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Asymptotics and Formulas for Cubic Exponential Sums

Ghaith A. Hiary [a]

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Abstract—Several asymptotic expansions and formulas for cubic exponential sums are derived. The expansions are most useful when the cubic coefficient is in a restricted range. This generalizes previous results in the quadratic case and helps to clarify how to numerically approximate cubic exponential sums and how to obtain upper bounds for them in some cases.

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1. INTRODUCTION

Let a denote an integer, q a positive integer, b an integer relatively prime to q and $e(x) := e^{2\pi i x}$. Bombieri and Iwaniec analyzed in [1] the cubic exponential sum

$$\sum_{N<n\le 2N} e\biggl(\frac{an+bn^2}{q}+\mu n^3\biggr), \qquad q\le N, \quad (b,q)=1, \quad 0<\mu\le N^{-2}. \tag{1.1}$$

This was part of their breakthrough method to bound the maximal size of the Riemann zeta function on the critical line. In view of the importance of these sums it is of interest to study the generalized sum

$$\sum_{n=0}^{N} e\biggl(\frac{an+bn^2}{2q}\biggr) e\bigl(\alpha n+\beta n^2+\mu n^3\bigr), \tag{1.2}$$

where α, β and μ are real numbers. We give asymptotic expansions and formulas for this sum that are perhaps most useful when the cubic coefficient μ is small enough, satisfying $\mu \ll N^{-2}$. Our motivation comes in part from an algorithm to compute the zeta function derived in [5] where the essential ingredient was a method for numerically evaluating sums of the form

$$\sum_{n=0}^{N} e\bigl(\alpha n+\beta n^2+\mu n^3\bigr), \qquad \mu\ll N^{-2}. \tag{1.3}$$

In particular, those asymptotics could improve the practicality of this method by enabling the use of an explicit asymptotic expansion instead of precise numerical computations when appropriate. Furthermore, as a by-product we obtain upper bounds for cubic sums. Our results are influenced by the work of Bombieri and Iwaniec in [1], and the work of Fiedler, Jurkat and Körner in [3]. The latter obtained asymptotic expansions for quadratic exponential sums that yield a rough approximation for such sums (typically accurate to within the square root of the length). See also [4, 6].

We introduce some notation first. Let $\langle x\rangle := \lfloor x+1/2\rfloor$ denote the nearest integer to x, $\operatorname{sgn}(x) := 1$ or -1 according to whether $x\ge 0$ or $x<0$, and $\mathbb{1}_{\mathcal{C}}$ be the indicator function of whether the condition $\mathcal{C}$ is satisfied. For integer k, define $k^* := -k\bar{k}^2$ where $k\bar{k}\equiv 1 \pmod q$ subject to the additional condition that $4\mid \bar{k}$ if q is odd, and let $\delta := 0$ or 1 according to whether bq is even or odd. These definitions of k^* and δ come directly from the formula for a complete Gauss sum in [3, Lemma 1]. Furthermore, let $\delta_1 = 0$ or 1 according to whether $bq+a$ is even or odd.

[a] Department of Mathematics, The Ohio State University, 231 West 18th Ave., Columbus, OH 43210, USA.
E-mail address: hiary.1@osu.edu

Define the Gauss sum

$$g(b,q) := \frac{1}{2\sqrt{q}} \sum_{h=0}^{2q-1} e\left(\frac{bh^2}{2q}\right), \tag{1.4}$$

which has modulus 1 or 0 according to whether bq is even or not. It is well known that this sum has a closed-form evaluation in terms of the Kronecker symbol. In addition, define

$$H_N(\alpha,\beta,\mu) := \sum_{n=0}^{N} e\big(\alpha n + \beta n^2 + \mu n^3\big), \tag{1.5}$$

where the summation is taken over $N \le n \le 0$ if N is negative. Note that in contrast to the sum (1.2), $H_N(\alpha,\beta,\mu)$ does not incorporate rational approximations for the linear and quadratic arguments explicitly.

We write $h_1(x) = O(h_2(x))$, or equivalently $h_1(x) \ll h_2(x)$, when there is an absolute constant C_1 such that $|h_1(x)| \le C_1 h_2(x)$ for all values of x under consideration (which will usually make a set of the form $x \ge x_0$).

Using conjugation if necessary, we may restrict ourselves to $\mu \ge 0$. In fact, we will assume $\mu > 0$, since otherwise the problem reduces to the quadratic sums already treated in [3]. With this in mind, the basic results are given in Propositions 8.1, 8.2 and 8.7 in Section 8. These propositions furnish transformation formulas for cubic exponential sums including explicit error bounds. As special cases, we give in the next few paragraphs several formulas and asymptotic expansions for cubic sums that are meant to be interesting specializations of these propositions.

Theorem 1.1 below is a specialization of Proposition 8.1. The theorem isolates a main term for the cubic sum $H_N(\alpha,\beta,\mu)$ and says, roughly, that $H_N(\alpha,\beta,\mu)$ splits into the product of an "arithmetic factor" which is (mostly) determined by rational approximations to α and β, times an "analytic factor" determined by the error in the said approximation and by μ and N.

Theorem 1.1. *Suppose $2\beta = b/q + 2\eta$ where $|\eta| \le 1/(8qN)$ and $0 < q \le 4N$ with $(b,q) = 1$, $2\alpha = a/q + 2\epsilon$ where $-1/(4q) \le \epsilon < 1/(4q)$, and $6\mu q N^2 < 1$. Define $u := \langle 2q(\epsilon - \eta^2/(3\mu))\rangle$, $v := \langle 2q(\epsilon + 2\eta N + 3\mu N^2)\rangle$, and let*

(i) $\Omega := \{0, v\}$ *if* $\eta \ge 0$ *or* $\eta \le -3\mu N$,

(ii) $\Omega := \{0, u, v\}$ *if* $-3\mu N < \eta < 0$.

Then

$$H_N(\alpha,\beta,\mu) = \sum_{\substack{\ell\in\Omega \\ \text{distinct } \ell}} D_\ell(a,b,q) \int_0^N e\left(\epsilon t + \eta t^2 + \mu t^3 - \frac{\ell t}{2q}\right) dt + O\big(\sqrt{q}\log(2q)\big),$$

where the arithmetic factor $D_\ell(a,b,q)$ is given by

$$D_\ell(a,b,q) := \mathbb{1}_{\ell \equiv \delta_1 \pmod 2} \frac{g(b+\delta q, q)}{\sqrt{q}}\, e\left(\frac{b^*(a+\ell)^2}{8q}\right).$$

The Diophantine conditions on α and β appearing in the theorem can always be fulfilled via the Dirichlet approximation theorem and using a continued fractions algorithm (though the denominator q that arises for a generic β can be of the same order as N). Hence, the theorem can be applied with any α and β, provided that μ is small enough. If $\eta \ge 0$ or $\eta \le -3\mu N$, on the one hand, then exactly one of the D_ℓ terms can possibly be nonzero. Moreover, if $\eta \ge 0$ then $\Omega \subset \{0,1\}$, while if $\eta \le -3\mu N$ then $\Omega \subset \{0,-1\}$. On the other hand, if $-3\mu N < \eta < 0$, then at most two of the D_ℓ terms can possibly be nonzero and $\Omega \subset \{0,1,-1\}$. For example, if $\eta \ge 0$ and $v = 0$, which is a typical situation, then

$$H_N(\alpha,\beta,\mu) = D_0(a,b,q)\int_0^N e\big(\epsilon t+\eta t^2+\mu t^3\big)\,dt + O\big(\sqrt{q}\log(2q)\big). \tag{1.6}$$

Note that if $\delta_1 = 1$ then $D_0(a,b,q)=0$, and so there is no main term in this case; in particular, $H_N(\alpha,\beta,\mu) \ll \sqrt{q}\log(2q) \ll \sqrt{N}\log(N+2)$.

Remark. If we let $f(x)=\epsilon x+\eta x^2+\mu x^3$ for a minute, then in the notation of the next section $u=\langle 2qf'(-\omega)\rangle$, $v=\langle 2qf'(N)\rangle$ and $0=\langle 2qf'(0)\rangle$. Also, $g(b,q)=G(0,b;2q)$ and $H_N(\alpha,\beta,\mu)=C(N;0,0,1;f)$.

Theorem 1.2 is also a specialization of Proposition 8.1 and provides a van der Corput type iteration for H_N. Using the periodicity relation $e(z+1)=e(z)$, we may restrict α, β and μ to the interval $[-1/2,1/2)$, where we have $0<\mu$ as before. In view of this, the length $|N'|$ of the transformed sum below will be smaller than the length N of the original sum provided that β and μ are small enough.

