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Torus Actions of Complexity 1 and Their Local Properties

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Abstract—We consider an effective action of a compact $(n-1)$-torus on a smooth $2n$-manifold with isolated fixed points. We prove that under certain conditions the orbit space is a closed topological manifold. In particular, this holds for certain torus actions with disconnected stabilizers. There is a filtration of the orbit manifold by orbit dimensions. The subset of orbits of dimensions less than $n-1$ has a specific topology, which is axiomatized in the notion of a sponge. In many cases the original manifold can be recovered from its orbit manifold, the sponge, and the weights of tangent representations at fixed points. We elaborate on the introduced notions using specific examples: the Grassmann manifold $G_{4,2}$, the complete flag manifold F_3, and quasitoric manifolds with an induced action of a subtorus of complexity 1.

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1. INTRODUCTION

An action of a compact torus G on a topological space X is a classical object of study [4]. For a point $x \in X$ let $G_x \subset G$ denote the stabilizer subgroup and Gx the orbit of x. Let $p\colon X \to X/G$ be the projection to the orbit space. Let $S(G)$ denote the set of all closed subgroups of G endowed with the lower interval topology. There is a continuous map

$$\widetilde{\lambda}\colon X/G \to S(G),$$

which sends an orbit $x \in X/G$ to the stabilizer subgroup G_x (see [6]).

The classical idea in the study of torus actions is the following. It is assumed that the projection map $p\colon X \to X/G$ admits a continuous section. Then, given the orbit space $Q = X/G$ and a continuous map $\widetilde{\lambda}\colon Q \to S(G)$, one builds a topological model

$$X_{(Q,\widetilde{\lambda})} = (Q \times G)/{\sim},$$

which is equivariantly homeomorphic to the original space X. The method of constructing model spaces was used by Davis and Januszkiewicz [11] for the classification of manifolds which are now called quasitoric [5]. This idea of model spaces traces back to the works of Vinberg [21].

The method can be naturally extended to the locally standard actions of $G \cong T^n$ on $2n$-manifolds [22]. In this case the projection may not admit the global section; however, it always admits a section locally. This allows one to construct a model space using principal G-bundles.

Buchstaber and Terzić [7–9] developed a theory of $(2n,k)$-manifolds in order to study the orbit spaces of more general torus actions and to obtain topological models for such actions. Grassmann manifolds and flag manifolds are important families of $(2n,k)$-manifolds. In this theory a manifold is subdivided into strata X_σ so that the action has the same stabilizer T_σ for all points of

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a stratum. It is essential in the definition of $(2n, k)$-manifold that there is a convex polytope P^k and a T^k-equivariant generalized moment map $X^{2n} \to P^k$. Every stratum X_σ is then represented as a principal T/T_x-bundle over the product $P_\sigma^\circ \times M_\sigma$, where P_σ is a certain subpolytope of P^k and M_σ is an auxiliary space of dimension $2(n-k)$, called the space of parameters. Therefore, the orbit space X^{2n}/T^k is represented as the union $\bigsqcup_\sigma (P_\sigma \times M_\sigma)$. The theory of $(2n, k)$-manifolds provides specific methods to describe the topology of this union.

Whenever a compact k-tours acts effectively on a $2n$-manifold, we call the number $n-k$ the *complexity of the action*. While actions of complexity zero are well studied, the actions of positive complexity are harder to analyze. The actions of complexity ≥ 2 are extremely complicated in general. The actions of complexity 1 occupy an intermediate position: they were studied from several different viewpoints. An algebraic theory of complexity 1 actions was developed in the works of many authors; in particular, the classification of such actions, including the nonabelian case, was given by Timashev [19, 20]. The Hamiltonian complexity 1 actions on symplectic manifolds are also well studied (see, e.g., the work of Karshon and Tolman [14] and references therein). A circle action on a 4-manifold is a classical subject (see, e.g., [10, 12, 17, 18]).

In this paper we study complexity 1 actions from the topological viewpoint. Our approach is different from the one used in [6] and [8]. Instead of trying to stratify the manifold so that the action on each stratum admits a section, we partition the manifold by orbit types. Under two restrictions we prove that the orbit space $Q = X/T$ is a topological manifold (see Theorem 2.10 for the precise statement). Note that for this result it is not required that the stabilizers of the action be connected. Such a restriction was imposed in the theory of $(2n, k)$-manifolds; however, there exist natural examples of actions with finite stabilizers for which the orbit space is still a manifold.

We make a remark on the main difference from the situations considered in toric topology: the typical action of complexity 1 does not admit a section, even locally.

In the study of complexity 1 actions we are guided by the following important examples:

(1) the T^3 action on the complex Grassmann manifold $G_{4,2}$;
(2) the T^2 action on the manifold F_3 of full complex flags in $\mathbb{C}^3$;
(3) quasitoric manifolds $X^{2n}_{(P,\Lambda)}$ with induced action of a generic subtorus $T \subset G$, $\dim T = n-1$;
(4) the space of isospectral periodic tridiagonal Hermitian matrices of size $n \geq 3$.

Using the theory of $(2n, k)$-manifolds, Buchstaber and Terzić proved [8] that the orbit space of the Grassmann manifold $G_{4,2}$ is S^5 and the orbit space of the flag manifold F_3 is S^4. These two examples motivated our study. In Section 5 we prove that the orbit space of a quasitoric manifold with respect to the action of an $(n-1)$-torus T is also homeomorphic to a sphere: $X_{(P,\Lambda)}/T \cong S^{n+1}$ (see Theorem 5.1). The recent result of Karshon and Tolman [15] (see also [14]) states that for any Hamiltonian action of complexity 1 in general position the orbit space is homeomorphic to a sphere. This general statement includes the cases of $G_{4,2}$ and F_3 and a series of other natural examples.

A space of isospectral tridiagonal $n \times n$ matrices is a more interesting object, in particular due to the fact that the torus action on this space is not Hamiltonian. This space is studied in detail in [2]. This space depends on the spectrum, and for some degenerate spectra it is not smooth. However, we prove that if it is a smooth manifold, then its orbit space is homeomorphic to the product $S^4 \times T^{n-3}$. In [2] we describe the non-free part of the torus action using the regular permutohedral tiling of the Euclidean space. This allows us to understand the topology of the whole space rather than just of its orbit space.

The study of the space of periodic tridiagonal matrices raised several questions about actions of complexity 1. One of the questions is the topological classification of such actions. In this paper we prove that under certain restrictions the space X with a complexity 1 action is determined by the orbit manifold $Q = X/T$, the set of non-free orbits $Z \subset Q$, and the weights of tangent representations at fixed points (see Theorem 5.5 and Proposition 5.7). The set of non-free orbits has a specific topology, which is axiomatized in the notion of *sponge*. Sponges seem to be the objects of independent interest.

2. APPROPRIATE ACTIONS OF COMPLEXITY 1

In what follows, T denotes a compact torus of dimension $n-1$ and G denotes compact tori of other dimensions. We refer to the classical monograph of Bredon [4] for general information about group actions on manifolds.

Let us specify the type of actions to be considered in the paper. For a smooth action of G on a smooth manifold X, define the *fine partition* on X by orbit types:

$$X = \bigsqcup_{H \in S(G)} X^H.$$

Here H runs over all closed subgroups of G and $X^H = \widetilde{\lambda}^{-1}(H) = \{x \in X \mid G_x = H\}$.

