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On the Analytic Complexity of Hypergeometric Functions

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Abstract—Hypergeometric functions of several variables resemble functions of finite analytic complexity in the sense that the elements of both classes satisfy certain canonical overdetermined systems of partial differential equations. Otherwise these two sets of functions are very different. We investigate the relation between the two classes of functions and compute the analytic complexity of certain bivariate hypergeometric functions.

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1. INTRODUCTION

Hypergeometric functions of several complex variables constitute a wide and important class of special functions. Numerous special functions of mathematical physics turn out to be hypergeometric. Multivariate hypergeometric functions can be defined as solutions to certain overdetermined systems of linear partial differential equations with polynomial coefficients. Such systems of equations are of substantial independent interest and appear in numerous applications. The simplest ordinary differential equation of this kind is the Gauss hypergeometric equation. Any second-order linear differential equation with three regular singularities in the Riemann sphere can be reduced to the Gauss equation by means of a suitable change of the variables.

The notion of analytic complexity stems from Hilbert's 13th problem on the possibility to represent multivariate functions through compositions of functions of at most two variables. For continuous functions, the positive answer is given by the Kolmogorov–Arnold theorem [1]. In the analytic category, this question leads to the notion of classes of analytic complexity [2, 3] and to the corresponding differential membership criteria [4, 5, 12] for holomorphic functions. A necessary and sufficient condition for a bivariate holomorphic function to be representable through a given number of compositions of arbitrary univariate functions and a given "canonical" bivariate function is that it solves a certain (nonlinear) system of partial differential equations with constant coefficients [6].

Thus hypergeometric functions as well as holomorphic functions of finite analytic complexity belong to the class of differentially algebraic functions. This appears to be the only obvious similarity between them. However, numerous examples and computer experiments suggest the existence of deeper connections between these classes of multivariate analytic functions. The present paper exposes results on the analytic complexity of hypergeometric functions of two complex variables.

2. HYPERGEOMETRIC SYSTEMS OF EQUATIONS AND THEIR SOLUTIONS

Throughout the paper, we use the following notation and definitions.

Definition 2.1. A formal Laurent series

$$\sum_{s\in\mathbb{Z}^n} \varphi(s)\, x^s \tag{2.1}$$

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is called *hypergeometric* if for any $j = 1, \dots, n$ the quotient of its adjacent coefficients $\varphi(s + e_j)/\varphi(s)$ is a rational function of the indices of summation $s = (s_1, \dots, s_n)$. Throughout the paper we denote this rational function by $P_j(s)/Q_j(s + e_j)$. Here $\{e_j\}_{j=1}^n$ is the standard basis of the lattice $\mathbb{Z}^n$. By the *support* of this series we mean the subset of $\mathbb{Z}^n$ on which $\varphi(s) \neq 0$. We say that such a series is *fully supported* if the convex hull of its support contains (a translation of) an open n-dimensional cone.

A *hypergeometric function* is a (multivalued) analytic function obtained by means of analytic continuation of a hypergeometric series with a nonempty domain of convergence along all possible paths.

Theorem 2.2 (O. Ore and M. Sato [10, 19]). *The coefficients of a hypergeometric series are given by the formula*

$$\varphi(s) = t^s\, U(s) \prod_{i=1}^{m} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i), \tag{2.2}$$

where $t^s = t_1^{s_1} \dots t_n^{s_n}$, $t_i, c_i \in \mathbb{C}$, $\mathbf{A}_i = (A_{i,1}, \dots, A_{i,n}) \in \mathbb{Z}^n$, $i = 1, \dots, m$, *and* $U(s)$ *is a product of a certain rational function and a periodic function* $\phi(s)$ *such that* $\phi(s + e_j) = \phi(s)$ *for all* $j = 1, \dots, n$.

Given the above data $(t_i, c_i, \mathbf{A}_i, U(s))$ that determine the coefficient of a hypergeometric series, it is straightforward to compute the rational functions $P_j(s)/Q_j(s + e_j)$ by means of the Γ-function identity. The converse requires solving a system of difference equations which is only solvable under some compatibility conditions on P_j and Q_j. A careful analysis of this system of difference equations was performed in [14].