Theorem 1.2. *Let $N' = \langle \alpha+2\beta N+3\mu N^2\rangle$. Suppose that $\alpha,\beta\in[-1/2,1/2)$ and $0<6\mu N^2<1$. If $|\beta|>1/N$, then*

$$\begin{aligned}H_N(\alpha,\beta,\mu) &= \frac{c_2}{\sqrt{2|\beta|}}\,H_{N'}\bigg(\frac{\alpha}{2\beta}+\frac{3\alpha^2\mu}{8\beta^3},-\frac{1}{4\beta}-\frac{3\alpha\mu}{8\beta^3},\frac{\mu}{8\beta^3}\bigg)\\ &\quad + O\bigg(\frac{\mu N^2+\mu^2N^5}{\sqrt{|\beta|}}+\frac{\mathbb{1}_{\beta>0}}{\sqrt{\beta}}+\frac{\mathbb{1}_{\beta<0}}{\sqrt{-(\beta+3\mu N)}}+\log(|N'|+2)\bigg),\end{aligned}$$

where

$$c_2 := e\bigg(\frac{\operatorname{sgn}(\beta)}{8}-\frac{\alpha^2}{4\beta}-\frac{\alpha^3\mu}{8\beta^3}\bigg).$$

The term $\mathbb{1}_{\beta>0}/\sqrt{\beta}$ in the remainder arises from estimating boundary terms $\mathcal{B}$ in Proposition 8.1 when $\beta>0$, while $\mathbb{1}_{\beta<0}/\sqrt{-(\beta+3\mu N)}$ arises from estimating $\mathcal{B}$ when $\beta<-1/N$ (see the proof of the theorem in Section 9). Additionally, one can replace $1/\sqrt{\beta}$ with $\min\{N,1/\sqrt{\beta}\}$ and similarly for $1/\sqrt{-(\beta+3\mu N)}$. Of course, both of these terms can be removed if $\mathcal{B}$ is included explicitly in the theorem. The term $(\mu N^2+\mu^2N^5)/\sqrt{|\beta|}$ in the remainder comes from trivially estimating the derivatives of $H_N(\alpha,\beta,\mu)$ with respect to α. Therefore, if one is interested in understanding the rough size of $H_N(\alpha,\beta,\mu)$ rather than deriving an asymptotic expansion, then it is better to bound these derivatives using the second mean value theorem for the Riemann integral, which yields

Corollary 1.3.

$$\begin{aligned}|H_N(\alpha,\beta,\mu)| &\le c_{\beta,\mu,N}H_{N'}^{\max}\bigg(\frac{\alpha}{2\beta}+\frac{3\alpha^2\mu}{8\beta^3},-\frac{1}{4\beta}-\frac{3\alpha\mu}{8\beta^3},\frac{\mu}{8\beta^3}\bigg)\\ &\quad + O\bigg(\frac{\mathbb{1}_{\beta>0}}{\sqrt{\beta}}+\frac{\mathbb{1}_{\beta<0}}{\sqrt{-(\beta+3\mu N)}}+\log(|N'|+2)\bigg),\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}H_N^{\max}(\alpha,\beta,\mu) &:= \max_{N_1\in[0,N]}\bigg|\sum_{n=N_1}^{N} e\big(\alpha n+\beta n^2+\mu n^3\big)\bigg|,\\ c_{\beta,\mu,N} &:= \frac{1+(c_3\mu N+c_4\mu^2N^4)|\beta|^{-1}}{\sqrt{2|\beta|}}\end{aligned} \tag{1.8}$$

and c_3 and c_4 are absolute nonnegative constants.

Interestingly, one can apply estimate (1.7) repeatedly until one of the conditions required by Theorem 8.2 fails. This could yield useful bounds for H_N in some applications. Also, if desired, the last theorem and corollary can both be written in a more symmetric form by using the change of variable $\widetilde{H}_N(\alpha,\beta,\mu) := H_N(\alpha,\beta/2,\mu)$. This enables absorbing the various powers of 2 that accompany β.

Our last example is a corollary of Proposition 8.7. This proposition furnishes a transformation formula for cubic sums when $\alpha=\beta=0$ and with rational approximations included explicitly, which was the type of sum considered in [1].

Corollary 1.4.

$$\sum_{n=0}^{N} e\biggl(\frac{an+bn^2}{q}+\mu n^3\biggr) \ll \mu^{1/2}N^{3/2}q^{1/2}+\min\{N,\mu^{-1/3}\}q^{-1/2}+\mu N q^{1/2}+q^{1/2}\log(\mu N^2 q+2q).$$

Proofs of Theorems 1.1 and 1.2 are given in Section 9. In Section 10, we suggest few improvements to these theorems. The remaining sections are devoted to proving the propositions in Section 8.

2. AN INITIAL TRANSFORMATION

Given a sequence of complex numbers $\{a_n\}$ and a set $\mathcal{S}\subset\mathbb{Z}$, we follow the notation in [3] and define

$$\sum_{n\in\mathcal{S}} a_n := \lim_{M\to\infty}\sum_{n=-M}^{M} a_n \mathbb{1}_{n\in\mathcal{S}}, \tag{2.1}$$

where $\mathbb{1}_{n\in\mathcal{S}}=1$ if $n\in\mathcal{S}$ and $\mathbb{1}_{n\in\mathcal{S}}=0$ otherwise. Let

$$f(x) := \mu x^3+\beta x^2+\alpha x$$

where α, β and μ are real numbers. Let $C(N;a,b,q;f)$ denote the cubic exponential sum

$$C(N;a,b,q;f) := \sum_{n=0}^{N} e\biggl(\frac{an+bn^2}{2q}\biggr)e(f(n)). \tag{2.2}$$

To analyze this sum, we will make heavy use of a truncated Airy–Hardy integral

$$\mathrm{AH}(\omega,N;\mu,s) := \int_{\omega}^{\omega+N} e(\mu t^3-3st)\,dt \tag{2.3}$$

and of the completed integrals

$$\begin{aligned}\mathrm{AH}(\mu,s) &:= \mathrm{AH}(0,\infty;\mu,s)=\int_0^\infty e(\mu t^3-3st)\,dt,\\ \mathrm{AI}(\mu,s) &:= \mathrm{AH}(\mu,s)+\overline{\mathrm{AH}(\mu,s)}=\int_{-\infty}^{\infty} e(\mu t^3-3st)\,dt.\end{aligned} \tag{2.4}$$

We will also make use of the complete Gauss sum

$$G(a,b;2q) := \frac{1}{2\sqrt{q}}\sum_{h=0}^{2q-1} e\biggl(\frac{ah+bh^2}{2q}\biggr). \tag{2.5}$$

We begin by applying the Poisson summation to decompose $C(N;a,b,q;f)$. In doing so, only half of the boundary terms at $n=0$ and $n=N$ are included, giving the term $(1+B)/2$ in equation (2.9) below.

Lemma 2.1. *Let $\mathcal{E} := \{m\in\mathbb{Z} \mid bq+a+m\in 2\mathbb{Z}\}$. Define*

$$\omega := \frac{\beta}{3\mu}, \qquad s_m := \mu\omega^2 + \frac{m/2-q\alpha}{3q}, \qquad g(m) := \frac{b^*(a+m)^2}{8q} \tag{2.6}$$

and

$$c_0 := e\biggl(\frac{2\omega^2\beta}{3} - \omega\alpha\biggr), \qquad c_1 = c_0\, G(0,b+\delta q;2q). \tag{2.7}$$

Also define

$$B := e\biggl(\frac{aN+bN^2}{2q} + f(N)\biggr), \qquad B_m := e\biggl(\frac{\omega m}{2q} + g(m)\biggr)\,\mathrm{AH}(\omega,N;\mu,s_m). \tag{2.8}$$

Then

$$C(N;a,b,q;f) = \frac{1+B}{2} + \frac{c_1}{\sqrt{q}}\sum_{m\in\mathcal{E}} B_m. \tag{2.9}$$

Remark. To avoid notational clutter, we suppressed some parameter dependencies; e.g., we have $s_m = s_m(\alpha,\beta,\mu,q)$, $g(m)=g(a,b,q;m)$ and $B=B(N;a,b,q;f)$.