Definition 2.1. An effective action of G on a compact smooth manifold X is said to be *appropriate* if

- the fixed point set X^G is finite;
- (adjoining condition) the closure of every connected component of any element X^H, $H \neq G$, of a fine partition contains a point x' with $\dim G_{x'} > \dim H$.

If, moreover, the stabilizer subgroup of every point is a torus, we call the action *strictly appropriate*.

Remark 2.2. The adjoining condition implies that whenever a subset X^H is closed in the topology of X, one has $H = G$.

Remark 2.3. A closed subgroup H of a torus has the form $H_{\mathrm{t}} \times H_{\mathrm{f}}$, where H_{t} is a torus and H_{f} is a finite abelian group. For strictly appropriate actions the finite components H_{f} of all stabilizers vanish. In other words, a strictly appropriate action is an action with all stabilizers being connected.

Example 2.4. Let an algebraic torus $(\mathbb{C}^\times)^k$ act algebraically on a smooth complex variety X with finitely many fixed points. Then the induced action of a compact subtorus $T^k \subset (\mathbb{C}^\times)^k$ on X is appropriate, as follows from the Bialynicki-Birula method [3]. Indeed, given a point $x \in X \setminus X^T$, consider the one-dimensional algebraic torus $\mathbb{C}^\times \subset (\mathbb{C}^\times)^k$ which acts on x nontrivially. Consider the point $x' = \lim_{t\to 0} tx$, where $0 < t \leq 1$, $t \in \mathbb{R}^\times \subset \mathbb{C}^\times$. The point x' is connected with x and has stabilizer of greater dimension (since x is stabilized by $(\mathbb{C}^\times)^k_x$ as well as by $\mathbb{C}^\times$). Iterating this procedure, we arrive at some fixed point.

In particular, the action of a compact torus on a complex GKM-manifold (see [13]) is appropriate.

Example 2.5. The effective action of T^{n-1} on F_n, the manifold of complete complex flags in $\mathbb{C}^n$, is strictly appropriate. The effective action of T^{n-1} on the Grassmann manifold $G_{n,k}$ of complex k-planes in $\mathbb{C}^n$ is also strictly appropriate.

Example 2.6. Let the action of $G \cong T^n$ on a smooth manifold X^{2n} be locally standard (see the definition in Section 5 below). The orbit space $P = X^{2n}/G$ is a manifold with corners. This action is appropriate whenever every face of P contains a vertex. If it is appropriate, then it is strictly appropriate. In particular, quasitoric manifolds provide examples of strictly appropriate torus actions.

Example 2.7. Let the action of G on X be appropriate and the induced action of a subtorus $T \subset G$ on X have the same fixed points as the G-action. Then the T-action is also appropriate.

Indeed, the partition element $(X')^K$ of the T-action for $K \subseteq T$ has the form

$$(X')^K = \bigcup_{H \subseteq G,\, H \cap T = K} X^H.$$

Therefore, the adjoining condition for the G-action implies the adjoining condition for the induced T-action.

From now on we restrict ourselves to actions of complexity 1, that is, to the case of $\dim T = n-1$ and $\dim X = 2n$. Let $x \in X^T$ be a fixed point and

$$\alpha_1, \dots, \alpha_n \in N = \mathrm{Hom}(T, S^1) \cong \mathbb{Z}^{n-1}$$

be the weights of the tangent representation at x. This means that

$$T_x X \cong V(\alpha_1) \oplus \dots \oplus V(\alpha_n),$$

where $V(\alpha)$ is the standard one-dimensional complex representation given by

$$tz = \alpha(t)z, \qquad t \in T, \quad z \in \mathbb{C}.$$

If there is no complex structure on X, then there is ambiguity in the choice of the signs of α_i. These signs are inessential in the following definitions.

Definition 2.8. A representation of T^{n-1} on $\mathbb{C}^n$ is said to be *in general position* if every $n-1$ of its n weights are linearly independent. An action of $T = T^{n-1}$ on $X = X^{2n}$ is called an *action in general position* if its tangent representation at any fixed point is in general position.

Remark 2.9. For an n-tuple of weights $\alpha_1, \dots, \alpha_n$, there is a relation $c_1\alpha_1 + \dots + c_n\alpha_n = 0$ in $N \cong \mathbb{Z}^{n-1}$. The action is in general position if and only if $c_i \neq 0$ for all $i = 1, \dots, n$.

Theorem 2.10. *Consider an appropriate action of $T = T^{n-1}$ on $X = X^{2n}$ and assume it is in general position. Then the orbit space $Q = X/T$ is a topological manifold.*

Proof. First we prove the local statement near fixed points.

Lemma 2.11. *For a representation of $T = T^{n-1}$ on $\mathbb{C}^n$ in general position we have $\mathbb{C}^n/T \cong \mathbb{R}^{n+1}$.*

Proof. Consider the standard action of $G = T^n$ on $\mathbb{C}^n$ which rotates the coordinates. The weights $e_1, \dots, e_n$ of the standard action form the standard basis of the character lattice $\mathrm{Hom}(G, S^1) \cong \mathbb{Z}^n$. Consider the lattice homomorphism $\phi\colon \mathbb{Z}^n \to N$ given by $\phi(e_i) = \alpha_i$, $i = 1, \dots, n$. This homomorphism is induced by some homomorphism $\phi^*\colon T \to G$ of tori. The given action of T is the composition of ϕ^* with the standard action.

So we may assume that there is an action of a subtorus $T' = f(T) \subset G$ where G acts in a standard way. The torus T' is given by $\{t_1^{c_1} \dots t_n^{c_n} = 1\}$, where $(c_1, \dots, c_n)$ is the set of coefficients of a linear relation on the weights α_i and $\gcd\{c_i\} = 1$. The condition of general position implies that $c_i \neq 0$ for all i. Hence the intersection of T' with each coordinate circle in G is a finite subgroup.

Let us denote the space $\mathbb{C}^n/T = \mathbb{C}^n/T'$ by Q. We have the map $g\colon Q \to \mathbb{C}^n/G \cong \mathbb{R}^n_{\geq 0}$, which sends a T-orbit to its G-orbit. For every $x \in \mathbb{R}^n_{>0}$ the preimage $g^{-1}(x)$ is a circle G/T'. For every $x \in \partial\mathbb{R}^n_{\geq 0}$, the preimage $g^{-1}(x)$ is a single point, since the product of T' with any nontrivial coordinate subtorus generates the whole torus G. Therefore, we have $Q = (\mathbb{R}^n_{\geq 0} \times S^1)/{\sim}$, where factoring by $\sim$ collapses circles over $\partial\mathbb{R}^n_{\geq 0}$. We have

$$\bigl((\mathbb{R}^n_{\geq 0} \times S^1)/{\sim}\bigr) \cong \bigl((\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} \times S^1)/{\sim}\bigr) \cong \mathbb{R}^{n-1} \times \mathbb{C},$$

which proves the lemma. $\square$

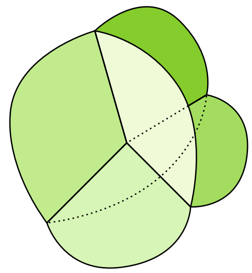

Fig. 1. Local structure of the sponge for $n = 4$.

We now prove the theorem by induction on the dimension of the stabilizer subgroup. If $\dim H = n-1$, that is, $H = T$, Lemma 2.11 shows that X/T is a manifold near the fixed point set X^T/T. Now let $[x] \in X/T$ be an orbit such that $T_x = H$, that is, $x \in X^H$. Due to the adjoining condition, there exists a point x' such that the local representations at x and x' coincide and x' is close to a partition element $X^{H'}$ with $\dim H' > \dim H$. Here by the local representation we mean a representation of T_x on the normal space $T_xX/T_x(Tx)$ to the orbit.

