta \le 3\pi/5$, so this alternative of the assertion of Lemma 5 works with $\sin\beta \ge w/2d$ for sure. That is, we have

$$\left|\frac{p'}{p}(\zeta)\right| + \left|\frac{p'}{p}(\zeta')\right| \ge \frac{3\sin\beta}{8d}n \ge \frac{3}{16}\frac{w}{d^2}n \qquad \forall\,\zeta\in\mathcal{A}_-,\quad \forall\,\zeta'\in\mathcal{A}_+.$$

In particular, if there is any point $\zeta \in \mathcal{A}_-$ with $|p'(\zeta)/p(\zeta)| \le (3/32)wd^{-2}n$, then we must have $|p'(\zeta')/p(\zeta')| \ge (3/32)wd^{-2}n$ *all over* $\mathcal{A}_+$, and, conversely, if there is such a “small value point” on $\mathcal{A}_+$, then we must have this lower estimation all over $\mathcal{A}_-$. So, either both subarcs $\mathcal{A}_-$ and $\mathcal{A}_+$ satisfy this lower estimation, or at least one of them must satisfy it at all of its points. So assume, as we may, that $\mathcal{A}_+$ satisfies this lower estimation; this yields

$$\int\limits_{\mathcal{A}_+}|p'|^q \ge \left(\frac{3}{32}\frac{w}{d^2}n\right)^q\int\limits_{\mathcal{A}_+}|p|^q.$$

Recall that $|\mathcal{A}_+| \ge 4rd/w$, while $|\mathcal{A}| \le 24rd/w$, and that $u \le |p(z)| \le v \le 2u$ holds all over $\mathcal{A}$. These furnish

$$\int\limits_{\mathcal{A}_+}|p|^q \ge |\mathcal{A}_+|u^q \ge \frac{4rd}{w}2^{-q}v^q \ge \frac{|\mathcal{A}|}{8}2^{-q}v^q \ge 2^{-q-3}\int\limits_{\mathcal{A}}|p|^q \ge 2^{-q-7}\int\limits_{\Gamma}|p|^q,$$

using also (6.5), established at the beginning of the case under consideration.

So in all, we are led to

$$\|p'\|_q^q \ge \int\limits_{\mathcal{A}_+}|p'|^q \ge \left(\frac{3}{32}\frac{w}{d^2}n\right)^q\int\limits_{\mathcal{A}_+}|p|^q \ge \left(\frac{3}{64}\frac{w}{d^2}n\right)^q\frac{1}{2^7}\|p\|_q^q.$$

So in this case, we arrive at

$$\|p'\|_q \ge \frac{3}{64\cdot 2^{7/q}}\frac{w}{d^2}n\|p\|_q > \frac{3}{10^4}\frac{w}{d^2}n\|p\|_q. \tag{6.8}$$

Collecting the above estimates (6.8), (6.4), and (6.7), we arrive at

$$\|p'\|_q \ge \min\biggl(\frac{3}{10^4}\frac{w}{d^2}n,\ 10^{-4}\frac{w}{d^2}n,\ \frac{1}{24\cdot 10^4}\frac{w^2}{d^3}\frac{n}{\log n}\biggr)\|p\|_q = \frac{1}{24\cdot 10^4}\frac{w^2}{d^3}\frac{n}{\log n}\|p\|_q$$

provided that all our conditions are met: $r = r(n) \le r_1 := 10^{-4}w^2/d$ and $n \ge n_1 := \max(73, 6\log(d/w))$. Note that here $r(n) := 300d^2w^{-1}n^{-1}\log n$ depends on n, decreases to zero, and it suffices to find an index $n_0 \ge n_1$ such that at $n = n_0$, and hence for all $n \ge n_0$ too, the inequality $r(n) \le r_1$ holds. That is, we want

$$300\frac{d^2}{w}\frac{\log n}{n} \le 10^{-4}\frac{w^2}{d}, \qquad \text{or equivalently} \qquad \frac{n}{\log n} \ge 3\cdot 10^6\frac{d^3}{w^3}.$$

For example, if $n_0 := \max(10^{21}, d^5/w^5)$ and $n \ge n_0$, then we certainly have $n/\log n \ge n^{11/12} > n^{13/42}n^{3/5} \ge 10^{6.5}d^3/w^3 > 3\cdot 10^6 d^3/w^3$. Therefore, for any $n \ge n_0$—which is much larger than the previously found bound n_1—the required estimate $r(n) < r_1$ is satisfied, whence all the above arguments hold true and Theorem 1′ follows. □

7. CONCLUDING REMARKS

The proof of our main result shows that we can even reach the cn order of oscillation, and even pointwise estimates, apart from the set of critical small elementary arcs, where the intersection of the (tilted) normal with K is very small, smaller than $c\log n/n$, but not zero (as in the case of $\zeta = D$ we still have an order n lower estimate at ζ, see Lemma 4(ii)). However, when $n \to \infty$, the quantity $\log n/n$ tends to zero, and in fact we see that for most domains the set of critical elementary small arcs becomes empty for n large.

