ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2018, Vol. 300, Suppl. 1, pp. S49–S55.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2016, Vol. 22, No. 4, pp. 87–93.

Moving Object in $\mathbb{R}^2$ and a Group of Observers

V. I. Berdyshev[1]

Received September 15, 2016

Abstract—We formulate an extremal problem of constructing a trajectory of a moving object that is farthest from a group of observers with fixed visibility cones. Under some constraints on the arrangement of the observers, we give a characterization and a method of construction of an optimal trajectory.

Keywords: moving object, observer, optimal trajectory.

DOI: 10.1134/S0081543818020062

Let M be a fixed set in $\mathbb{R}^2$ such that it is the closure of an open set, and let t be a moving object. The set M hinders the motion and visibility of the object. A continuous non-self-intersecting trajectory $\mathcal{T}_0$, $\mathcal{T}_0 \cap M = \varnothing$, connecting some points $t_* \neq t^*$ is given in $\mathbb{R}^2$. The object t moves inside a corridor

$$Y = \bigcup_{t \in \mathcal{T}_0} V_r(t),$$

where $r = r(t) = \min\{\|t - m\| \colon m \in M\}$ and $V_r(t)$ is the closed ball of radius r centered at t. We assume that $V_r(t_*) \cap V_r(t^*) = \varnothing$. Denote by $\mathbb{T}$ the set of continuous trajectories

$$\mathcal{T} = \{t(\tau) : \ 0 \leq \tau \leq 1, \ t(0) = t_*, \ t(1) = t^*\} \subset Y. \tag{1}$$

Let $\operatorname{bd} Y$ be the boundary of the corridor Y, and let $\Gamma = (\operatorname{bd} Y) \backslash (V_r(t_*) \cup V_r(t^*))$. The set Γ is decomposed into two parts: the left part Γ^l and the right part Γ_r with respect to the object moving from t_* to t^* along $\mathcal{T}_0$. It is assumed that a finite group of observers $\mathbb{S} = \{S\}$, $S \notin \mathring{Y}$, is given. For simplicity, we assume that $\mathbb{S} \subset \Gamma$. Each observer S has a fixed visibility cone $K(S)$, which is the union of S and a convex open cone at the vertex S. The intersection of $K(S)$ and Y may consist of several connected components. We denote by $K_Y(S)$ the component containing S. For every S, the cone $K(S)$ is such that each trajectory $\mathcal{T} \in \mathbb{T}$ intersects $K_Y(S)$. The groups of observers belonging to Γ^l and Γ_r will be denoted by $\mathbb{S}^l$ and $\mathbb{S}_r$, respectively.

We define the "distance" from a point $t \in Y$ to S as follows:

$$\rho(t, S) = \begin{cases} \|t - S\| & \text{for } \ t \in K_Y(S), \\ +\infty & \text{for } \ t \notin K_Y(S). \end{cases}$$

The problem consists in finding a trajectory $\mathcal{T}^* = \mathcal{T}(\mathbb{S})$ (1) implementing the maximum

$$\mathbb{M} = \mathbb{M}(\mathbb{S}) \overset{\text{def}}{=} \max_{\mathcal{T} \in \mathbb{T}} \min\{\rho(t, S) : \ t \in \mathcal{T}, \ S \in \mathbb{S}\} = \min\{\rho(t, S) : \ t \in \mathcal{T}^*, \ S \in \mathbb{S}\}. \tag{2}$$

[1]Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620990 Russia
email: bvi@imm.uran.ru

In the present paper, we establish characteristic properties of optimal (best) trajectories and give a method of construction of the trajectories. It is easy to see that there are many such trajectories. We give a method of construction of optimal trajectories consisting of segments of straight lines, circular arcs, and segments of the boundary of the corridor Y, which are defined by the arrangement of observers and cones $K(S)$, $S \in \mathbb{S}$.

A similar problem was considered in [1] without studying methods of construction of an optimal trajectory.

Denote by $L(x, y)$ the straight line containing points $x \neq y$ and by $\overline{Q}$ the closure of a set Q.

Consider special cases of problem (2).

I. For $S \in \mathbb{S}^l$ (for $S \in \mathbb{S}_r$), we denote by $p = p(S)$ the point of Γ_r (of Γ^l) nearest to S and define

$$M(S) = \rho(p(S), S). \tag{3}$$

The following statement is obvious.