By the induction hypothesis, the space X/T is a manifold near $X^{H'}/T$; therefore, there exists a neighborhood of $[x']$ homeomorphic to $\mathbb{R}^{n+1}$. Therefore, there exists a neighborhood of $[x]$ homeomorphic to $\mathbb{R}^{n+1}$. Indeed, both neighborhoods are homeomorphic to the orbit space of the local representation according to the slice theorem. $\square$

Construction 2.12. Let $v_1, \dots, v_{n-1} \in \mathbb{R}^{n-1}$ be a basis of a vector space and let $v_n = -v_1 - \dots - v_{n-1}$. Consider the subset C of $\mathbb{R}^{n-1}$ given by

$$C = \bigcup_{I \subset [n],\ |I| = n-2} \operatorname{Cone}(v_i \mid i \in I).$$

This subset is homeomorphic to the $(n-2)$-skeleton of the standard nonnegative cone $\mathbb{R}^n_{\geq 0}$.

The subset C is the $(n-2)$-skeleton of a simplicial fan Δ_{n-1} corresponding to the toric variety $\mathbb{C}\mathrm{P}^{n-1}$; it comes equipped with the filtration

$$C_0 \subset \dots \subset C_{n-2} = C$$

where C_k is the union of k-dimensional cones of Δ_{n-1}. This filtration can be defined topologically: we say that $x \in \mathbb{R}^{n-1}$ has type k if C cuts a small disc B_x around x into $n-k$ chambers. Then C_k consists of all points of type $\leq k$.

Next we introduce a notion of the subspace in a topological manifold which is locally modeled by the subset $C \subset \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$. Assume we are given a topological manifold Q of dimension $n+1$ and a subset $Z \subset Q$.

Definition 2.13. A subset $Z \subset Q$ is called a *sponge* if, for any point $x \in Z$, there is a neighborhood $U_x \subset Q$ such that $(U_x, U_x \cap Z)$ is homeomorphic to $(V \times \mathbb{R}^2, (V \cap C) \times \mathbb{R}^2)$, where V is an open subset of the space $\mathbb{R}^{n-1} \supset C$.

Every sponge is filtered in a natural way compatible with the filtration of C. We say that a point $x \in Z \subset Q$ has type k if $H^2(U_x \setminus Z; \mathbb{Z}) \cong \mathbb{Z}^{n-k-1}$ for a small disc neighborhood $U_x \ni x$, $U_x \subset Q$. Then Z_k consists of all points of type at most k. Note that $\dim Z_k = k$. Informally speaking, the sponge set is a collection of $(n-2)$-manifolds with corners, and the corners are stacked together like maximal cones in C. The case $n = 4$ is shown in Fig. 1.

Construction 2.14. For an arbitrary action of a torus G, $\dim G = m$, on a space X we can consider the *coarse filtration*

$$X_0 \subset X_1 \subset \dots \subset X_m = X,$$

where $X_i = \bigcup_{\dim H \geq m-i} X^H$ is the union of all orbits of dimension at most i. In particular, the set $X \setminus X_{m-1}$ is the locus of an almost free action (i.e., an action that has only finite stabilizers). There is an induced coarse filtration on $Q = X/G$:

$$Q_0 \subset Q_1 \subset \ldots \subset Q_m.$$

Remark 2.15. The terms "fine partition" and "coarse filtration" refer to the following fact: a fine partition distinguishes different subgroups of the torus; however, a coarse filtration distinguishes only the dimensions of the subgroups.

Proposition 2.16. *For an appropriate action in general position of T^{n-1} on X^{2n} we get a topological manifold $Q = X/T$. The coarse filtration on Q has the form*

$$Z_0 \subset Z_1 \subset \ldots \subset Z_{n-2} = Z \subset Q,$$

where $Z \subset Q$ is a sponge. The coarse filtration coincides with the sponge filtration defined topologically.

Proof. The local statement near fixed points is proved in Lemma 2.11. The proof in the global case uses the adjoining condition and is similar to that of Theorem 2.10. □

3. CHARACTERISTIC DATA

Assume there is an appropriate action of $T = T^{n-1}$ in general position on $X = X^{2n}$. We allow X to have a boundary; however, in this case we require that the action is free on ∂X. The same arguments as before show that $Q = X/T$ is a topological manifold with boundary and its boundary ∂Q is $\partial X/T$.

In this section we assume that the actions are strictly appropriate. This means that there are no finite components in the stabilizer subgroups and, therefore, the face partition of a coarse filtration coincides with the fine partition on Q. With a strictly appropriate action in general position we assign the characteristic data (Q, Z, μ, e) consisting of the following elements:

- $Q = X/T$, the orbit space;
- $Z \subset Q^\circ$, the sponge subset determined by the action:

  $$Z_0 \subset Z_1 \subset \ldots \subset Z_{n-2} = Z,$$

  where $Z_i \subset Q$ is the set of orbits of dimension at most i;
- μ, a *characteristic map*,

  $$\mu\colon \mathcal{F} \to \{\text{One-dimensional subgroups of } T^{n-1}\} = \operatorname{Hom}(S^1, T^{n-1}) \cong \mathbb{Z}^{n-1},$$

  which is defined on the set $\mathcal{F} = \{F_1, \ldots, F_m\}$ of all facets of Z and sends a facet F_k to the oriented stabilizer subgroup T_{F_k} of any of its interior points (we introduce orientation arbitrarily; see Section 4 for details);
- $e \in H^2(Q \setminus Z; H^2(BT))$, the *Euler class* of the free part of the action.

Let us describe some of these elements in more detail.

The closure of a connected component of $Z_i \setminus Z_{i-1}$ is called a *face* of Z of dimension i. The faces of dimension $n-2$ are called *facets*. Every face of dimension i is contained in exactly $\binom{n-i}{2}$ different facets. The stabilizer remains the same for all points in the interior of any given face F, since no finite components are allowed in the stabilizers. This stabilizer will be denoted by T_F.

For any face F of dimension i we have $F = \bigcap_{i \in I} F_i$ for a certain subset $I \subset [m]$, $|I| = \binom{n-i}{2}$. The stabilizer T_F is the product of $\mu(F_i) = T_{F_i}$ inside the torus T^{n-1}. Note that this product is generally not free, since it has dimension $n-1-i<|I|$. However, it can be seen that characteristic map μ determines the stabilizers of all points.

Every orbit in $Q\setminus Z$ is full-dimensional and there are no finite stabilizer subgroups; therefore, the free part of the action is a principal T-bundle $p\colon X^{\mathrm{free}}\to Q\setminus Z$. This bundle is classified by the homotopy class of a map

$$Q\setminus Z\to BT\cong(\mathbb{C}\mathrm{P}^\infty)^{n-1}\cong K(\mathbb{Z}^{n-1};2).$$

Such maps also classify the second cohomology group of $Q\setminus Z$. Therefore, we have the classifying element

$$e\in H^2(Q\setminus Z;\mathbb{Z}^{n-1}),\qquad \text{where}\quad \mathbb{Z}^{n-1}\cong H_2(BT;\mathbb{Z})\cong H_1(T;\mathbb{Z}).$$

Remark 3.1. Note that unlike the half-dimensional torus actions the characteristic data of complexity 1 actions cannot be arbitrary. It will be shown in this and the next section that the Euler class e and the characteristic function μ determine each other to much extent. Moreover, the Euler class of complexity 1 actions is always nontrivial.