Proof of Lemma 2.1. Divide the sum along residue classes modulo $2q$, which gives

$$C(N;a,b,q;f) = \sum_{h=0}^{2q-1} e\biggl(\frac{ah+bh^2}{2q}\biggr) \sum_{-h/(2q)\le r\le (N-h)/(2q)} e(f(h+2qr)). \tag{2.10}$$

Apply the Poisson summation formula (see, e.g., [2, p. 14]) to each inner sum, followed by the change of variable $t \leftarrow h+2qt$. The inner sum is thus equal to

$$\frac{\mathbb{1}_{h=0} + B\cdot \mathbb{1}_{h\equiv N \pmod{2q}}}{2} + \frac{1}{2q}\sum_{m\in\mathbb{Z}} e\biggl(\frac{mh}{2q}\biggr)\int_0^N e\biggl(f(t)-\frac{mt}{2q}\biggr)\,dt. \tag{2.11}$$

Substituting (2.11) into (2.10) and then recalling the Gauss sum definition (2.5), we obtain

$$C(N;a,b,q;f) = \frac{1+B}{2} + \frac{1}{\sqrt{q}}\sum_{m\in\mathbb{Z}} G(a+m,b;2q)\int_0^N e\biggl(f(t)-\frac{mt}{2q}\biggr)\,dt. \tag{2.12}$$

Furthermore, by [3, Lemma 1],

$$G(a+m,b;2q) = e(g(m))\,G(0,b+\delta q;2q)\cdot \mathbb{1}_{m\in\mathcal{E}}. \tag{2.13}$$

The integral on the right-hand side of (2.12) has a saddle point (i.e., a point $t\in[0,N]$ such that $f'(t)-m/(2q)=0$) whenever $-\omega\pm\sqrt{s_m/\mu}\in[0,N]$, where $\omega=\beta/(3\mu)$. To isolate the contribution of this saddle point, we follow [1] expanding $f(t)-mt/(2q)$ around $t=-\omega$. This has the advantage that $f''(-\omega)=0$ and will help unify the subsequent analysis in terms of the Airy–Hardy integral.

With this in mind, let $y = t + \omega$, appeal to the identity

$$f(t) - \frac{mt}{2q} = \left(\frac{\omega m}{2q} + \frac{2\omega^2\beta}{3} - \alpha\omega\right) + \left(\alpha - \frac{m}{2q} - 3\mu\omega^2\right)y + \mu y^3, \tag{2.14}$$

which is a Taylor expansion of the left-hand side around $t = -\omega$, and use the change of variable $y \leftarrow t + \omega$. This leads to the formula

$$\int_0^N e\left(f(t) - \frac{mt}{2q}\right)dt = e\left(\frac{\omega m}{2q} + \frac{2\omega^2\beta}{3} - \omega\alpha\right)\mathrm{AH}(\omega, N; \mu, s_m). \tag{2.15}$$

Substituting formulas (2.15) and (2.13) into (2.12) and recalling the definitions of B_m and c_1 immediately yield the lemma. □

Remark. The formula in [3, p. 132] gives an explicit evaluation of $G(0, b + \delta q; 2q)$ in terms of the Kronecker symbol. In particular, $|G(0, b + \delta q; 2q)| = 1$ or 0.

3. ANALYSIS OF THE TRANSFORMED SUM

The integral $\mathrm{AH}(\omega, N; \mu, s_m)$ in (2.9) is treated according to the following cases.

(1) If the integrand in $\mathrm{AH}(\omega, N; \mu, s_m)$ contains one interior saddle point, i.e., if the derivative $3\mu t^2 - 3s$ vanishes exactly once over $t \in (\omega, \omega + N)$, then the main term in our evaluation of $\mathrm{AH}(\omega, N; \mu, s_m)$ will be given by the completed Airy–Hardy integral $\mathrm{AH}(\mu, s)$ or its conjugate $\overline{\mathrm{AH}(\mu, s)}$.

(2) If there are two interior saddle points, then the main term will be given by the completed Airy integral $\mathrm{AI}(\mu, s) = \mathrm{AH}(\mu, s) + \overline{\mathrm{AH}(\mu, s)}$.

(3) If there are saddle points at the edge of the integration interval (i.e., at $t = \omega$ or $t = \omega + N$), then a special treatment is required.

(4) Last, in the absence of a saddle point, $\mathrm{AH}(\omega, N; \mu, s_m)$ will be estimated via Lemmas 4.2 and 4.3 in [7], or using integration by parts.

With this in mind, let

$$\|f'\|_N^+ := \max_{0\le x\le N} f'(x), \qquad \|f'\|_N^- := \min_{0\le x\le N} f'(x) = -\|-f'\|_N^+. \tag{3.1}$$

(Note that $\|\cdot\|_N^-$ is not a norm since it does not satisfy the usual triangle inequality, but rather a "reversed" inequality.) The quadratic polynomial $f'(x)$ achieves its minimum at $x = -\omega$. Using this and the earlier assumption $\mu > 0$ (so $f'(x) \to +\infty$ as $x \to \pm\infty$), we deduce that

$$\|f'\|_N^+ = \begin{cases} f'(N) & \text{if } \ \omega \ge -\dfrac{N}{2}, \\ f'(0) & \text{if } \ \omega < -\dfrac{N}{2}. \end{cases} \tag{3.2}$$

Also,

$$\|f'\|_N^- = \begin{cases} f'(0) & \text{if } \ \omega > 0, \\ f'(-\omega) & \text{if } \ -N \le \omega \le 0, \\ f'(N) & \text{if } \ \omega < -N. \end{cases} \tag{3.3}$$

Note that $\|f'\|_N^-$ and $\|f'\|_N^+$ are continuous in ω.

Let us split the range of summation in (2.9) into three intervals determined by the points

$$M_1 := \langle 2q\|f'\|_N^-\rangle \qquad \text{and} \qquad M_2 := \langle 2q\|f'\|_N^+\rangle, \tag{3.4}$$

noting that by definition $M_1 \le M_2$. In addition, we will make use of

$$M := M_2 - M_1. \tag{3.5}$$

It might enlighten matters at this point to refer to the van der Corput iteration in [5]. If $\omega > 0$, then the derivative $f'(x)$ is strictly increasing on $[0, N]$. So, for $a = b = 0$ in the cubic sum $C(N; a, b, q; f)$, the van der Corput iteration reads

$$\sum_{n=0}^{N} e(f(n)) = e^{\pi i/4} \sum_{f'(0)\le m\le f'(N)} \frac{e(f(x_m) - m x_m)}{\sqrt{|f''(x_m)|}} + \mathcal{R}_{N,f}, \tag{3.6}$$

where x_m is the (unique) solution of $f'(x) = m$ in $0 \le x \le N$ and $\mathcal{R}_{N,f}$ is a remainder term (see [5]). Similarly, if $\omega < -N$, then the derivative $f'(x)$ is strictly decreasing on $[0, N]$, in which case the iteration (3.6) is modified to have $e^{-\pi i/4}$ (instead of $e^{\pi i/4}$) in front, and the range of summation becomes $f'(N) \le m \le f'(0)$. In either case, and after allowing for a and b not necessarily zero, we find that there is a single saddle point if $m \in (M_1, M_2)$ and no saddle point if $m \notin [M_1, M_2]$, with $m = M_1$ or M_2 being boundary cases.

In contrast, when $-N < \omega < 0$, the form of the van der Corput iteration is significantly different because $f'(x)$ is not strictly monotonic but has a minimum at $x = -\omega$, so both saddle points $-\omega \pm \sqrt{s_m/\mu}$ could fall in $(0, N)$. Explicitly, if $0 \le s_m \le \mu \min\{\omega^2, (\omega + N)^2\}$, then the integral $\mathrm{AH}(\omega, N; \mu, s_m)$ has two saddle points (counted with multiplicity). Now, recalling that $s_m = \mu\omega^2 + (m/2 - q\alpha)/(3q)$, we find that the condition $s_m \ge 0$ is met precisely when $m \ge 2qf'(-\omega) = 2q\|f'\|_N^-$, and so certainly when $m > M_1$. Moreover, since $-N < \omega < 0$ by assumption,

$$\min\{\omega^2, (\omega + N)^2\} = \begin{cases} \omega^2 & \text{if } \omega \ge -\dfrac{N}{2}, \\[2mm] (\omega + N)^2 & \text{if } \omega < -\dfrac{N}{2}. \end{cases} \tag{3.7}$$

Hence, the condition $s_m \le \mu \min\{\omega^2, (\omega + N)^2\}$ is met precisely when

$$m \le \begin{cases} 2qf'(0) & \text{if } \omega \ge -\dfrac{N}{2}, \\[2mm] 2qf'(N) & \text{if } \omega < -\dfrac{N}{2}. \end{cases} \tag{3.8}$$

This motivates defining

$$M^* = \begin{cases} \langle 2qf'(0)\rangle & \text{if } \omega \ge -\dfrac{N}{2}, \\[2mm] \langle 2qf'(N)\rangle & \text{if } \omega < -\dfrac{N}{2}. \end{cases} \tag{3.9}$$

So, for $-N < \omega < 0$ the integral $\mathrm{AH}(\omega, N; \mu, s_m)$ has two saddle points if $m \in (M_1, M^*)$, a single saddle point if $m \in (M^*, M_2)$, and no saddle point if $m \notin [M_1, M_2]$.