In the present paper the Ore–Sato coefficient (2.2) plays the role of a primary object which generates everything else: the series, the system of differential equations, the algebraic hypersurface containing the singularities of its solutions, as well as the analytic complexity of solutions. Throughout the paper we will assume that $m \geq n$, since otherwise the corresponding hypergeometric series (2.1) is just a linear combination of hypergeometric series in fewer variables (times an arbitrary function of the remaining variables, which makes the system nonholonomic) and n can be reduced to meet the inequality.

Definition 2.3 (Horn system associated with an Ore–Sato coefficient). A (formal) Laurent series $\sum_{s \in \mathbb{Z}^n} \varphi(s) x^s$ whose coefficient satisfies the relations $\varphi(s + e_i)/\varphi(s) = P_j(s)/Q_j(s + e_j)$ is a (formal) solution to the following system of partial differential equations of hypergeometric type:

$$(x_j P_j(\theta) - Q_j(\theta)) f(x) = 0, \qquad j = 1, \dots, n. \tag{2.3}$$

Here $\theta = (\theta_1, \dots, \theta_n)$ and $\theta_j = x_j \frac{\partial}{\partial x_j}$. This system will be referred to as the *Horn hypergeometric system defined by the Ore–Sato coefficient* $\varphi(s)$ (cf. [10]) and denoted by $\mathrm{Horn}(\varphi)$. In this paper we treat only holonomic Horn hypergeometric systems if not otherwise specified, i.e., $\mathrm{rank}(\mathrm{Horn}(\varphi))$ is always assumed to be finite. A necessary and sufficient condition for the system $\mathrm{Horn}(\varphi)$ to be holonomic was established in [7, Theorem 6.3].

We will often be dealing with the important special case of an Ore–Sato coefficient (2.2) with $t_i = 1$ for all $i = 1, \dots, n$ and $U(s) \equiv 1$. The Horn system associated with such an Ore–Sato coefficient will be denoted by $\mathrm{Horn}(A, c)$, where A is the matrix with the rows $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{Z}^n$ and $c = (c_1, \dots, c_m) \in \mathbb{C}^m$. In this case the following operators $P_j(\theta)$ and $Q_j(\theta)$ explicitly determine system (2.3):

$$P_j(s) = \prod_{i\colon A_{i,j} > 0} \prod_{\ell_j^{(i)}=0}^{A_{i,j}-1} \big(\langle \mathbf{A}_i, s\rangle + c_i + \ell_j^{(i)}\big), \qquad Q_j(s) = \prod_{i\colon A_{i,j} < 0} \prod_{\ell_j^{(i)}=0}^{|A_{i,j}|-1} \big(\langle \mathbf{A}_i, s\rangle + c_i + \ell_j^{(i)}\big).$$

Definition 2.4. The Ore–Sato coefficient (2.2), the corresponding hypergeometric series (2.1), and the associated hypergeometric system (2.3) are called *nonconfluent* if

$$\sum_{i=1}^{m} \mathbf{A}_i = 0. \tag{2.4}$$

Definition 2.5. A Puiseux polynomial solution to the hypergeometric system $\mathrm{Horn}(A,c)$ is called *persistent* if its support remains finite under arbitrary small perturbations of the vector of parameters c.

In the present paper we only consider holomorphic solutions to differential equations or systems of such equations defined in a domain in a complex space. By a nontrivial solution to a hypergeometric system we will mean a solution that is not identically zero.

3. ANALYTIC COMPLEXITY OF BIVARIATE HOLOMORPHIC FUNCTIONS

Definition 3.1. By a *differential polynomial* over a field K of characteristic zero with an unknown function $f = f(x)\colon \mathbb{C}^n \to \mathbb{C}$ depending on the variables $x = (x_1,\dots,x_n) \in \mathbb{C}^n$, we will mean a polynomial

$$P(f) := P\big(f, f'_{x_1},\dots,f'_{x_n}, f''_{x_1x_2},\dots,f''_{x_{n-1}x_n},\dots\big) \in K\big[f, f'_{x_1},\dots,f'_{x_n}, f''_{x_1x_2},\dots,f''_{x_{n-1}x_n},\dots\big]$$

in the elements of the jets of $f(x)$.

Throughout the paper we will adopt the following definition.