Let $x\in Z\subset Q$ be a point of type $k\leq n-2$. Let U_x be a small neighborhood of x in Q homeomorphic to an open disc. Let $i_x\colon U_x\to Q$ be the inclusion map. We have an induced homomorphism

$$H^2(Q\setminus Z;H_1(T))\to H^2(U_x\setminus Z;H_1(T)).$$

The image of $e\in H^2(Q\setminus Z;H_1(T))$ under this homomorphism is denoted by

$$e_x\in H^2(U_x\setminus Z;H_1(T))\cong\mathbb{Z}^{n-k-1}\otimes H_1(T)$$

and called the *local Euler class at* x. Recall that the type of the point is defined by the rank of the second cohomology of $U_x\setminus Z$ (see Section 2).

In particular, if x has type $n-2$ (i.e., x lies in the interior of a facet), the neighborhood U_x can be chosen in such a way that $U_x\cap Z\cong\mathbb{R}^{n-2}$. In this case we have $U_x\setminus Z\cong\mathbb{R}^{n+1}\setminus\mathbb{R}^{n-2}$ and

$$H^2(U_x\setminus Z;H_1(T))\cong H^2(\mathbb{R}^{n+1}\setminus\mathbb{R}^{n-2};H_1(T))\cong H^1(T).$$

The last isomorphism is canonical provided Q (and hence U_x) is oriented.

Definition 3.2. The Euler class e and characteristic function μ are said to be *compatible* if the following condition holds: for any $x\in Z$, the map $H_1(T)\to H_1(T/T_x)$ induced by the quotient map $T\to T/T_x$ sends $e_x\in H^1(T)$ to zero.

Proposition 3.3. *Assume there is an appropriate action in general position of $T=T^{n-1}$ on a manifold $X=X^{2n}$. Then its characteristic data e and μ are compatible.*

Proof. As before, let $Q=X/T$ be the orbit space, $Z\subset Q$ the set of orbits of dimensions $\leq n-2$, and $p\colon X\to Q$ the projection map. Take any point $x\in Z\subset Q$. We can choose a small neighborhood $U_x\ni x$, $U_x\subset Q$, such that the stabilizers of any point $y\in U_x$ are contained in T_x and $U_x\cong\mathbb{R}^{n+1}$. Consider the map

$$f\colon\ p^{-1}(U_x)/T_x\xrightarrow{T/T_x}U_x$$

taking the remaining quotient. Since all stabilizers of points in U_x are contained in T_x, the map f is a principal T/T_x-bundle. It is a trivial bundle since U_x is contractible; therefore, the induced T/T_x-bundle over $U_x\setminus Z$ is also trivial. Hence its Euler class vanishes. However, this Euler class is the image of $e_x\in H^2(U_x\setminus Z;H_1(T))$ under the induced map $H_1(T)\to H_1(T/T_x)$, which proves the statement. $\square$

Remark 3.4. For a point x in the interior of a facet F_i, the stabilizer T_x is one-dimensional. In this case the compatibility condition states that e_x is proportional to the fundamental class of $T_x = \mu(F_i)$:

$$e_x = k_i\mu(F_i) \in H_1(T;\mathbb{Z}) \cong \mathrm{Hom}(S^1, T).$$

The constants $k_i \in \mathbb{Z}$ can be determined from the weights of the tangent representation at any fixed point adjacent to F_i. It will be shown in Section 4 that all these constants are actually ± 1 for strictly appropriate actions.

Construction 3.5. Let us construct a topological model space given abstract compatible characteristic data. Assume a topological $(n+1)$-manifold Q is given, and let $Z \subset Q$ be a sponge with facets $F_1, \dots, F_m$. Let μ be a map assigning a one-dimensional subgroup of $T = T^{n-1}$ to any facet F_i with the following property: if a k-dimensional face F of the sponge lies in facets F_i with labels $i \in I$, $|I| = \binom{n-k}{2}$, then

$$\dim \prod_{i\in I} \mu(F_i) = k.$$

The subgroup $\prod_{i\in I}\mu(F_i)$ will be denoted by T_x if x lies in the interior of F. If $x \in Q \setminus Z$, we set $T_x = \{1\} \subset T$. Finally, fix a class $e \in H^2(Q\setminus Z; H_1(T))$ compatible with μ. With all this information fixed, introduce a space $Y = Y_{(Q,Z,\mu,e)}$. We define Y as a set by putting

$$Y = \bigsqcup_{x\in Q} T/T_x.$$

The topology is introduced in two steps.

1. The topology on a subset

$$Y^{\mathrm{free}} = \bigsqcup_{x\in Q\setminus Z} T/T_x = \bigsqcup_{x\in Q\setminus Z} T \subset Y$$

is introduced in such a way that the natural projection $Y^{\mathrm{free}} \to Q \setminus Z$ is the principal T-bundle classified by $e \in H^2(Q\setminus Z; H_1(T))$.

2. For a point y in $\bigsqcup_{x\in Z} T/T_x$ we specify the base of topology. Let $x \in Z$ and $t_x \in T/T_x \subset Y$. To define the base of topology near t_x, we fix a small open neighborhood $U_x \subset Q$ of x and for each $x' \in U_x$ take a projection of tori $p_{x'}\colon T/T_{x'} \to T/T_x$. This is well defined since U_x is assumed to be small enough so that T_x contains any other stabilizer $T_{x'}$. Let V be a neighborhood of t_x in T/T_x. The subsets of the form

$$\bigsqcup_{x'\in U_x} p_{x'}^{-1}(V)$$

constitute the base of topology around t_x.

Note that since e and μ are compatible, we have a trivial principal T/T_x-bundle over $A \to U_x \setminus Z$; therefore, the topology defined at step 2 is compatible with the one defined at step 1 on a subset $U_x \setminus Z$.

Finally, define the T-action on each fiber T/T_x as given by the projection $T \to T_x$. It can be seen that Y is a compact Hausdorff topological space carrying the continuous action of T. Its orbit space is homeomorphic to Q.

The constructed space $Y = Y_{(Q,Z,\mu,e)}$ is not necessarily a manifold.

Example 3.6. Assume $e_x = 0$ for some point x lying in the interior of a facet F_j. Then Y is not a manifold over x. Indeed, by construction, a neighborhood of x in Y is homeomorphic to

$(U_x \times T)/{\sim}$, where $(x', t') \sim (x'', t'')$ whenever $x' = x'' \in Z$ and $t'(t'')^{-1} \in \mu(F_j)$. This subset is not a manifold, which can be shown by computing its local homology groups for points lying over Z.