Proposition 1. *Let a group of observers $\mathbb{S}$ be such that $K_Y(S^l) \cap K_Y(S_r) \cap Y = \varnothing$ for every $S^l \in \mathbb{S}^l$ and $S_r \in \mathbb{S}_r$. An optimal trajectory $\mathcal{T}^* \in \mathbb{T}$ is characterized by the following properties:*

$p(S) \in \mathcal{T}^*$ for all S implementing the minimum $M = \min\{M(S)\colon S \in \mathbb{S}\}$;

$\rho(S, \mathcal{T}^*) \geq M$ for all $S \in \mathbb{S}$.

Every trajectory $\mathcal{T}$ containing all segments of the boundaries $K_Y(S^l) \cap \Gamma_r$ and $K_Y(S_r) \cap \Gamma^l$ and satisfying the condition $\rho(S, \mathcal{T}) \geq M$ is optimal.

II. Let $\mathbb{S} = \{S^l, S_r\}$ be a pair of observers such that $(K_Y(S^l) \cap K_Y(S_r)) \neq \varnothing$. Define

$$Q = \{x \in \overline{K}_Y(S_l) \cap \overline{K}_Y(S_r)\colon \|x - S^l\| = \|x - S_r\|\}.$$

There are two possible subcases: (II$_1$) $Q \neq \varnothing$ and (II$_2$) $Q = \varnothing$.

Every trajectory intersects the sets $K_Y(S)$, $S \in \mathbb{S}$. The best trajectory $\mathcal{T}^*$ must intersect them as far as possible from the vertices and can be arbitrary outside the set $\overline{K}_Y(S_l) \cup \overline{K}_Y(S_r)$ by the definition of the distance $\rho(t, S)$.

In subcase II$_1$, the trajectory $\mathcal{T}^*$ obviously intersects the set $\overline{K}_Y(S_l) \cap \overline{K}_Y(S_r)$; more exactly, it contains a point $p = p(S^l, S_r) \in Q$ implementing the minimum

$$M(S^l, S_r) \overset{\text{def}}{=} \min_{p \in \overline{K}_Y(S^l) \cap \overline{K}_Y(S_r)} \max\left\{\|S^l - p\|, \|S_r - p\|\right\} = \max\left\{\|S^l - p(S^l, S_r)\|, \|S_r - p(S^l, S_r)\|\right\}, \tag{4}$$

and

$$M(S^l, S_r) = \|S^l - p\| = \|S_r - p\|. \tag{5}$$

Let the point p belong to the boundary of one of the cones $K(S^l)$ and $K(S_r)$; for example, let $p \in \operatorname{bd} K(S^l)$. Then the following inequality holds for points t from this cone that lie between the arcs $C'(S^l)$ and $C'(S_r)$ intersecting $K(S^l) \cap K(S_r)$ with the end point p and radius $M(S^l, S_r)$ and centered at S^l and S_r, respectively:

$$\min\{\rho(t, S^l), \rho(t, S_r)\} \geq M(S^l, S_r). \tag{6}$$

This inequality also holds for points located inside the cone $K(S_r)$ between the arc $C'(S_r)$, where $C'(S_r) \cap K(S^l) = \varnothing$, and the segment $[p, S^l]$. Constructing the trajectory $\mathcal{T}^*$, we will use

(N$'$) the arcs $C'(S^l)$ and $C'(S_r)$ and the segment $[p, S^l]$; they contain points t satisfying inequality (6).

If the point p belongs to the interior of the set $K_Y(S^l) \cap K_Y(S_r)$, then (6) also holds for points t from this intersection lying between the circles $C'(S^l)$ and $C'(S_r)$.

In what follows, for a point $p = (\cdot, \cdot)$, we will use the notation $p'(S^l, S_r)$ in which the first argument is the vertex for which the boundary of its cone contains the point p. If $p \in (\operatorname{bd} K(S^l)) \cap (\operatorname{bd} K(S_r))$ or p is contained in the interior of the set $K_Y(S^l) \cap K_Y(S_r)$, then the order of vertices that are arguments in $p'(\cdot, \cdot)$ is not fixed.

In subcase II$_2$, the best trajectory also contains the point $p(S^l, S_r)$ that is a solution of problem (4). Assume for definiteness that

$$\|S^l - p\| < \|S_r - p\| \ \forall\, p \in K(S^l) \cap K(S_r). \tag{7}$$

Then

$$M(S^l, S_r) = \|S_r - p\|. \tag{8}$$

Inequality (6) holds for points t located in $K(S_r)$ between the segment $[p, S^l]$ and the arc $C''(S_r)$ of radius $\|S_r - p\|$ centered at S_r. Constructing the trajectory $\mathcal{T}^*$, we will use

(N$''$) the segment $L(p, S^l) \cap Y$ and the arc C''.