Definition 3.2. By a *root of a differential polynomial* P at a point $x^{(0)} \in \mathbb{C}^n$ we will mean a germ of an (in general multivalued) analytic function $f \in \mathcal{O}(U_{x^{(0)}})$ such that $P(f) = 0$ in a neighborhood $U_{x^{(0)}}$ of $x^{(0)}$.

A function is called *differentially algebraic over the field* K if it is a root of a differential polynomial with the coefficients in K on some nonempty set. A function that is not a root of any differential polynomial over the field K is called *differentially transcendental over* K. While "most" analytic functions are differentially transcendental, the present paper aims at investigation of a relatively small class of differentially algebraic functions that are roots of differential polynomials of a rather special form. Throughout the paper we will be using the following definitions.

Definition 3.3. The *algebraic degree* of the differential monomial

$$\prod_I \left(\frac{\partial^{|I|} f(x_1,\dots,x_n)}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\right)^{p_I} \tag{3.1}$$

is defined to be the sum $\sum_I p_I$ of the exponents of all the derivatives that appear in it. Here $I = (i_1,\dots,i_n)$ is a multi-index with the values in some finite subset of $\mathbb{N}_0^n$, $|I| = i_1 + \ldots + i_n$, and $p_I \in \mathbb{N}$. A differential polynomial is said to be *algebraically homogeneous of degree* k if all its differential monomials have the same algebraic degree k.

Definition 3.4. By the *differential order* of the differential monomial (3.1) we will mean $\max_I |I|$. The differential order of a differential polynomial is defined to be the maximal differential order of its monomials, that is, the order of the corresponding differential equation.

The analytic complexity of a holomorphic function $f(x,y)$ of two complex variables is its numeric invariant with the values in $\mathbb{N} \cup \{\infty\}$. It reflects the structure of the most concise representation of $f(x,y)$ in terms of compositions of univariate functions of a complex variable and a fixed bivariate function $s(x,y)$. Everywhere below we will choose the latter to be the sum of the variables:

$s(x,y) = x + y$. In [2] a hierarchy $\mathrm{Cl}_0 \subset \mathrm{Cl}_1 \subset \ldots \subset \mathrm{Cl}_n \subset \ldots \subset \mathrm{Cl}_\infty$ of classes of analytic complexity for functions of two complex variables was introduced. In this hierarchy, the trivial class of functions of zero analytic complexity comprises the functions that depend on at most one of the variables. The building blocks of functions of higher (i.e., nonzero) analytic complexity are the univariate functions and the function $s(x,y)$, whose finite compositions form, by definition, the set of functions of finite analytic complexity. This set is stratified by counting the number of uses of the function $s(x,y)$ that is necessary to obtain a given function. More precisely, we will be using the following inductive definition.

Definition 3.5 (see [2])**.** The class Cl_0 of functions of analytic complexity zero is defined to comprise the functions that depend on at most one of the variables. A function $f(x,y)$ is said to belong to the class Cl_n of functions with analytic complexity $n > 0$ if there exists a point $(x_0, y_0) \in \mathbb{C}^2$ and a germ $\mathfrak{f}(x,y) \in \mathcal{O}(U(x_0,y_0))$ of this function holomorphic at (x_0,y_0) such that $\mathfrak{f}(x,y) = c(a(x,y) + b(x,y))$ for some germs of holomorphic functions $a, b \in \mathrm{Cl}_{n-1}$ and $c \in \mathrm{Cl}_0$. If there is no such representation for any finite n, then the function f is said to be of infinite analytic complexity.

The above definition is correct in the sense that the analytic complexity of a bivariate holomorphic function does not depend on the choice of a germ of this function. The most concise representation in Definition 3.5 is necessarily valid for each of its germs. This follows from the fact that for any class of analytic complexity there exists a differential membership criterion which identifies the functions of given analytic complexity with the set of roots of a system of differential polynomials with integer coefficients. By the conservation principle the analytic continuation of a root of such a system along any path is also a root and therefore has the same analytic complexity. We will say that a holomorphic function $f(x,y)$, $x, y \in \mathbb{C}$, has analytic complexity n if $f(x,y) \in \mathrm{Cl}_n$ but $f(x,y) \notin \mathrm{Cl}_{n-1}$. By the complexity of a function we will always mean its analytic complexity.