Proposition 3.7. *Let $X = X^{2n}$ be a manifold with strictly appropriate action of $T = T^{n-1}$ in general position. Let (Q, Z, μ, e) be its characteristic data. Let Y be the model space constructed from the data (Q, Z, μ, e). Then there is a T-equivariant homeomorphism $h\colon Y \to X$ which induces the identity homeomorphism on the orbit space Q:*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p_Y"] & X \arrow[d, "p_X"] \\ Q \arrow[r, equal] & Q \end{tikzcd}$$

Proof. The equivariant homeomorphism over $Q \setminus Z$ follows immediately, since both $p_X^{-1}(Q \setminus Z)$ and $p_Y^{-1}(Q \setminus Z)$ are principal T-bundles classified by e. For a point $x \in Z \subset Q$, the equivariant homeomorphism $h\colon p_Y^{-1}(U_x \setminus Z) \to p_X^{-1}(U_x \setminus Z)$ is extended uniquely to the equivariant homeomorphism $h\colon p_Y^{-1}(U_x) \to p_X^{-1}(U_x)$. Indeed, there is a unique equivariant homeomorphism $\widetilde{h}\colon p_Y^{-1}(U_x)/T_x \to p_X^{-1}(U_x)/T_x$ which extends the homeomorphism $h/T_x\colon p_Y^{-1}(U_x \setminus Z)/T_x \to p_X^{-1}(U_x \setminus Z)/T_x$, since both spaces are trivial T/T_x-bundles over U_x (according to the compatibility condition) and $U_x \setminus Z$ is dense in U_x. For a point $t_x \in T/T_x \subset p_Y^{-1}(U_x)$ over x there is a unique point $\alpha \in p_X^{-1}(U_x)$ such that $\widetilde{h}([t_x]) = [\alpha]$, since the projection map $p_X^{-1}(U_x) \to p_X^{-1}(U_x)/T_x$ is a bijection over x. Hence we can extend h by putting $h(t_x) = \alpha$.

This procedure defines an equivariant continuous bijection between compact spaces Y and X. Since X is compact and Y is Hausdorff, it is an equivariant homeomorphism. $\square$

4. ORIENTATION ISSUES AND DETAILS

Consider a representation of the torus $T = T^{n-1}$ on $\mathbb{C}^n$ in general position. The weights $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(T, S^1)$ are defined up to sign.

Definition 4.1. An *omniorientation* is a choice of the orientation of T (and hence of the orientation of the lattice $N = \mathrm{Hom}(T, S^1)$) and a choice of signs of all vectors α_i.

Construction 4.2. Assume there is a fixed basis in the lattice N, so that α_j is written in coordinates as $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n-1})$. For each $i = 1, \ldots, n$ consider the determinant of the matrix formed by α_j with $j \neq i$:

$$\widetilde{c}_i = (-1)^i \alpha_1 \wedge \ldots \wedge \widehat{\alpha}_i \wedge \ldots \wedge \alpha_n \in \Lambda^{n-1} N \cong \mathbb{Z}.$$

Since α_i are in general position, we have $\widetilde{c}_i \neq 0$ for all $i = 1, \ldots, n$. Cramer's rule implies

$$\widetilde{c}_1 \alpha_1 + \ldots + \widetilde{c}_n \alpha_n = 0.$$

Let $c_{\mathrm{gcd}} = \gcd(\widetilde{c}_1, \ldots, \widetilde{c}_n)$ and $c_i = \widetilde{c}_i / c_{\mathrm{gcd}}$. Let $G = T^n$ act on $\mathbb{C}^n$ in a standard way:

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n),$$

and let T' be a subtorus

$$T' = \{t_1^{c_1} \ldots t_n^{c_n} = 1\} \subset G. \tag{4.1}$$

The proof of Lemma 2.11 implies that the orbit space of the representation of T on $\mathbb{C}^n$ coincides with the orbit space of the induced action of T' on $\mathbb{C}^n$; therefore, we need not distinguish these two cases.

Lemma 4.3. *The representation action of $T = T'$ on $\mathbb{C}^n$ in general position is strictly appropriate if and only if $c_i = \pm 1$, that is, all parameters $\widetilde{c}_i$ coincide up to sign.*

Proof. The point $(0, \dots, 0, 1, 0, \dots, 0)$ with unit at the jth position has the stabilizer $T' \cap G_j$, where T' is given by (4.1) and G_j is the jth coordinate circle of $G \cong T^n$. This stabilizer is isomorphic to the cyclic group $\mathbb{Z}_{|c_j|}$. If the action is strictly appropriate, then there are no finite components in the stabilizer subgroups, so c_i is necessarily ± 1.

The converse statement is proved in a similar way. The stabilizers of the T'-action on $\mathbb{C}^n$ have the form $T' \cap G_I$ for all possible coordinate subtori $G_I \in G$, $I \subseteq [n]$. This group has a finite component of order $\gcd(c_i \mid i \in I)$. Hence, if all c_i are ± 1, these finite components are trivial. $\square$

Recall that $C \subset \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ denotes the $(n-2)$-skeleton of the fan Δ_{n-1} corresponding to the toric variety $\mathbb{C}\mathrm{P}^{n-1}$. This space is the sponge of an appropriate representation action of $T = T^{n-1}$ on $\mathbb{C}^n$.

In the following we only consider strictly appropriate actions. The facets $\{F_{i,j} \mid 1 \le i < j \le n\}$ of Z are labeled in such a way that $F_{i,j}$ is "spanned" by all weights except α_i and α_j. Let us fix an orientation on the one-dimensional stabilizers of the action (this corresponds to a choice of the signs of the characteristic values $\mu(F_{i,j}) \in \mathrm{Hom}(S^1, T)$). These orientations determine the orientation of the orbit $Tx \cong T/\mu(F_{i,j})$ for $x \in F_{i,j}^\circ$. The preimage of $F_{i,j}^\circ$ under the projection map has the form $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j = 0,\ z_k \ne 0 \text{ for } k \ne i, j\}$. This space has a canonical orientation determined by the complex structure on $\mathbb{C}^n$. Therefore, the orientations of the stabilizer circles determine the orientations of facets $F_{i,j}$. Finally, since the orientation on $\mathbb{C}^n/T \cong \mathbb{R}^{n+1}$ is fixed, the orientation of $F_{i,j}$ determines the orientation of a small 2-sphere $S_{i,j}^2$ around $F_{i,j}$. Let us describe the Euler class of the free part of the action.

Proposition 4.4. *The Euler class $e \in H^2(Q \setminus Z; H_1(T))$ of a strictly appropriate representation action of T on $\mathbb{C}^n$ is given by the condition*

$$\langle e, [S_{i,j}^2] \rangle = \frac{c_i}{c_j} \mu(F_{i,j}) \in H_1(T) \cong \mathrm{Hom}(S^1, T)$$

for a small 2-sphere around the facet $F_{i,j}$, $1 \le i < j \le n$.

The constants c_i were defined earlier in this section. Lemma 4.3 shows that for strictly appropriate actions $c_i = \pm 1$. Note that $c_i/c_j = \widetilde{c}_i/\widetilde{c}_j$ and the parameters $\widetilde{c}_i$ and $\widetilde{c}_j$ can be computed from the weight vectors.

Proof of Proposition 4.4. Assume $i = 1$ and $j = 2$ for simplicity. The preimage of a sphere $S_{1,2}^2$ in the space $\mathbb{C}^n$ has the form

$$M = \bigl\{(z_1, \dots, z_n) \in \mathbb{C}^n \bigm| z_1|^2 + |z_2|^2 = \varepsilon,\ |z_k| = \varepsilon,\ k > 2\bigr\} \qquad \text{for small } \varepsilon > 0.$$

The subtorus $T = \{t_1^{c_1} \dots t_n^{c_n} = 1\} \subset G$ acts freely on M. The stabilizer $T_x = \mu(F_{1,2})$ for $x \in F_{1,2}^\circ$ has the form

$$T_x = \{t_1^{c_1} t_2^{c_2} = 1,\ t_k = 1,\ k > 2\}.$$

The induced action of T/T_x on M/T_x is a trivial principal bundle; therefore, the Euler class of the T-action on M coincides with the image of the Euler class of the T_x-action on M under the inclusion map $i_x \colon T_x \to T$. The T_x-action on M is the Hopf bundle if c_1 and c_2 have the same sign, and it is the "anti-Hopf" bundle if c_1 and c_2 have different signs. Its Euler class is $\mu(F_{1,2}) \in H^2(S_{1,2}^2; H_1(T))$ in the first case and $-\mu(F_{1,2})$ in the second case. $\square$