Last, we will use the boundary set

$$\Omega = \{M_1, M^*, M_2\}, \tag{3.10}$$

corresponding to the terms in (2.9) that might contain a saddle point at the edge. We observe that if $\omega \notin (-N,0)$ then $M^* = \langle 2q\|f'\|_N^-\rangle$. Thus, $M^* = M_1$ and $\Omega = \{M_1, M_2\}$ in this case. Also, if $\omega = -N/2$ then $M^* = M_2$, and so $\Omega = \{M_1, M_2\}$ over a neighborhood of $\omega = -N/2$.

We will use the following lemmas in the sequel.

Lemma [7, Lemma 4.2]. *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic and either $F'(x) \ge m > 0$ or $F'(x) \le -m < 0$ throughout the interval $[a,b]$. Then*

$$\left|\int_a^b e^{iF(x)}\,dx\right| \le \frac{4}{m}.$$

Moreover, if $G(x)$ is a monotonic function over $[a,b]$ such that $|G(x)| \le G$ over $[a,b]$, then

$$\left|\int_a^b G(x)e^{iF(x)}\,dx\right| \le \frac{4G}{m}.$$

Lemma [7, Lemma 4.4]. *Let $F(x)$ be a real twice differentiable function such that either $F''(x) \ge r > 0$ or $F''(x) \le -r < 0$ throughout the interval $[a,b]$. Then*

$$\left|\int_a^b e^{iF(x)}\,dx\right| \le \frac{8}{\sqrt{r}}.$$

4. TERMS WITH NO SADDLE POINT

We will have two treatments for the terms with no saddle point. Define the first tail of the sum in (2.9) by

$$\Upsilon_1 := \sum_{\substack{m\in\mathcal{E}\\ m\notin[M_1-q,M_2+q]}} e\biggl(\frac{\omega m}{2q} + g(m)\biggr)\,\mathrm{AH}(\omega,N;\mu,s_m) \tag{4.1}$$

and the second tail by

$$\Upsilon_2 := \sum_{\substack{m\in\mathcal{E},\ m\notin[M_1,M_2]\\ m\in[M_1-q,M_2+q]}} e\biggl(\frac{\omega m}{2q} + g(m)\biggr)\,\mathrm{AH}(\omega,N;\mu,s_m). \tag{4.2}$$

In Lemmas 4.1 and 4.3 we bound Υ_1, and in Lemma 4.2 we bound Υ_2. In both cases, we will use the following integration by parts formula: Let

$$\phi_x(\mu,s) := \frac{1}{6\pi i}\,\frac{e(\mu x^3 - 3sx)}{\mu x^2 - s}; \tag{4.3}$$

then

$$\mathrm{AH}(u,v;\mu,s) = \phi_{u+v}(\mu,s) - \phi_u(\mu,s) + \frac{1}{3\pi i}\int_u^{u+v} \frac{\mu t e(\mu t^3 - 3st)}{(\mu t^2 - s)^2}\,dt \tag{4.4}$$

provided that $\mu t^2 - s \ne 0$ for $t \in [u, u+v]$. Starting with Υ_1 and taking $u = \omega$ and $v = N$ in the above formula, we are motivated to write

$$\Upsilon_1 = \widetilde{\Phi}_{\omega+N} - \widetilde{\Phi}_\omega + \bigl(\Upsilon_1 - \widetilde{\Phi}_{\omega+N} + \widetilde{\Phi}_\omega\bigr),$$

where, after simplification,

$$\widetilde{\Phi}_{\omega+x} = \frac{qe(f(x))}{\pi i c_0} \sum_{\substack{m\in\mathcal{E}\\ m\notin[M_1-q,M_2+q]}} \frac{e(g(m)-xm/2q)}{2qf'(x)-m} \tag{4.5}$$

and the summation is done by pairing the terms for m and $-m$ whenever possible (as was decreed in Section 2). This sum is convergent at $x=\omega$ and $x=\omega+N$, which is seen on dividing the sum along residue classes modulo $2q$ and using the periodicity of $e(g(m))$ and $\mathbb{1}_{m\in\mathcal{E}}$ modulo $2q$ (see the proof of Lemma 4.3 below for details).

Lemma 4.1.

$$\bigl|\Upsilon_1 - \widetilde{\Phi}_{\omega+N} + \widetilde{\Phi}_{\omega}\bigr| \le \frac{128(2|\beta|+3\mu N)}{\pi^2}\,\frac{q^3(2q+9)}{(2q+1)^3}.$$

Proof. If $m\notin[M_1,M_2]$, then $\mu t^2 - s_m$ does not vanish over $t\in[\omega,\omega+N]$, which is seen on noting that $6q(\mu(\omega+u)^2-s_m) = 2qf'(u)-m$ and using the definitions of M_1 and M_2. Therefore, we can apply the integration by parts formula (4.4) to $\mathrm{AH}(\omega,N;\mu,s_m)$. The function $1/(\mu t^2-s_m)^2$ that arises is monotonic over $\omega\le t\le 0$ and $0\le t\le\omega+N$ separately. This is seen on noting that the derivative $\frac{d}{dt}(\mu t^2-s_m)^2 = 4\mu t(\mu t^2-s_m)$ has a single root at $t=0$ over $[\omega,\omega+N]$. So we can apply the second mean value theorem for the Riemann integral to each of the intervals $[\omega,0]$ and $[0,\omega+N]$ in turn. We thus find that the integral on the right-hand side of (4.4) is bounded by

$$\frac{2}{3\pi}\max_{\omega\le t\le\omega+N}\frac{1}{|\mu t^2-s_m|^2} \times\left(\max_{\omega\le\omega_1<\omega_2\le 0}\left|\int_{\omega_1}^{\omega_2}\mu t\, e(\mu t^3-3s_m t)\,dt\right| + \max_{0\le\omega_1<\omega_2\le\omega+N}\left|\int_{\omega_1}^{\omega_2}\mu t\, e(\mu t^3-3s_m t)\,dt\right|\right), \tag{4.6}$$

where the extra 2 in front is because we consider the real and imaginary parts of $e(\mu t^3-3s_m t)$ separately when applying the second mean value theorem for the Riemann integral. Using the second mean value theorem once again (this time to remove the t from each integral in (4.6)), we find, on applying Lemma 4.2 in [7], that the expression in (4.6) is not greater than

$$\frac{4}{9\pi^2}\max_{\omega\le t\le\omega+N}\frac{\mu(2|\omega|+N)}{|\mu t^2-s_m|^3}. \tag{4.7}$$

Writing $t=\omega+u$ with $0\le u\le N$ and recalling the definitions of s_m and ω give $\mu t^2 - s_m = (2qf'(u)-m)/(6q)$. So

$$\max_{\omega\le t\le\omega+N}\frac{1}{|\mu t^2-s_m|} = \max_{0\le u\le N}\frac{6q}{|2qf'(u)-m|}. \tag{4.8}$$

Combining (4.8) and the observation $\mu(2|\omega|+N) = 2|\beta|/3+\mu N$, we see that the expression in (4.7) is bounded by

$$\frac{32}{\pi^2}\max_{0\le u\le N}\frac{q^3(2|\beta|+3\mu N)}{|2qf'(u)-m|^3}. \tag{4.9}$$

Now, by definition, $M_1-1/2\le 2qf'(u)\le M_2+1/2$ over $0\le u\le N$. Moreover, $m\in\mathcal{E}$ is either always odd or always even. Hence,

$$\sum_{\substack{m\in\mathcal{E}\\ m\notin[M_1-q,M_2+q]}}\max_{0\le u\le N}\frac{q^3(2|\beta|+3\mu N)}{|2qf'(u)-m|^3} \le 2(2|\beta|+3\mu N)\sum_{j\ge 0}\frac{q^3}{(q+2j+1/2)^3}. \tag{4.10}$$