More precisely, one can do the following. Let $\mathcal{E} := \mathcal{E}(\varphi, r)$ be the union of all (closed) elementary small arcs $\widetilde{\zeta D}$, defined in Subsection 6.2.1. Then obviously $\mathcal{E}(\varphi, r)$ is a decreasing set function of the parameter $r > 0$; also it is clarified above that it consists of $k \le 4$ connected arc pieces of Γ. Each elementary small arc has a total variation of the tangent angle function at least φ, so in each connected component the same holds. Moreover, as discussed in connection with the construction of $\mathcal{L}$, any such connected component has to be at most of length $24rd/w$. Taking the limit when $r \to 0$, i.e., taking $\mathcal{E}^* := \bigcap_{r>0}\mathcal{E}(\varphi, r)$, we find that either $\mathcal{E}^* = \varnothing$ or $\mathcal{E}^*$ consists of a few (at most four) isolated points, where the variation (jump) of the tangent angle function α reaches φ. In the case $\mathcal{E}^* = \varnothing$, we can still conclude that for n large enough (even if this largeness ineffectively depends on the geometry of the domain K) this critical part of the proof can be skipped and there holds a constant times n oscillation estimate. This is not much different from the phenomenon described in [14]. If, on the other hand, $\mathcal{E}^* \ne \varnothing$, then we know that ∂K has vertices with (almost) right angle jumps of the tangents. With a suitable choice of φ we can thus prove a sharpening of the result of Theorem F in the extent that the dependence of the occurring constant is better than the one in Theorem F.

Actually, we have the following direct corollary of the results of Section 4.

Corollary 1. *Assume that the compact convex domain K does not have any boundary points where the jump of the tangent directional function α would reach $\varphi = \pi/2 - \arctan(w/d)/40$. Then we have $\mathcal{E}(\varphi, r) = \varnothing$ for small enough r, and, as a result, $|p'(\zeta)/p(\zeta)| \ge cwd^{-2}n$ at every boundary point $\zeta \in \mathcal{H}$ for $n \ge n_0(K)$. Furthermore, then we also have*

$$\|p'\|_q \ge c\frac{w}{d^2}n\|p\|_q \qquad \textit{for} \quad n \ge n_0(K).$$

The above suggests that the truth in general could be as large as $cwd^{-2}n$, with an absolute constant c—the same order of magnitude as was found for the $\|\cdot\|_\infty$ case in [24]. It is easy to obtain polynomials with as small oscillation as C/dn (see Theorem G), but recently Yu. S. Goryacheva in her master's thesis [16] has worked out a construction with even smaller oscillation: according to her work, the oscillation of order $Cwd^{-2}n$ is possible. Based on these observations and the maximum norm case, there seems to be enough evidence to further sharpen our Conjecture 1.

Conjecture 2. *There exists an absolute constant $c > 0$ such that for all compact convex domains $K \Subset \mathbb{C}$ and for any $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\|_{L^q(\partial K)} \geq c\frac{w}{d^2}n\|p\|_{L^q(\partial K)}.$$

Finally, let us analyze the strength of the arguments of the paper. Clearly our considerations are more involved than the ones used in [24] to derive Theorem D, but from the end result neither this (for $q = \infty$) nor the sharper special cases of Theorem F or E (for $1 \leq q < \infty$) follow. However, from one of the key elements, namely from Lemma 4, a numerical improvement of Theorem D follows.

Corollary 2. *Let $K \Subset \mathbb{C}$ be any compact convex domain. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\|_K \geq 10^{-3}\frac{w}{d^2}n\|p\|_K. \tag{7.1}$$

Proof. Choose $\zeta \in \partial K$ with $\|p\|_K = |p(\zeta)|$, draw any tangent, and apply Lemma 4. If we are in case (i), then an even better result is obtained. If, on the other hand, we have some positive $\delta_\pm$, then the actual value of $\delta_\pm$ becomes irrelevant as $\log(\|p\|_K/|p(\zeta)|) = \log 1 = 0$. $\square$

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