Remark 4.5. Note that there exist relations on the cycles $[S_{i,j}] \in H_2(Q \setminus C; \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. Every triple of indices i, j, k determines an $(n-3)$-face $F_{i,j,k} \in C$ which lies in the facets $F_{i,j}$, $F_{j,k}$,

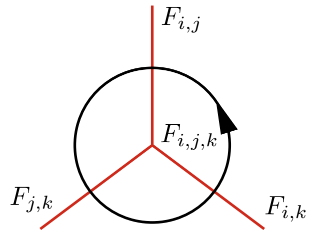

Fig. 2. Orienting three facets with a common face of codimension 1.

and $F_{i,k}$. If we choose a small circle around $F_{i,j,k} \subset \mathbb{R}^{n-1}$ and orient the facets $F_{i,j}$, $F_{j,k}$, and $F_{i,k}$ consistently (see the schematic Fig. 2), we get a relation in $H_2(Q \setminus C; \mathbb{Z})$:

$$[S_{i,j}] + [S_{j,k}] + [S_{i,k}] = 0.$$

It implies the cocycle relation for stabilizers:

$$\frac{c_i}{c_j}\mu(F_{i,j}) + \frac{c_j}{c_k}\mu(F_{j,k}) + \frac{c_i}{c_k}\mu(F_{i,k}) = 0. \tag{4.2}$$

This relation is not surprising. Indeed, the product of the circle subgroups $\mu(F_{i,j})$, $\mu(F_{j,k})$, and $\mu(F_{i,k})$ inside the torus T has dimension 2; therefore, there should be exactly one linear relation for their fundamental classes.

Proposition 4.4 implies that for strictly appropriate torus actions we have $e_x = \pm[\mu(F)]$ for $x \in F^\circ$, since this holds in the local chart around a fixed point.

5. REDUCTIONS OF LOCALLY STANDARD ACTIONS

A smooth manifold $X = X^{2n}$ with an action of $G = T^n$ is said to be *locally standard* if the action is locally modeled by the standard representation of $G = T^n$ on $\mathbb{C}^n$. Since $\mathbb{C}^n/T^n \cong \mathbb{R}^n_{\geq 0}$, the orbit space $P = X/G$ has a natural structure of a manifold with corners. Manifolds with locally standard actions are classified up to equivariant homeomorphism (see [22]) by the following *characteristic data*:

(1) the manifold with corners P, $\dim P = n$, such that each of its k-dimensional faces is contained in precisely $n-k$ facets (such manifolds with corners are called *nice* in [16] or just *manifolds with faces*);

(2) the characteristic function λ which maps a facet F of P into a circle subgroup of G, the stabilizer of any interior point of F;

(3) the Euler class $e \in H^2(P; H_1(G)) \cong H^2(P^\circ; H_1(G))$, which classifies the principal G-bundle $X^{\mathrm{free}} \to X^{\mathrm{free}}/G = P^\circ$, where X^{free} is the free part of the G-action.

The characteristic function λ satisfies the celebrated $(*)$-condition: whenever facets $F_1, \dots, F_k$ intersect in P, the subgroups $\lambda(F_1), \dots, \lambda(F_k)$ form a direct product inside G. Since every circle subgroup of G determines a primitive integral vector in $\mathrm{Hom}(S^1, G) \cong \mathbb{Z}^n$ up to sign, it will be convenient to assume that λ takes values in the lattice $\mathbb{Z}^n$.

In the following we assume that every face of P contains a vertex so that the action is *appropriate*.

A manifold X with a locally standard action of G is called a *quasitoric manifold* if $P = X/G$ is isomorphic to a simple polytope as a manifold with corners. The free part of the action is a trivial G-bundle, since P is contractible. So the Euler class vanishes for quasitoric manifolds.

A fixed point v of a locally standard action of G on X corresponds to a vertex v of P (we denote them by the same letter). We have $v = F_1 \cap \ldots \cap F_n$ for some facets $F_i \subset P$. Then the weights $\alpha_1, \dots, \alpha_n \in \mathrm{Hom}(G, S^1) = N$ of the tangent representation at v form the dual basis to $\lambda(F_1), \dots, \lambda(F_n) \in \mathrm{Hom}(S^1, G) = N^*$.

Let $\{\alpha_{v,i}\}$ be the collection of all weights at all fixed points. We can choose a generic homomorphism of the lattices

$$\phi\colon \mathrm{Hom}(G, S^1) \cong \mathbb{Z}^n \to \mathbb{Z}^{n-1}$$

such that the images $\phi(\alpha_{v,1}), \dots, \phi(\alpha_{v,n}) \in \mathbb{Z}^{n-1}$ are in general position for any fixed point v. The homomorphism ϕ is determined by some homomorphism of tori $\phi^*\colon T^{n-1} \to G$. Therefore, the action of the subtorus $T = \phi^*(T^{n-1}) \subset G$ on X is in general position.

Theorem 5.1. *Let $X = X^{2n}$ be a quasitoric manifold with an action of $G \cong T^n$. Let $T \subset G$ be a subtorus of dimension $n-1$ such that the induced action of T on X is an action in general position. Then $X/T \cong S^{n+1}$.*

Proof. Denote the orbit space X/T by Q and the orbit space X/G by P. By the definition of a quasitoric manifold, P is a simple polytope, $\dim P = n$. We have a map $g\colon Q \to P$, which sends a T-orbit to the G-orbit in which it lies. For any point x in the interior of P we have $g^{-1}(x) \cong S^1$. Since the action is in general position, the preimage of a point $x \in \partial P$ is a single point (this fact was actually proved in Lemma 2.11 for a local chart). Since P is contractible, the map $g\colon Q \to P$ admits a section over P°. Therefore, we have

$$Q \cong (P \times S^1)/{\sim},$$

where factoring by $\sim$ collapses circles over ∂P. Since P is homeomorphic to the n-disc D^n, we have

$$Q \cong ((D^n \times S^1)/{\sim}) \cong \partial(D^n \times D^2) \cong S^{n+1},$$

which completes the proof. $\square$

We further investigate the characteristic data of the induced action of $T \cong T^{n-1}$ on a quasitoric manifold. The arguments in the proof of Theorem 5.1 imply the following statement.

Proposition 5.2. *The sponge of the T-action on a quasitoric manifold X has the form*

$$Z \subset S^{n-1} \subset \Sigma^2 S^{n-1} \cong Q,$$

where S^{n-1} is identified with the boundary of the polytope P and Z is its $(n-2)$-skeleton. The facets of Z are exactly the faces of P of codimension 2.

Note that the characteristic function λ of the G-action determines the characteristic function μ of the T-action. Let F be a codimension 2 face of P (hence a facet of Z). Then $F = F_1 \cap F_2$, where F_1 and F_2 are facets of P. We have

$$\mu(F) = (\lambda(F_1) \times \lambda(F_2)) \cap T.$$

Here $\lambda(F_1) \times \lambda(F_2)$ is a 2-torus in G, and since T is a codimension 1 subtorus of G in general position, the intersection $(\lambda(F_1) \times \lambda(F_2)) \cap T$ is a one-dimensional subgroup, which is the stabilizer of the T-action on the interior of F. If we want this subgroup to be a circle (recall that the definition of a strictly appropriate action requires that the stabilizers do not have finite components), then the subgroup $T \subset G$ is subject to some additional restrictions. Namely, the subgroup $T \subset G$ determines a character $\alpha_T \in \mathrm{Hom}(G, S^1)$, $\alpha_T\colon G \to G/T$. The next lemma follows easily from Lemma 4.3.