For a point p, we will use the notation $p = p''(S^l, S_r)$, where the first argument is the vertex at which $\min\{\|S^l - p\|, \|S_r - p\|\}$ is attained.

The point p is an end point of the arcs $C'(S^l) \cap \overline{K}(S^l)$ and $C'(S_r) \cap \overline{K}(S_r)$. Denote the other end points of these arcs by q^l and q_r, respectively.

The following statement holds (see (2)–(8)).

Proposition 2. *In case* II, $\mathbb{M}(\mathbb{S}) = \min\{M(S^l, S_r), M(S^l), M(S_r)\}$. *In subcase* II$_1$, *the required optimal trajectory is formed by the segment* $[p', S^l] \cap Y$, *the arc* $C'(S^l) \cap \overline{K}(S^l)$ *(or the arc* $C'(S_r) \cap \overline{K}(S^l)$*), and the segment* $(L(S^l, q^l)\backslash[S^l, q^l]) \cap Y$ *(or the segment* $(L(S^l, q_r)\backslash[S^l, q_r]) \cap Y$*) and is completed by parts of the boundaries* Γ^l *and* Γ_r. *In subcase* II$_2$, *the optimal trajectory is formed by the segment* $L(S^l, p'') \cap Y$ *and is completed by parts of the boundaries* Γ^l *and* Γ_r.

III. Let a triple of observers $\mathbb{S} = \{S_1^l, S_2^l, S_r\}$ be given such that $S_1^l, S_2^l \in \Gamma^l$, $S_r \in \Gamma_r$, and

$$\Big(K_Y(S_1^l) \cap K_Y(S_r)\Big) \cap \Big(K_Y(S_2^l) \cap K_Y(S_r)\Big) = \varnothing. \tag{9}$$

Proposition 3. *The following equality holds*:

$$\mathbb{M}(\mathbb{S}) = \min\{M(S_1^l, S_r),\ M(S_2^l, S_r),\ M(S_1^l),\ M(S_2^l),\ M(S_r)\}, \tag{10}$$

and there exists an optimal trajectory containing the points $p(S_1^l, S_r)$ *and* $p(S_2^l, S_r)$. *This trajectory is formed by the arcs and segments listed in* (N$'$) *and* (N$''$) *and by parts of the boundaries* Γ^l *and* Γ_r.

Proof. If the points $p(S_1^l, S_r)$ and $p(S_2^l, S_r)$ have the form $p''(S_r, S_1^l)$ and $p''(S_r, S_2^l)$, then they lie on the same side of the cone $\overline{K}(S_r)$. The part of this side belonging to Y and completed by parts of the boundaries Γ^l and Γ_r forms the trajectory $\mathcal{T}^*$. If these points have the form $p''(S_1^l, S_r)$ and $p''(S_2^l, S_r)$ and

$$\|S_r - p''(S_2^l, S_r)\| < \|S_r - p''(S_1^l, S_r)\|,$$

then we include in $\mathcal{T}^*$ the part of the side of the cone $\overline{K}(S_2^l)$ that lies in Y and the part of the side of the cone $K(S_1^l)$ from the point $p''(S_1^l, S_r)$ to the point of its intersection with Γ_r. The remaining part of the trajectory belongs to $\operatorname{bd} Y$.

Let the points $p(S_r, S_1^l)$ and $p(S_r, S_2^l)$ have the form $p''(S_1^l, S_r)$ and $p''(S_r, S_2^l)$. Then we include in $\mathcal{T}^*$ the part of the side of the cone $K(S_r)$ that lies in Y and contains the point $p''(S_r, S_2^l)$ as well