We isolate the term with $j = 0$ in the last sum and note that the function $1/(q + 2x + 1/2)^3$ is decreasing. This gives, on comparing the sum to an integral, that the sum on the right-hand side of (4.10) is bounded by

$$\frac{q^3}{(q+1/2)^3} + \int_0^\infty \frac{q^3}{(q+2x+1/2)^3}\,dx = \frac{2q^3(9+2q)}{(2q+1)^3}. \tag{4.11}$$

Substituting this into (4.10) and then back into (4.9) yields the lemma. □

Lemma 4.2.

$$|\Upsilon_2| \le \frac{32}{\pi} q + \frac{8}{\pi} q\log(2q-1).$$

Proof. Note that $\mu t^2 - s_m$ is monotonic over each of $[\omega, 0]$ and $[0, \omega + N]$. So we can apply Lemma 4.2 in [7] in each interval separately to deduce that

$$|\mathrm{AH}(\omega, N; \mu, s_m)| \le \frac{4}{\pi} \max_{\omega\le t\le\omega+N} \frac{1}{3|\mu t^2 - s_m|}. \tag{4.12}$$

Write $t = \omega + u$, where $0 \le u \le N$, and then proceed as in the proof of Lemma 4.1 to arrive at the same formula (4.8), and ultimately at the estimate

$$|\Upsilon_2| \le \frac{8}{\pi} \sum_{0\le j\le (q-1)/2} \frac{2q}{(2j+1/2)}. \tag{4.13}$$

Next, we isolate the term corresponding to $j = 0$ and bound the remaining sum by an integral; i.e., we obtain the bound

$$4q + 2q \int_0^{(q-1)/2} \frac{1}{2x+1/2}\,dx = 4q + q\log(2q-1). \tag{4.14}$$

Substituting this into (4.13) proves the claim. □

We now consider the sizes of $\widetilde{\Phi}_\omega$ and $\widetilde{\Phi}_{\omega+N}$. To this end, let us introduce the quantity

$$M_{\max} = 2q|f'(0)| + 2q|f'(N)| + |M_1| + |M_2| + q, \tag{4.15}$$

which will serve to "symmetrize" the summation interval below. This choice of $M_{\max}$ is a little arbitrary since we need only ensure that $M_{\max} \ge |M_1 - q|$ and $M_{\max} \ge |M_2 + q|$.

Lemma 4.3.

$$|\widetilde{\Phi}_{\omega+N}| + |\widetilde{\Phi}_\omega| \le \frac{28q}{\pi} + \frac{4q}{\pi}\log\biggl(\frac{M_{\max}}{q+1/2} + 1\biggr).$$

Proof. Recalling the formula (4.5) for $\widetilde{\Phi}_{\omega+x}$, we wish to replace the summation condition $m \notin [M_1 - q, M_2 + q]$ in this formula by the symmetric condition $|m| > M_{\max}$. Note that $f'(0) = \alpha$, $e(g(m))$ is periodic modulo $2q$, and $\mathbb{1}_{m\in\mathcal{E}}$ is periodic modulo 2. So dividing the sum along residue classes modulo $2q$, we obtain

$$\begin{aligned}\widetilde{\Phi}_\omega = {}& \frac{q}{\pi i c_0} \sum_{h=0}^{2q-1} e(g(h)) \sum_{\substack{m\in\mathcal{E},\ m>M_{\max}\\ m\equiv h \pmod{2q}}} \biggl(\frac{1}{2q\alpha - m} + \frac{1}{2q\alpha + m}\biggr)\\ &+ \frac{q}{\pi i c_0} \sum_{h=0}^{2q-1} e(g(h)) \sum_{\substack{m\in\mathcal{E}\cap\mathcal{T}\\ m\equiv h \pmod{2q}}} \frac{1}{2q\alpha - m}\end{aligned} \tag{4.16}$$

where $\mathcal{T} = [-M_{\max}, M_1 - q) \cup (M_2 + q, M_{\max}]$.

We start by bounding the double sum on the second line of (4.16). If $m > M_2 + q$, then, depending on the correct parity of m, either $m = M_2 + q + 2m' + 1$ or $m = M_2 + q + 2m' + 2$ for some nonnegative integer m'. Additionally, since m belongs to a fixed residue class modulo $2q$, m' must increment by a multiple of q as m progresses, say $m' = jq$. So, considering that $M_2 \ge 2qf'(0) - 1/2 = 2q\alpha - 1/2$, we deduce $|2q\alpha - m| \ge q + 2jq + 1/2$. By similar reasoning, if $m < M_1 - q$, then $|2q\alpha - m| \ge q + 2jq + 1/2$. (Here, we used the bound $M_1 \le 2q\alpha + 1/2$.) Therefore, the double sum under consideration is of size not exceeding

$$\frac{4q^2}{\pi} \sum_{0\le j\le M_{\max}/(2q)} \frac{1}{q + 2jq + 1/2}. \tag{4.17}$$

We isolate the term with $j = 0$ and compare the tail with an integral like $\int_0^X (q + 2xq + 1/2)^{-1}\, dx$, which yields that (4.17) is not greater than

$$\frac{8}{\pi}\frac{q^2}{q + 1/2} + \frac{2q}{\pi}\log\left(\frac{M_{\max}}{q + 1/2} + 1\right). \tag{4.18}$$

Next, we bound the double sum on the first line of (4.16). But first let us derive lower bounds for $M_{\max} - 2q\alpha$ and $M_{\max} + 2q\alpha$. To this end, consider that as m progresses in a fixed residue class modulo $2q$, we have $m \ge M_{\max} + 2jq + 1$ where j steps through the nonnegative integers. In addition, since $M_{\max} \ge 2q|\alpha| + M_2 + q$ and since by definition $M_2 \ge 2q\alpha - 1/2$, we have $M_{\max} - 2q\alpha \ge 2q|\alpha| + q - 1/2$. Similarly, since $M_{\max} \ge 2q|\alpha| + |M_1| + |M_2| + q$ and $|M_1| + |M_2| \ge 2q|\alpha| - 1/2$, we deduce that $M_{\max} + 2q\alpha \ge 2q|\alpha| + q - 1/2$. Therefore, on simplifying and applying the triangle inequality, the double sum under consideration is bounded by

$$\frac{2q^2}{\pi} \sum_{\substack{m\in\mathcal{E},\ m>M_{\max} \\ m\equiv h \pmod{2q}}} \frac{4q|\alpha|}{|(2q\alpha - m)(2q\alpha + m)|} \le \frac{2q^2}{\pi}\sum_{j\ge 0} \frac{4q|\alpha|}{(2q|\alpha| + q + 2jq + 1/2)^2}. \tag{4.19}$$

The last expression is estimated by isolating the term corresponding to $j = 0$ and comparing the rest to the integral $\int_0^\infty 4q|\alpha|(2q|\alpha| + q + 2xq + 1/2)^{-2}\, dx$. Doing so yields the bound

$$\frac{2q^2}{\pi}\left(\frac{4q|\alpha|}{(2q|\alpha| + q + 1/2)^2} + \frac{4|\alpha|q}{q + 2q^2 + 4|\alpha|q^2}\right) \le \frac{6q}{\pi}. \tag{4.20}$$

Finally, inserting estimates (4.20) and (4.18) into (4.16) shows that $\widetilde{\Phi}_\omega$ is bounded by 1/2 times the right-hand side expression in the statement of the lemma. The other 1/2 comes from $\widetilde{\Phi}_{\omega+N}$, which satisfies this same bound as $\widetilde{\Phi}_\omega$, as can be seen via the same method employed so far. □

5. TERMS WITH ONE SADDLE POINT

Lemma 5.1. *If* $\omega > 0$, *then*

$$\sum_{\substack{m\in\mathcal{E} \\ M_1<m<M_2}} |\mathrm{AH}(\omega, N; \mu, s_m) - \mathrm{AH}(\mu, s_m)| \le \frac{16}{\pi}q + \frac{4}{\pi}q\log(2M - 1)\cdot \mathbb{1}_{M>0}. \tag{5.1}$$

If $\omega < -N$, *then the same bound holds but with* $\overline{\mathrm{AH}(\mu, s_m)}$ *instead of* $\mathrm{AH}(\mu, s_m)$.