Lemma 5.3. *The induced action of T on a locally standard G-manifold X is strictly appropriate if and only if $\langle \alpha, \lambda(F_i) \rangle = \pm 1$ for all facets F_i of a manifold with corners P.*

Example 5.4. Let $c\colon \{F_1, \dots, F_m\} \to [n]$ be a proper coloring of facets of a simple polytope P. This means that whenever F_i and F_j are adjacent, their colors $c(F_i)$ and $c(F_j)$ are different. Given such a coloring, we can construct a special characteristic function $\lambda_c\colon \{F_1,\dots,F_m\}\to\mathbb{Z}^n$ which sends F_i to the basis vector $\lambda(F_i)=\epsilon_{c(F_i)}\in\mathbb{Z}^n$, where $\epsilon_1,\dots,\epsilon_n$ is a fixed basis of the lattice $\mathbb{Z}^n$. Such characteristic functions and corresponding quasitoric manifolds $X_{(P,\lambda_c)}$ were called *pullbacks of the linear model* in [11]. It can be seen that the induced action of the subtorus

$$T=\{t_1^{c_1}t_2^{c_2}\dots t_n^{c_n}=1\}\subset G,\qquad c_i=\pm 1,$$

on $X_{(P,\lambda_c)}$ is strictly appropriate.

Note that there exist other examples of strictly appropriate induced actions which do not come from colored characteristic functions.

The local Euler classes e_x of the induced action of T on a quasitoric manifold X determine its topology.

Theorem 5.5. *Let X' and X'' be two manifolds with strictly appropriate actions in general position,* $\dim X'=\dim X''=2n$, $n\geq 2$. *Let $(Q'\cong S^{n+1},Z',\mu',e')$ and $(Q''\cong S^{n+1},Z'',\mu'',e'')$ be their characteristic data. Suppose there is a homeomorphism of pairs $(Q',Z')\cong(Q'',Z'')$ and $e'_x=e''_x$ for any point x in a sponge. Then X' and X'' are equivariantly homeomorphic.*

Proof. Taking x in the interior of a facet F of a sponge $Z'\cong Z''$, we see that $\mu'(F)=\mu''(F)$, since e'_x is the fundamental class of $\mu'(F)$ and e''_x is the fundamental class of $\mu''(F)$. Hence $\mu'=\mu''$.

Let (Q,Z) be either (Q',Z') or (Q'',Z'') and let $U=\bigcup_{x\in Z}U_x$ be a neighborhood of Z in Q. As before, U_x denotes a small neighborhood of $x\in Z$ homeomorphic to an open ball. The local classes e_x determine the classes $e'_x\in H^3(U_x,U_x\setminus Z;\mathbb{Z}^{n-1})$ due to the exact sequence

$$\begin{array}{ccccccc} H^2(U_x;\mathbb{Z}^{n-1}) & \longrightarrow & H^2(U_x\setminus Z;\mathbb{Z}^{n-1}) & \longrightarrow & H^3(U_x,U_x\setminus Z;\mathbb{Z}^{n-1}) & \longrightarrow & H^3(U_x;\mathbb{Z}^{n-1}) \\ \| & & & & & & \| \\ 0 & & & e_x\longmapsto e'_x & & & 0 \end{array}$$

The classes $\{e'_x\mid x\in Z\}$ determine a unique element $e'\in H^3(U,U\setminus Z;\mathbb{Z}^{n-1})$ such that $i_x^*(e')=e'_x$ for an inclusion $i_x\colon U_x\hookrightarrow U$ according to the Mayer–Vietoris argument. By excision, we can view e' as an element in $H^3(Q,Q\setminus Z;\mathbb{Z}^{n-1})\cong H^3(U,U\setminus Z;\mathbb{Z}^{n-1})$. Recall that $Q\cong S^{n+1}$. From the exact sequence

$$\begin{array}{ccccccc} H^2(Q;\mathbb{Z}^{n-1}) & \longrightarrow & H^2(Q\setminus Z;\mathbb{Z}^{n-1}) & \longrightarrow & H^3(Q,Q\setminus Z;\mathbb{Z}^{n-1}) & \longrightarrow & H^3(Q;\mathbb{Z}^{n-1}) \\ \| & & & & & & \\ 0 & & & e\longmapsto e' & & & \end{array}$$

we extract a unique element $e\in H^2(Q\setminus Z;\mathbb{Z}^{n-1})$ which projects to e_x for any point $x\in Z$.

The characteristic data $(Q'\cong S^{n+1},Z',\mu',e')$ and $(Q''\cong S^{n+1},Z'',\mu'',e'')$ coincide; hence the spaces X' and X'' are equivariantly homeomorphic to the model space according to Proposition 3.7. Thus they are homeomorphic to each other. $\square$

Remark 5.6. Instead of the equality $e'_x=e''_x$ one can require the equality of characteristic functions $\mu'=\mu''$ and, for a small 2-sphere around each facet F, specify the type of its preimage (whether it is the Hopf or anti-Hopf bundle, see Proposition 4.4). If the types agree for X and X', then the equality $\mu'=\mu''$ implies the equality of local classes $e'_x=e''_x$.

In order to study certain examples, we need a modification of Theorem 5.5. Let M be a closed manifold of dimension $n-1$. Assume there is a regular simple cell subdivision on M, which means that a regular cell structure is given in which every k-dimensional cell is contained in exactly $n-k$ maximal cells. Its $(n-2)$-skeleton $Z_M=M_{n-2}$ is a sponge. Consider the manifold with boundary $Q_M=M\times D^2$. We consider M as a subset $M\times\{0\}\subset Q_M$.

Proposition 5.7. *Let $(X,\partial X)$ be a $2n$-dimensional manifold with boundary, and assume there is an appropriate action of $T=T^{n-1}$ on X with characteristic data (Q_M,Z_M,μ_M,e_M). We also* *assume that the action is free on the boundary ∂X and the principal T-bundle $\partial X \to \partial X/T = \partial Q_M \cong M \times \partial D^2$ is trivial. Then the class $e_M \in H^2(Q_m \setminus Z_M; \mathbb{Z}^{n-1})$ is uniquely determined by the local classes e_x, $x \in Z_M$.*

Proof. There is an exact sequence of the pair $(Q_M \setminus Z_M, \partial Q_M)$:

$$H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1}) \to H^2(Q_M \setminus Z_M; \mathbb{Z}^{n-1}) \to H^2(\partial Q_M; \mathbb{Z}^{n-1}).$$

The class $e \in H^2(Q_M \setminus Z_M; \mathbb{Z}^{n-1})$ maps to zero, since the free part of the action is a trivial T-bundle over ∂Q. Hence there exists a class

$$\widetilde{e} \in H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1})$$

which maps to e, and e is uniquely determined by the class $\widetilde{e}$. We have

$$(Q_M \setminus Z_M)/\partial Q_M \simeq \Sigma^2(M \setminus Z_M);$$

hence $H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1}) \cong \widetilde{H}^0(M \setminus Z_M)$. The space $M \setminus Z_M$ is the disjoint union of open top-dimensional cells of M. It can be seen that the cohomology classes of $H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1})$ are localized near Z_M and are thus completely determined by the local classes. □

Corollary 5.8. *Under the assumptions of Proposition* 5.7, *the equivariant homeomorphism type of X is determined by (Q_M, Z_M) and the weights of all tangent representations at all fixed points.*

Construction 5.9. The examples of the actions above can be constructed in the following way. We consider a manifold $P \cong M \times [0,1]$ with boundary $\partial P = \partial_0 P \sqcup \partial_1 P$, $\partial_i P = M \times \{i\}$, and endow it with the structure of a nice manifold with corners. Namely, we subdivide the boundary component $\partial_0 P$ according to the subdivision of M and do nothing with $\partial_1 P$ (this boundary component is considered as a single face of dimension $n-1$). Now we may take an abstract characteristic function satisfying the $(*)$-condition:

$$\lambda\colon \{\text{Facets of } \partial_0 P\} \to \operatorname{Hom}(S^1, G) \cong \mathbb{Z}^n,$$

and construct a topological manifold

$$X = (P \times G)/{\sim}$$

with boundary $\partial X = \partial_1 P \times G$. Here $G \cong T^n$ and factoring by $\sim$ collapses tori over $\partial_0 P$ according to the characteristic function (see [11, 22, 5] for details). These particular manifolds with boundary were studied in [1].