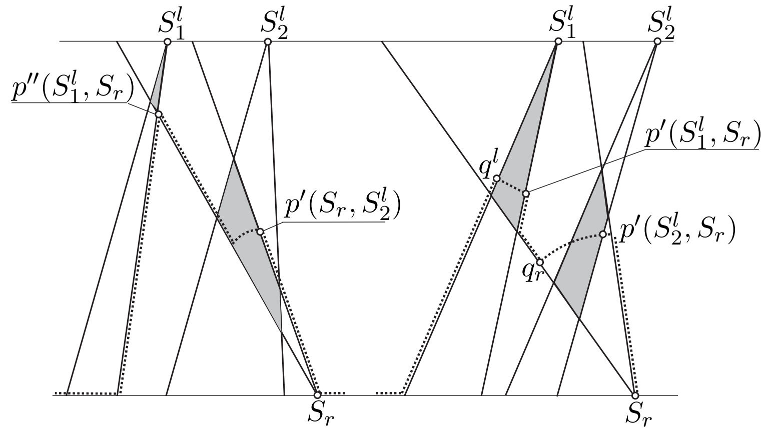

Fig. 1.

as the part of the side of the cone $K(S_1^l)$ containing the point $p''(S_1^l, S_r)$. Let these points have the form $p''(S_1^l, S_r)$ and $p'(S_r, S_2^l)$ (see Fig. 1). Then we include in $\mathcal{T}^*$ the arc $C'(S_r)$ and the following three segments: one on the side of the cone $K(S_r)$ from the point $p''(S_1^l, S_r)$ to this arc, one on the side of the cone $K(S_1^l)$ from the point $p''(S_1^l, S_r)$ to Γ_r, and one on the side of the cone $\overline{K}(S_r)$ from the point $p'(S_r, S_2^l)$ to L_r. Finally, consider the case of the points $p'(S_1^l, S_r)$ and $p'(S_r, S_2^l)$. By relation (9), we have

$$\|S_r - p(S_r, S_2^l)\| < \|S_r - p'(S_2^l, S_r)\|.$$

We include in $\mathcal{T}^*$ the arcs $C'(S_r) \cap K(S_r)$ and $C'(S_1^l) \cap K(S_1^l)$, the segment on the side of the cone $\overline{K}(S_1^l)$ from q^l to Γ_r, the segment on the side of the cone $K(S_r)$ that does not contain the point q_r from $C'(S_r)$ to the boundary L_r, and the segment on the side of the cone $\overline{K}(S_r)$ from q_r to $K(S_1^l)$ if $q_r \notin K(S_1^l)$. The constructed curve (shown by dots in the figure) is completed by parts of the boundary L_r. Proposition 3 is proved. □

IV. Consider the case of an arbitrary (finite) number of observers. It is natural to arrange them economically in a certain sense, in particular, to restrict from above the multiplicity of covering the corridor Y by the sets $K_Y(S)$, $S \in \mathbb{S}$. It is clear that a group of observers located on one "coast," for example, on Γ^l, provides a more complete cover near Γ_r than near Γ^l. The number of observers on both "coasts" must be roughly the same (in view of the openings of the cones $K_Y(S)$). We will number them from t_* to t^* using upper indices for the left boundary and lower indices for the right boundary. Thus, we have the sets of cones $\{K(S^i),\, S^i \in \mathbb{S}_l\}$ and $\{K(S_j),\ S_j \in \mathbb{S}_r\}$. Assume that

$$K_Y(S^i) \cap K_Y(S^n) = \varnothing, \quad K_Y(S_j) \cap K_Y(S_m) = \varnothing \ \text{ for } \ i \neq n,\ j \neq m. \tag{11}$$

Then

$$\big(K_Y(S^i) \cap K_Y(S_j)\big) \cap \big(K_Y(S^k) \cap K_Y(S_m)\big) = \varnothing \ \text{ for } \ (i,j) \neq (k,m), \tag{12}$$

which means that the multiplicity of covering the corridor Y by the cones $K(S)$ is at most 2.

In addition to requirements (11)–(12) on the set $\{K(S)\colon S \in \mathbb{S}\}$, we impose a regularity condition without which the general picture can be chaotic for a large number of observers. Let a pair of vertices (S^l, S_r) be such that

$$K_r^l \overset{\text{def}}{=} K_Y(S^l) \cap K_Y(S_r) \neq \varnothing.$$

The segment $[S^l, S_r]$ divides the corridor Y into two parts. We agree to call the part containing the point t_* left and the part containing the point t^* right. We will call the pair (S^l, S_r)

– left if $\overline{K}_r^l \cap [S^l, S_r] = \varnothing$ and the set K_r^l lies on the left-hand side of the corridor;

– right if $\overline{K}_r^l \cap [S^l, S_r] = \varnothing$ and the set K_r^l lies on the right-hand side of the corridor;

– mean if $\overline{K}_r^l \cap [S^l, S_r] \neq \varnothing$.