Proof. Assume that $\omega > 0$. In view of the identity

$$\mathrm{AH}(\omega, N; \mu, s_m) = \mathrm{AH}(\mu, s_m) - \mathrm{AH}(0, \omega; \mu, s_m) - \mathrm{AH}(\omega + N, \infty; \mu, s_m), \tag{5.2}$$

the left-hand side in (5.1) is not greater than

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M_2}} |\mathrm{AH}(0,\omega;\mu,s_m)| + \sum_{\substack{m\in\mathcal{E}\\ M_1<m<M_2}} |\mathrm{AH}(\omega+N,\infty;\mu,s_m)|. \tag{5.3}$$

We treat the sum involving $\mathrm{AH}(0,\omega;\mu,s_m)$ first. Appealing to Lemma 4.2 in [7], we find

$$|\mathrm{AH}(0,\omega;\mu,s_m)| \le \frac{2}{\pi}\max_{0\le t\le\omega}\frac{1}{3|\mu t^2 - s_m|}. \tag{5.4}$$

For all $m\in(M_1,M_2)\cap\mathcal{E}$ either $m = M_1+2j+1$ or $m = M_1+2j+2$ throughout, where j a nonnegative integer. The correct parity is determined by $\mathcal{E}$. Also, since $\omega>0$, $M_1 = \langle 2q\alpha\rangle \ge 2q\alpha - 1/2$. Therefore, in any case, we have $m \ge M_1+2j+1$ throughout and $s_m = \mu\omega^2 + (m/2-q\alpha)/3q \ge \mu\omega^2 + (j+1/4)/3q$. Combining this with the trivial bound $\max_{0\le t\le\omega}|\mu t^2| = \mu\omega^2$ and inserting into (5.4) now yield

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M_2}} |\mathrm{AH}(0,\omega;\mu,s_m)| \le \frac{2}{\pi}\sum_{0\le j\le (M_2-M_1-1)/2}\frac{q}{j+1/4}. \tag{5.5}$$

As for the sum involving $\mathrm{AH}(\omega+N,\infty;\mu,s_m)$, we write $t=\omega+u$ with $u\ge N$. Then, as before, we apply Lemma 4.2 in [7] to obtain

$$|\mathrm{AH}(\omega+N,\infty;\mu,s_m)| \le \frac{2}{\pi}\max_{t\ge\omega+N}\frac{1}{3|\mu t^2 - s_m|} = \frac{2}{\pi}\max_{u\ge N}\frac{2q}{|2qf'(u)-m|}. \tag{5.6}$$

We proceed analogously to the previous sum. Specifically, for all $m\in(M_1,M_2)\cap\mathcal{E}$ either $m = M_2-2j-1$ or $m = M_2-2j-2$ where j is a nonnegative integer. Also, $M_2 = \langle 2qf'(N)\rangle \le 2qf'(N)+1/2$. Hence, $m\le 2qf'(N)-2j-1/2$. Last, using that $\omega>0$ (so the minimum of $f'(u)$ occurs when $u=-\omega<0$), we obtain $\min_{u\ge N} 2qf'(u)\ge 2qf'(N)$. Therefore, summarizing, we conclude that

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M_2}} |\mathrm{AH}(\omega+N,\infty;\mu,s_m)| \le \frac{2}{\pi}\sum_{0\le j\le (M_2-M_1-1)/2}\frac{q}{j+1/4}. \tag{5.7}$$

The sums in (5.5) and (5.7) are bounded routinely. If $M_1=M_2$, then these sums are empty. And if $M_1<M_2$, then one isolates the term for $j=0$ and compares the remaining sum to an integral. Putting these bounds together yields the lemma when $\omega>0$.

The treatment of the case $\omega<-N$ is analogous except one starts with the identity

$$\mathrm{AH}(\omega,N;\mu,s_m) = \overline{\mathrm{AH}(\mu,s_m)} + \mathrm{AH}(\omega,-\infty;\mu,s_m) + \mathrm{AH}(0,\omega+N;\mu,s_m), \tag{5.8}$$

then continues as in the previous case, this time appealing to the bounds

$$\begin{aligned} |\mathrm{AH}(\omega,-\infty;\mu,s_m)| &\le \frac{2}{\pi}\max_{t\ge|\omega|}\frac{1}{3|\mu t^2 - s_m|},\\ |\mathrm{AH}(0,\omega+N;\mu,s_m)| &\le \frac{2}{\pi}\max_{0\le t\le|\omega+N|}\frac{1}{3|\mu t^2 - s_m|} = \frac{2}{\pi}\frac{2q}{|2qf'(N)-m|} \end{aligned} \tag{5.9}$$

and the formulas $M_1 = \langle 2qf'(N)\rangle$ and $M_2 = \langle 2q\alpha\rangle$, valid for $\omega<-N$. To handle the integral $|\mathrm{AH}(0,\omega+N;\mu,s_m)|$, one additionally uses that $|\omega+N| = |\omega|-N$ combined with the change of variable $t \leftarrow |\omega| - u$, $N \le u \le |\omega|$, and the observation that $3\mu(|\omega| - u)^2 - s_m = f'(u) - m/(2q)$ is decreasing in u over $N \le u \le |\omega|$. $\square$

Lemma 5.2. *If $-N/2 < \omega \le 0$, then*

$$\sum_{\substack{m\in\mathcal{E}\\ M^*<m<M_2}} |\mathrm{AH}(\omega, N; \mu, s_m) - \mathrm{AH}(\mu, s_m)| \le \frac{16}{\pi} q + \frac{4}{\pi} q \log(2M - 1) \cdot \mathbb{1}_{M>0}. \tag{5.10}$$

If $-N \le \omega \le -N/2$, then the same bound holds but with $\mathrm{AH}(\mu, s_m)$ replaced by its conjugate $\overline{\mathrm{AH}(\mu, s_m)}$.

Proof. The proof of the first bound, i.e., when $-N/2 < \omega \le 0$, follows analogously as in the proof of Lemma 5.1 for the case $\omega > 0$. The proof of the second bound, i.e., when $-N \le \omega \le -N/2$, also follows as in Lemma 5.1 but for the case $\omega < -N$. $\square$

6. TERMS WITH TWO SADDLE POINTS

Lemma 6.1. *If $-N \le \omega \le 0$, then*

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M^*}} |\mathrm{AH}(\omega, N; \mu, s_m) - \mathrm{AI}(\mu, s_m)| \le \frac{16}{\pi} q + \frac{4}{\pi} q \log(2M - 1) \cdot \mathbb{1}_{M>0}. \tag{6.1}$$

Proof. We start with the identity

$$\mathrm{AH}(\omega, N; \mu, s_m) = \mathrm{AI}(\mu, s_m) + \mathrm{AH}(\omega, -\infty; \mu, s_m) - \mathrm{AH}(\omega + N, \infty; \mu, s_m). \tag{6.2}$$

Let us first recall that $s_m = \mu\omega^2 + (m/2 - q\alpha)/(3q)$. Also, $M^* \le 2q\alpha + 1/2$ if $\omega \ge -N/2$, $M^* \le 2qf'(N) + 1/2$ if $\omega < -N/2$, and $f'(N) \le f'(0) = \alpha$ if $-N \le \omega < -N/2$. So we deduce, in all cases, that $m/2 - q\alpha < 0$ for $m < M^*$ and in particular $s_m < \mu\omega^2 = \min_{t\ge|\omega|} \mu t^2$.

Now, applying Lemma 4.2 in [7] to each term $\mathrm{AH}(\omega, -\infty; \mu, s_m)$ gives

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M^*}} |\mathrm{AH}(\omega, -\infty; \mu, s_m)| \le \frac{2}{\pi} \sum_{\substack{m\in\mathcal{E}\\ M_1<m<M^*}} \frac{2q}{|m - 2q\alpha|}. \tag{6.3}$$

Let $m = M^* - 2j - 1$ with $j \in \mathbb{Z}_{\ge 0}$. (The case $m = M^* - 2j - 2$ is easier since one obtains a tighter bound in this situation.) By the previous observations about M^*, we obtain $|2q\alpha - m| \ge 2j + 1/2$; hence the last sum is not greater than

$$\frac{2}{\pi} \sum_{0\le j\le (M^*-M_1-1)/2} \frac{q}{j + 1/4}. \tag{6.4}$$

We estimate this sum by an integral, as was done for the sum in (5.7).