We take a generic $(n-1)$-dimensional subtorus $T \subset G$ so that the induced action of T on X is strictly appropriate and is in general position. It can be seen that for the orbit space $Q = X/T$ we have the homeomorphism

$$Q \cong ((P \times S^1)/{\sim}) = ((M \times [0,1]) \times S^1)/{\sim},$$

where the circles are collapsed over $\partial_0 P = M \times \{0\}$. Therefore, $Q \cong M \times D^2$. The sponge of the T-action is the $(n-2)$-skeleton of $M = M \times \{0\} \subset M \times D^2$. Finally, the free T-action over $M \times \partial D^2$ is a trivial bundle, since the G-action is trivial over $\partial_1 P$.

6. GRASSMANN AND FLAG MANIFOLDS

Next we review two classical examples that motivated our study.

Example 6.1. The standard action of a compact torus T^4 on $\mathbb{C}^4$ induces the action of T^4 on the Grassmann manifold $G_{4,2}$ of complex 2-planes in $\mathbb{C}^4$. This action has non-effective kernel $\Delta(T^1) \subset T^4$; hence we have an effective action of $T = T^4/\Delta(T^1) \cong T^3$ on $G_{4,2}$, $\dim_{\mathbb{R}} G_{4,2} = 8$.

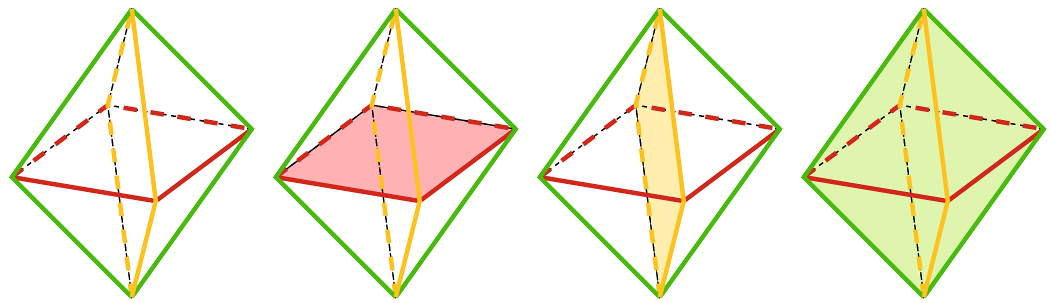

Fig. 3. The sponge of $G_{4,2}$ consists of the boundary of an octahedron with three squares attached along the equators.

There are six fixed points, and it is not difficult to find the weights of their tangent representations. The easiest way to do this is to look at the image of the moment map, which coincides with a regular octahedron $\Delta_{4,2}$. Its vertices correspond to the fixed points, and the primitive lattice vectors along the edges of the octahedron that emanate from a vertex correspond to the weights of the tangent representation. For example, the edges emanating from the top vertex $(0,0,1)$ of the octahedron are

$$\alpha_1 = (1,0,-1), \qquad \alpha_2 = (0,1,-1), \qquad \alpha_3 = (-1,0,-1), \qquad \alpha_4 = (0,-1,-1).$$

Every three of them are linearly independent; hence the action is in general position. The action is strictly appropriate.

It was proved in [8] that the orbit space $G_{4,2}/T$ is homeomorphic to S^5. The sponge Z of the action is obtained by taking the boundary of the octahedron $\partial\Delta_{4,2}$ and by attaching three squares along the equatorial cycles as shown in Fig. 3 (the squares must not intersect at non-boundary points). This description actually follows from the methods of [8].

Example 6.2. The standard action of T^3 on $\mathbb{C}^3$ induces the effective action of $T = T^3/\Delta(T^1)$ on the manifold F_3 of complete complex flags in $\mathbb{C}^3$. We have $\dim T = 2$ and $\dim F_3 = 6$. There are six fixed points, and the tangent representation at each point is in general position. The action is strictly appropriate.

Using the technique of [8] (see [2] for an alternative proof), one can show that the orbit space F_3/T is homeomorphic to S^4. The sponge of the action has dimension 1. This is simply the GKM-graph of the action, which is well known. This graph is shown in Fig. 4. As an abstract graph, it is a complete bipartite graph $K_{3,3}$. The figure on the right shows how to realize this graph as a 1-skeleton of a simple cell structure on a 2-torus $\mathcal{T}$. Actually, $\mathcal{T}$ can be embedded in $S^4 = F_3/T$ in a canonical way, and the preimage of its small neighborhood $U_{\mathcal{T}}$ under the projection map is described by Construction 5.9. This subject is covered in detail in [2].

Note an interesting geometrical difference of these two examples from the induced T-action on a quasitoric manifold. In the case of a T-action on a quasitoric manifold, the sponge, which is an $(n-2)$-dimensional complex, can be embedded in S^{n-1} (since it is the $(n-2)$-skeleton of

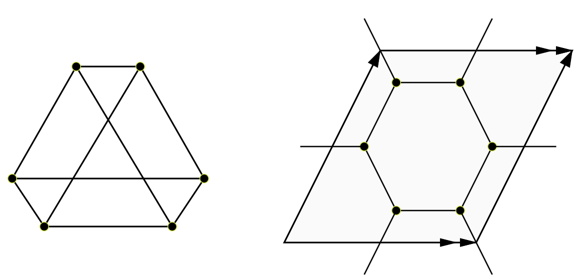

Fig. 4. Sponge of the complete flag manifold F_3.

a polytope). However the sponges of $G_{4,2}$ and F_3 do not embed in a sphere as codimension 1 complexes. In the case of F_3, the graph $K_{3,3}$ is well-known to be non-planar. The sponge of $G_{4,2}$, which is the octahedron with three squares attached, cannot be embedded in $\mathbb{R}^3$, as can be easily checked.

Remark 6.3. Whenever the orbit space $Q = X/T$ is a sphere S^{n+1}, the Alexander duality implies that $H^2(Q \setminus Z; R) \cong H_{n-2}(Z; R)$ for a sponge $Z \subset Q$. The homology class corresponding to $e \in H^2(Q \setminus Z; H_1(T))$ is represented by the chain

$$\sigma = \sum_{F \text{ is a facet of } Z} e_x \cdot [F] \in C_{n-2}(Z; H_1(T)).$$

Here $[F]$ is the fundamental chain of a facet F and $e_x \in H_2(U_x; H_1(T)) \cong H_1(T)$ is the local Euler class at an interior point $x \in F^\circ$. The chain σ is a cycle according to relation (4.2).

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