The regularity requirement consists in the following: the vertex set can be divided into groups of the form

$$(S^i, S^{i+1}, \dots, S^{i+n};\ S_j, S_{j+1}, \dots, S_{j+m}) \quad (n \geq 0,\ m \geq 0)$$

such that either every pair (S^{i+n_1}, S_{j+m_1}), $0 \leq n_1 \leq n$, $0 \leq m_1 \leq m$, is left or every such pair is right. If there are several such groups, then they alternate and there can be a group of mean pairs $(S^i, S_j), (S^{i+1}, S_{j+1}), \dots, (S^{i+k}, S_{j+k})$ between neighboring groups of left and right pairs.

Theorem. *The following equality holds*:

$$\mathbb{M}(\mathbb{S}) = \min\big\{M(S^i, S_j), M(S^i), M(S_j)\colon K(S^i) \cap K(S_j) \neq 0,\ S^i \in \mathbb{S}_l,\ S_j \in \mathbb{S}_r\big\}. \tag{13}$$

The best trajectory $\mathcal{T}^* \in \mathbb{T}$ is characterized by the properties:

(i) *$\mathcal{T}^*$ contains the points $p(S^i)$ and $p(S_j)$ for all singular observers S^i and S_j and the point $p(S^i, S_j)$ for all pairs (S^i, S_j) of observers from each group implementing the minimum* (13), *$S^j \in \mathbb{S}^l$, $S_j \in \mathbb{S}_r$*;

(ii) $\rho(S, \mathcal{T}^*) \geq \mathbb{M}$ *for all $S \in \mathbb{S}$.*

There exists the best trajectory containing all points $p(S^i, S_j)$ for $S^i \in \mathbb{S}^l$ and $S_j \in \mathbb{S}_r$ such that $K_Y(S^i) \cap K_Y(S_j) \neq \varnothing$.

Proof. Denote by D^i (by D_j) the closed domain in Y between the cones $K(S^i)$ and $K(S^{i+1})$ (the cones $K(S_j)$ and $K(S_{j+1})$, respectively), and consider the following sets of points:

$K_j^i = K_Y(S^i) \cap K_Y(S_j)$ (see (13)) is an open set with the multiplicity of covering equal to 2;

$K_Y(S^i) \cap \mathcal{D}_j$ and $K_Y(S^j) \cap \mathcal{D}^i$ are open-closed sets with the multiplicity of covering equal to 1;

$\mathcal{D}_j^i = \mathcal{D}^i \cap \mathcal{D}_j$ is a closed set of points with zero multiplicity of covering.

Note that $\rho(t, S) = +\infty\ \forall\ t \in \mathcal{D}_j^i\ \forall\ S \in \mathbb{S}$. Hence, there are no constraints on the position of trajectories $\mathcal{T}^*$ in the set $\mathcal{D}_j^i$.

An optimal trajectory in neighborhoods of the sets K_j^i was constructed in cases I–III. The construction is based on a solution $p(S^i, S_j)$ of problem (4), which, in the two possible subcases II_1 and II_2, was denoted by $p'(\cdot,\cdot)$ and $p''(\cdot,\cdot)$ with the order of the arguments S^i and S_j depending on the mutual arrangement of the cones $K(S^i)$ and $K(S_j)$.

Consider the group of left pairs (see Fig. 2). Fix an index i and consider the position of points $p(S^i, S_j)$ on the cone $K(S^i)$. If the point nearest to the vertex S^i has the form $p''(S^i, S_j)$, then there can be several consecutive points of the same form with indices j monotonically decreasing from the index j to some index $j(i) + 1$ with the growth of their distance to the vertex S^i. They all lie on the side of the cone $K(S^i)$ facing the segment $[S^i, S_{j(i)}]$. The point $p(S^i, S_{j(i)})$ next in order of distance from S^i belongs to the boundary of $K_{j(i)}^i$ and is either (a) a $p'(S_{j(i)}, S^i)$-point or a $p'(S^i, S_{j(i)})$-point or (b) a point of the form $p''(S_{j(i)}, S^i)$ lying on the side of the cone $K(S_{j(i)})$ facing the segment $[S^i, S_{j(i)}]$. In these cases (see I and II), the half-open segment of the straight line $L(S^i, p''(S^i, S_j))$ from S^i to the point of intersection with the set $\overline{K}_{j(i)}^i$ (we denote this segment by Δ^i, $S^i \notin \Delta^i$) can be included in the optimal trajectory. Let $p''(S_{j(i)}, S^m)$ be the point nearest to $S_{j(i)}$. Repeating the above argument, we see that the half-open segment of the straight line