The terms $\mathrm{AH}(\omega + N, \infty; \mu, s_m)$ are treated analogously, so we skip the details. Put together, this verifies the bound in the lemma. $\square$

Remark. We have

$$\mathrm{AI}(\mu, s) = \frac{2\pi}{(6\pi\mu)^{1/3}} \mathrm{Ai}\left(-\frac{(2\pi)^{2/3} s}{(3\mu)^{1/3}}\right)$$

where $\mathrm{Ai}(x) := (2\pi)^{-1} \int_{-\infty}^{\infty} e^{it^3/3 + ixt} dt$ is the usual Airy function satisfying $|\mathrm{Ai}(x)| \le 1/|x|^{1/4}$, and so one obtains $|\mathrm{AI}(\mu, s)| \le \sqrt{2\pi}/(3\mu|s|)^{1/4}$.

7. AN ALTERNATIVE BOUND FOR THE TAIL

We may consider the tails Υ_1 and Υ_2 in Section 4 together and apply the method of Lemma 4.1 to both of them. This has the effect of adding more terms to the function $\widetilde{\Phi}$ in Section 4 and gives an error term that still goes to zero as β and μ go to zero but with an extra factor of q^3.

Explicitly, rather than apply Lemma 4.2 in [7] to each term in Υ_2 immediately, we first apply integration by parts followed by an application of Lemma 4.2 in [7] and then proceed similarly to the proof of Lemma 4.1. This yields the following bound. Define

$$\Phi^{\mathrm{o}}_{\omega+x} = \frac{qe(f(x))}{\pi i c_0} \sum_{\substack{m\in\mathcal{E}\\ m\notin[M_1,M_2]}} \frac{e(g(m)-xm/2q)}{2qf'(x)-m}, \tag{7.1}$$

which is similar to (4.5) except that it involves the additional terms $m\in[M_1-q,M_1)$ and $m\in(M_2,M_2+q]$.

Lemma 7.1.

$$\left|\Upsilon_1+\Upsilon_2-\Phi^{\mathrm{o}}_{\omega+N}+\Phi^{\mathrm{o}}_{\omega}\right| \le \frac{576(2|\beta|+3\mu N)q^3}{\pi^2}.$$

One may also use integration by parts to execute the proofs of the lemmas in Sections 5 and 6. This would add yet more terms to the function $\Phi^{\mathrm{o}}_{\omega+x}$, enlarging the range of summation to all $m\notin\Omega=\{M_1,M_2,M^*\}$.

Lemma 7.2. *If* $\omega>-N/2$, *then*

$$\sum_{\substack{m\in\mathcal{E}\\ M^*<m<M_2}} \left|\mathrm{AH}(\omega,N;\mu,s_m)-\mathrm{AH}(\mu,s_m)-\phi_{\omega+N}(\mu,s_m)+\phi_\omega(\mu,s_m)-\phi_0(\mu,s_m)\right| \le \frac{576(2|\beta|+3\mu N)q^3}{\pi^2}. \tag{7.2}$$

If $\omega\le-N/2$, *then the same bound holds but with* $\mathrm{AH}(\mu,s_m)$ *replaced by its conjugate and* $-\phi_0(\mu,s_m)$ *replaced by* $\phi_0(\mu,s_m)$.

Lemma 7.3. *If* $-N\le\omega\le0$, *then*

$$\sum_{\substack{m\in\mathcal{E}\\ M_1<m<M^*}} \left|\mathrm{AH}(\omega,N;\mu,s_m)-\mathrm{AI}(\mu,s_m)-\phi_{\omega+N}(\mu,s_m)+\phi_\omega(\mu,s_m)\right| \le \frac{576(2|\beta|+3\mu N)q^3}{\pi^2}. \tag{7.3}$$

In view of the previous two lemmas, we are motivated to define

$$\Phi(x) := \frac{qe(f(x))}{\pi i c_0} \sum_{\substack{m\in\mathcal{E}\\ m\notin\Omega}} \frac{e(g(m)-xm/2q)}{2qf'(x)-m}, \tag{7.4}$$

which accounts for the terms ϕ_ω and $\phi_{\omega+N}$, and

$$Y(x) := \frac{\operatorname{sgn}(x+N/2)}{6\pi i} \sum_{\substack{m\in\mathcal{E}\\ M^*<m<M_2}} \frac{e(g(m)+xm/2q)}{s_m}, \tag{7.5}$$

which accounts for the term ϕ_0. The numerator in these definitions is inserted because ϕ_x will be multiplied by $e(g(m)+\omega m/2q)$ according to the formula in Lemma 2.1 (see (2.8)).

8. FORMULAS FOR THE TRANSFORMED SUM

In summary, we have proved the following. Define

$$T_m := e\left(\frac{\omega m}{2q} + g(m)\right) \mathrm{AH}(\mu, s_m), \tag{8.1}$$

and define $\overline{T}_m$ the same way as T_m except that $\mathrm{AH}(\mu, s_m)$ is replaced by its conjugate while $e(\omega m/2q + g(m))$ is kept the same. Moreover, define the boundary term

$$\mathcal{B} := \sum_{\substack{\ell \in \mathcal{E} \\ \text{distinct } \ell \in \Omega}} B_\ell. \tag{8.2}$$

To clarify the behavior of the main sum $\mathcal{M}$ below, we refer to Lemmas 8.3 and 8.4. Also, estimates for $\mathcal{B}$ are provided in Lemma 8.6.

Proposition 8.1.

$$C(N; a, b, q; f) = \frac{c_1}{\sqrt{q}}[\mathcal{M} + \mathcal{B} + \mathcal{R}_1] + \frac{1+B}{2}, \tag{8.3}$$

where the main sum $\mathcal{M}$ is equal to

$$\mathcal{M} = \sum_{\substack{m \in \mathcal{E} \\ M_1 < m < M^*}} (T_m + \overline{T}_m) + \begin{cases} \displaystyle\sum_{\substack{m \in \mathcal{E} \\ M^* < m < M_2}} T_m & \text{if } \omega \ge -\dfrac{N}{2}, \\ \displaystyle\sum_{\substack{m \in \mathcal{E} \\ M^* < m < M_2}} \overline{T}_m & \text{if } \omega \le -\dfrac{N}{2} \end{cases} \tag{8.4}$$

and the remainder term $\mathcal{R}_1$ satisfies the bound

$$|\mathcal{R}_1| \le \frac{128(2|\beta| + 3\mu N)}{\pi^2}\frac{q^3(2q+9)}{(2q+1)^3} + \frac{4q}{\pi}\log\left(\frac{M_{\max}}{q+1/2} + 1\right) + \frac{8}{\pi}q\log(2q-1) + \frac{8}{\pi}q\log(2M-1)\cdot \mathbb{1}_{M>0} + \frac{92}{\pi}q. \tag{8.5}$$

Note that the remainder $\mathcal{R}_1$ satisfies $\mathcal{R}_1 \ll q(|\beta| + \mu N + \log M_{\max} + \log(2q))$; in particular, $\mathcal{R}_1$ does not tend to zero as β and μ tend to zero. However, by incorporating more lower order terms using the lemmas in Section 7, we can obtain a remainder term that tends to zero with β and μ but that depends more heavily on q; namely, we obtain a remainder of size $\ll (|\beta| + \mu N)q^3$.

Proposition 8.2.

$$C(N; a, b, q; f) = \frac{c_1}{\sqrt{q}}\left[\mathcal{M} + \mathcal{B} + \Phi(N) - \Phi(0) + Y(\omega) + \mathcal{R}_2\right] + \frac{1+B}{2}, \tag{8.6}$$

and

$$|\mathcal{R}_2| \le \frac{1728(2|\beta| + 3\mu N)q^3}{\pi^2}. \tag{8.7}$$

Lemma 8.3. *If $s > 0$, then*

$$\left|\mathrm{AH}(\mu, s) - \frac{1}{(36\mu s)^{1/4}} e\left(\frac{1}{8} - \frac{2s\sqrt{s}}{\sqrt{\mu}}\right)\right| \le \frac{1}{\pi s}. \tag{8.8}$$

Proof. By a change of variable $t \leftarrow \mu^{1/3}t$, we obtain

$$\mathrm{AH}(\mu, s) = \frac{1}{\mu^{1/3}} \int_0^\infty e\biggl(t^3 - \frac{3st}{\mu^{1/3}}\biggr)\, dt. \tag{8.9}$$

A close examination of the proof of [1, Lemmas 2.5, 2.6] (applied with $y = s/\mu^{1/3}$) gives the result. $\square$

Lemma 8.4. *If $m > M_1$, then $s_m > 0$. Specifically,*

$$s_m \ge \begin{cases} \mu\omega^2 + \dfrac{m - M_1 - 1/2}{6q} & \textit{if } \ \omega > 0, \\[2mm] \dfrac{m - M_1 - 1/2}{6q} & \textit{if } \ -N \le \omega \le 0, \\[2mm] \mu(\omega + N)^2 + \dfrac{m - M_1 - 1/2}{6q} & \textit{if } \ \omega < -N. \end{cases}$$