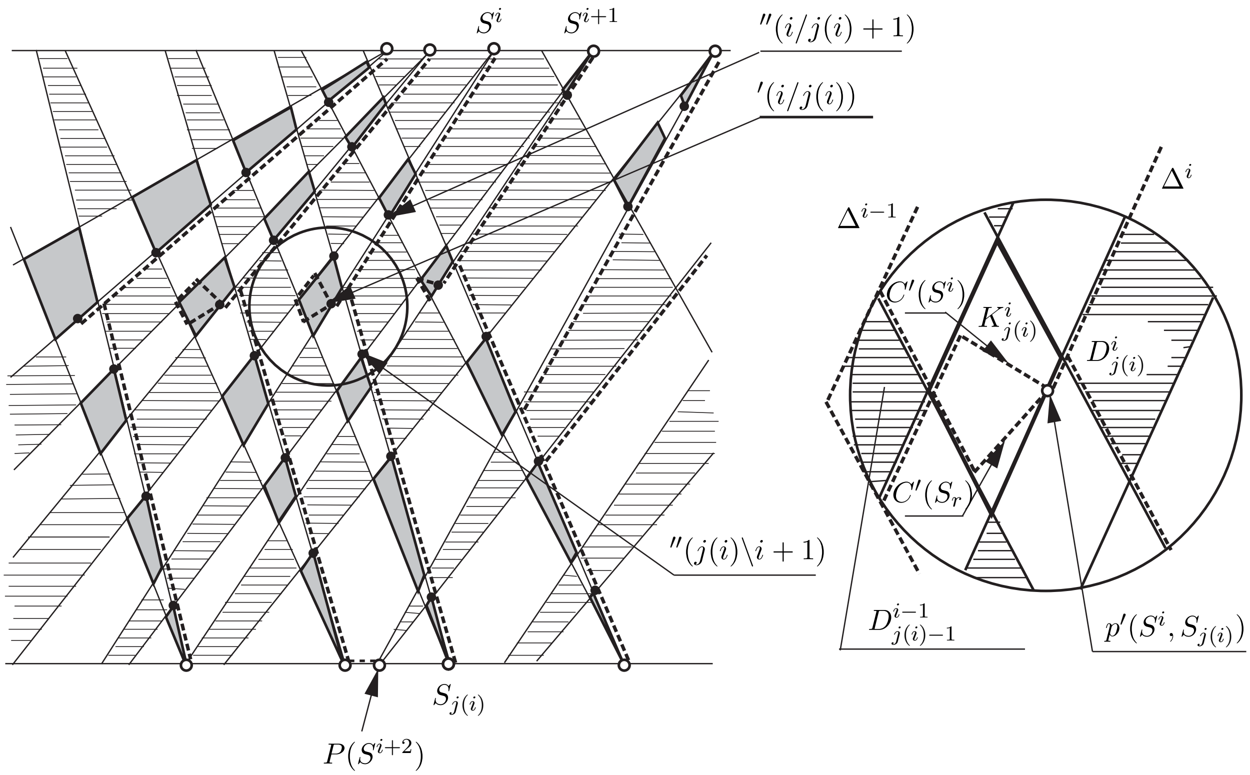

Fig. 2.

$L(S_{j(i)}, p''(S_{j(i)}, S^m))$ from the point $S_{j(i)}$ to the point of intersection with the set $\overline{K}^i_{j(i)}$ (we denote this segment by $\Delta_{j(i)}$, $S_{j(i)} \notin \Delta_{j(i)}$) can be included in the optimal trajectory. It is clear that the segments Δ^i and $\Delta_{j(i)}$ have a common end point (denoted by v_i) satisfying the inclusion

$$v_i \in (\mathrm{bd}\,\overline{K}^i_{j(i)}) \cap \mathcal{D}^i_{j(i)}. \tag{14}$$

Thus, the two-link polygonal line $\Delta^i \cup \Delta_{j(i)}$ (shown with dots in Fig. 2) can be included in the optimal trajectory so that the value $\rho(t, \mathbb{S})$, $t \in \mathcal{T}$, remains not less than the minimum (13).

In case (a), the segment $[p(S^i, S_{j(i)}), v_i]$ (see $\mathrm{II}_1, (\mathrm{N}')$) can be included in the trajectory $\mathcal{T}^*$. In case (b), according to (N$''$) from II_2, the whole segment $[p''(S_{j(i)}, S^i), S_{j(i)}]$ is included in $\mathcal{T}^*$.

Now, assume that, for given i, the point nearest to S^i has the form $p'(S_j, S^i)$ or $p'(S^i, S_j)$ for some j. Similarly to the above case, we should consider the position of points $p(S_j, S^i)$ in the cone $K(S_j)$ for different i and fixed j.

Thus, all points $p''(S^i, S_j)$ lie in the set of two-link polygonal lines of the form $\Delta^i \cup \Delta_{j(i)}$ for i such that the point $p(S^i, S_j)$ nearest to S^i is a $p''(S^i, S_j)$-point and of polygonal lines of the form $\Delta_j \cup \Delta^{i(j)}$ for j such that the point $p(S_j, S^i)$ nearest to S_j is a $p''(S_j, S^i)$-point. As shown above, the points $p'(S^i, S_{j(i)})$ and $p'(S_{j(i)}, S^i)$ lie on the boundary of the set $\overline{K}^i_{j(i)}$, and one can verify similarly that the points $p'(S_j, S^{i(j)})$ and $p'(S^{i(j)}, S_j)$ lie on the boundary of the set $\overline{K}^{i(j)}_j$. In case (a), using the arcs C' (see (N$'$)), we can connect the set $\mathcal{D}^{i-1}_{j(i)-1}$ and, hence, by (14), the polygonal line $\Delta^{i-1} \cup \Delta_{j(i)-1}$ with the set $\mathcal{D}^i_{j(i)}$ and, hence, with the polygonal line $\Delta^i \cup \Delta_{j(i)}$. In case (b), the segments Δ^i and Δ^{i-1} are connected by the segment one end point of which is $S_{j(i)}$ and the other is the intersection point of the segment Δ^{i-1} with the straight line $L(S_{j(i)}, p''(S_{j(i)}, S^i))$. The point $p(S^i)$ (the point $p(S_j)$), see case I, can be connected by the segment of the boundary Γ^l (of the boundary Γ_r) with the segment $\Delta^{n(i)}$ (segment $\Delta_{m(j)}$) nearest on the left.

Thus, we have constructed the trajectory $\mathcal{T}^*$ composed from straight line segments, fragments of the boundary Γ, and circular arcs so that the following inequality holds:

$$\rho(S, \mathcal{T}^*) \geq \min\left\{M(S^i, S_j),\ M(S^i),\ M(S_j)\colon S^i \in \mathbb{S}^l,\ S_j \in \mathbb{S}_r\right\} \quad \forall\ S \in \mathbb{S}.$$

A trajectory for the group of right pairs is constructed similarly. The problem of construction of a trajectory for two neighboring groups one of which contains left pairs and the other contains right pairs or for three neighboring groups consisting of left pairs, mean pairs, and right pairs, respectively, reduces to the problem of construction of a trajectory for two neighboring pairs that are either left and right pairs, or left and mean pairs, or mean and right pairs, which is easily solved by the methods presented in case II. The theorem is proved. □

In Fig. 2, we use the following notation for brevity: $p'(S^i, S_j) =' (i/j)$, $p''(S^i, S_j) ='' (i/j)$, and $p''(S_j, S^i) ='' (j\backslash i)$.

The right hand side depicts an enlarged fragment of the whole picture.

An optimal trajectory may contain not all of the points $p(S^i, S_j)$, $p(S^i)$, and $p(S_j)$. The above theorem enables a simplification of the problem of constructing an optimal trajectory from the parts mentioned above: circular arcs, parts of the boundary $\operatorname{bd} Y$, and segments of the boundaries of cones $K(S)$ in the case when the minimization problem (13) has a large number of solutions.

Remark. The constructed trajectory contains sections with backward motion. Find a shortest optimal trajectory is an important problem.

REFERENCES

1. V. I. Berdyshev and V. B. Kostousov, "Extremal problems in planning the route of a moving object under observation," in *Modern Problems in Mathematics and Its Applications*: *Proceedings of the 47th International Youth School–Conference, Yekaterinburg, Russia, 2016*, pp. 32–41. http://ceur-ws.org/Vol-1662/opt1.pdf

Translated by M. Deikalova