Proof. This follows from the definitions of M_1, M_2 and M^*. $\square$

Lemma 8.5. *Suppose that $q = 1$, $a = 0$ and $3|m - \alpha|\mu/\beta^2 \le 1 - \epsilon_1 < 1$. If $\beta > 0$ or $\beta < -1/N$, then*

$$\frac{2\beta^3}{27\mu^2} - \frac{\beta\alpha}{3\mu} + \frac{\beta m}{3\mu} - \operatorname{sgn}(\beta)\frac{2s_{2m}\sqrt{s_{2m}}}{\sqrt{\mu}} = -\frac{\alpha^2}{4\beta} - \frac{\alpha^3\mu}{8\beta^3} + \biggl(\frac{\alpha}{2\beta} + \frac{3\alpha^2\mu}{8\beta^3}\biggr)m + \biggl(-\frac{1}{4\beta} - \frac{3\alpha\mu}{8\beta^3}\biggr)m^2 + \frac{\mu}{8\beta^3}m^3 + O_{\epsilon_1}\biggl(\frac{\mu^2|m-\alpha|^4}{\beta^5}\biggr).$$

Moreover,

$$\frac{1}{(36\mu s_{2m})^{1/4}} = \frac{1}{\sqrt{2|\beta|}} + O_{\epsilon_1}\biggl(\frac{1}{\sqrt{|\beta|}}\frac{\mu|m-\alpha|}{\beta^2}\biggr). \tag{8.10}$$

Lemma 8.6. *We have*

$$|\mathcal{B}| \le \begin{cases} \min\Bigl\{2N, \dfrac{16}{\sqrt{12\pi\mu\omega}}\Bigr\} & \textit{if } \ \omega > 0, \\[2mm] \min\Bigl\{3N, \dfrac{48}{(12\pi\mu)^{1/3}}\Bigr\} & \textit{if } \ -N \le \omega \le 0, \\[2mm] \min\Bigl\{2N, \dfrac{16}{\sqrt{-12\pi\mu(\omega+N)}}\Bigr\} & \textit{if } \ \omega < -N. \end{cases}$$

Proof. The bounds when $\omega > 0$ or $\omega < -N$ follow from Lemma 4.4 in [7] and the fact that two terms contribute to $\mathcal{B}$ in these cases. When $-N \le \omega \le 0$, there are at most three terms contributing to $\mathcal{B}$. Write $\mathrm{AH}(\omega, N; \mu, s) = \int_\omega^0 e(\mu t^3 - 3s)\, dt + \int_0^{\omega+N} e(\mu t^3 - 3s)\, dt$ and then treat each integral separately; e.g., $\int_\omega^0 e(\mu t^3 - 3s)\, dt = \int_\omega^\delta e(\mu t^3 - 3s)\, dt + \int_\delta^0 e(\mu t^3 - 3s)\, dt$, where we bound the first integral using Lemma 4.4 in [7], bound the second integral trivially, and then optimize the choice of $\delta = 4/(12\pi\mu)^{1/3}$. $\square$

Proposition 8.7. *Let* $w := \langle 6\mu q N^2\rangle$. *If* $w \neq 0$, *then*

$$\sum_{n=0}^{N}{}' e\left(\frac{an+bn^2}{2q}+\mu n^3\right) = \frac{e(1/8)\,g(b+\delta q,q)}{(6\mu q)^{1/4}} \sum_{\substack{0<m<w\\ m\equiv \delta_1 \pmod 2}} \frac{1}{m^{1/4}}\, e\left(\frac{b^*(a+m)^2}{8q}-\frac{2m^{3/2}}{6q\sqrt{6\mu q}}\right)$$

$$+ \mathbb{1}_{\delta_1=0}\, \frac{g(b+\delta q,q)}{\sqrt{q}}\, e\left(\frac{b^* a^2}{8q}\right) \int_0^N e(\mu t^3)\,dt$$

$$+ \mathbb{1}_{\delta_1\equiv w \pmod 2}\, \frac{g(b+\delta q,q)}{\sqrt{q}}\, e\left(\frac{b^*(a+w)^2}{8q}\right) \int_0^N e\left(\mu t^3-\frac{wt}{2}\right)dt$$

$$+ O\big(\mu N q^{1/2}+q^{1/2}\log(w+2q)\big)$$

where the prime on the sum means that the boundary terms at $n=0$ *and* N *are weighted by* $1/2$. *If* $w=0$, *i.e., if* $\mu < 1/(12qN^2)$, *then the two integrals on the right-hand side are equal and one of them is dropped.*

Proof. Apply Proposition 8.1 with $\alpha=\beta=0$, followed by Lemmas 8.3 and 8.4. □

9. PROOFS

Proof of Theorem 1.1. This is a special case of Proposition 8.1 when the intervals (M_1, M^*) and (M^*, M_2) are empty, so $\mathcal{M}=0$ and the only terms that survive are the boundary terms $\mathcal{B}$. □

Proof of Theorem 1.2. Consider the case $q=1$ and $a=0$. Then necessarily $b=0$ and $C(N;a,b,q;f)=H_N(\alpha,\beta,\mu)$. Moreover, $b^*=0$, $b+\delta q=0$ and $m\in\mathcal{E}$ is equivalent to $m\in 2\mathbb{Z}$. Therefore, in the situation $q=1$ and $a=0$, we have $g(m)\equiv 0$, $G(0,b+\delta q;2q)=1$ and we need only consider even m in Proposition 8.1. In addition, since $g(m)\equiv 0$, we have $c_1=c_0=e(2\beta^3/27\mu^2-\beta\alpha/3\mu)$. Suppose further that $|\beta|>1/N$ and that $0<6N^2\mu<1$. Then $|\omega|>N$ and so $M^*=M_1$. In particular, Proposition 8.1 involves only T_{2m} if $\beta>1/N$, and only $\overline{T}_{2m}$ if $\beta<-1/N$. Therefore, after simplifying using Lemmas 8.3 and 8.4, we see that the terms that need to be considered in Proposition 8.7 are of the form

$$\frac{e(2\beta^3/27\mu^2-\beta\alpha/3\mu)}{(36\mu s_{2m})^{1/4}}\, e\left(\frac{\beta m}{3\mu}+\frac{\operatorname{sgn}(\beta)}{8}-\operatorname{sgn}(\beta)\frac{2s_{2m}\sqrt{s_{2m}}}{\sqrt{\mu}}\right)+O\left(\frac{1}{s_{2m}}\right).$$

This motivates considering the formula appearing in Lemma 8.5. Note that the conditions required by this lemma are satisfied due to our assumptions on μ and β. Indeed, if we substitute these expansions into Lemma 8.4, then back into Proposition 8.1, and use Lemma 8.6 to estimate the boundary terms $\mathcal{B}$, then we obtain the result. □

10. SUGGESTED IMPROVEMENTS

One might be able to remove the $\log(|N'|+2)$ term appearing in the O-notation in Theorem 1.2 by using Proposition 8.2 instead of Proposition 8.1 to derive the theorem. The former proposition incorporates the secondary terms $\Phi(x)$ and $Y(x)$, which may be estimated more precisely, and it has a remainder $\mathcal{R}_2$ that tends to zero with β and μ. Similarly, one might be able to remove the $\log(2q)$ factor from the remainder in Theorem 1.1 by using Proposition 8.2 instead of Proposition 8.1. Both improvements will require careful and substantial analysis of the functions $\Phi(x)$ and $Y(x)$. For example, one should probably divide the sum in $\Phi(x)$ along arithmetic progressions modulo $2q$ so as to express $\Phi(x)$ as a linear combination of Hurwitz–Lerch zeta functions and then apply known asymptotics for the latter.

Additionally, it might be desirable to derive a version of the bound (1.7) where instead of $H_N^{\max}(\alpha, \beta, \mu)$ we use the function

$$\max_{N_2 \in [0,N]} \left| \sum_{n=0}^{N_2} e\big(\alpha n + \beta n^2 + \mu n^3\big) \right|, \tag{10.1}$$

which offers some advantages; e.g., if we start with $\alpha = 0$, then the new α (in the transformed sum) will still be zero. Finally, although we have not done so for the results stated in the Introduction, all the implicit constants appearing there can be made explicit if desired by using the explicit error bounds in Section 8.

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