According to the results of [2, 3], every class of analytic complexity can be defined as the set of roots of a set of differential polynomials with integer coefficients. The algebraic degree and the differential order of these polynomials increase rapidly as the analytic complexity grows. The explicit calculation of such polynomials is a task of formidable difficulty.

4. THE HOLONOMIC RANK OF A HYPERGEOMETRIC SYSTEM AND THE ANALYTIC COMPLEXITY OF ITS SOLUTIONS

Recall that an ideal J in the Weyl algebra (as well as the corresponding system of differential equations) is called *holonomic* if the complex dimension of its *characteristic variety*

$$\operatorname{char}(J) = \bigl\{(x,z) \in \mathbb{C}^{2n} \colon\ \sigma(P)(x,z) = 0 \ \text{ for all } \ P \in J\bigr\}$$

is equal to the dimension of the space of variables, i.e., n. Here $\sigma(P)$ denotes the principal symbol of the differential operator P. The space of analytic solutions to a holonomic system of differential equations in a neighborhood of a generic point is necessarily finite-dimensional.

In the one-dimensional case the Horn system, as well as any other ordinary differential equation, has a finite number of linearly independent solutions. In the case of two variables the Horn system can be nonholonomic if each operator generating the system contains the same right factor. Nevertheless, this can only happen on a set of zero measure (namely, on some algebraic hypersurface) in the space of parameters of the system, whereas a bivariate Horn system with generic parameters is necessarily holonomic. For three or more variables, a Horn system might be nonholonomic for a generic choice of its parameters. This fact distinguishes the bivariate case from other dimensions and motivates the study of the ways to compute the analytic complexity of a bivariate hypergeometric function.

Theorem 4.1. *For any $N \in \mathbb{N}$ there exists a bivariate hypergeometric function whose analytic complexity equals N.*

Proof. It suffices to show that any bivariate polynomial belongs to the intersection of the kernels of suitable differential operators of the form (2.3).

Let $p(x,y) = \sum_{(\alpha,\beta)\in S} c_{\alpha,\beta} x^\alpha y^\beta$ be a bivariate polynomial with support $S \subset \mathbb{N}^2$. We denote by $\#S$ the cardinality of S. For $(s,t) \in \mathbb{C}^2$ define the Ore–Sato coefficient $\varphi(s,t)$ by

$$\varphi(s,t) = \frac{\prod_{(\alpha,\beta)\in S}(s+t-\alpha-\beta)}{\prod_{(\alpha,\beta)\in S}(s-\alpha)(t-\beta)}. \tag{4.1}$$

By Definition 2.3, the action of the first differential operator in the Horn system defined by this Ore–Sato coefficient on the polynomial $p(x,y)$ is given by

$$\Biggl(x\prod_{(\alpha,\beta)\in S}(\theta_x+\theta_y-\alpha-\beta) - \prod_{(\alpha,\beta)\in S}(\theta_x-\alpha)\Biggr)\sum_{(\gamma,\delta)\in S} c_{\gamma,\delta}x^\gamma y^\delta$$
$$= \sum_{(\gamma,\delta)\in S} c_{\gamma,\delta}\Biggl(\Biggl(x\prod_{(\alpha,\beta)\in S}(\theta_x+\theta_y-\alpha-\beta)\Biggr)x^\gamma y^\delta - \Biggl(\prod_{(\alpha,\beta)\in S}(\theta_x-\alpha)\Biggr)x^\gamma y^\delta\Biggr) \equiv 0,$$

since $\mathbb{C}[\theta_x,\theta_y]$ is a commutative subring in the Weyl algebra and since $(\theta_x+\theta_y-\alpha-\beta)x^\alpha y^\beta = (\theta_x-\alpha)x^\alpha y^\beta \equiv 0$. A similar calculation shows that the polynomial $p(x,y)$ belongs to the kernel of the second operator in the Horn system. We remark that the hypergeometric system defined by the Ore–Sato coefficient (4.1) is nonconfluent and has holonomic rank $(\#S)^2$. □

Lemma 4.2. *A nontrivial solution to a hypergeometric system with holonomic rank* 1 *and generic parameters belongs to the third class of analytic complexity.*

Proof. A hypergeometric system of holonomic rank 1 cannot admit persistent polynomial solutions, since by [18, Theorem 3.7] for generic parameters its only solution can be expanded in fully supported series. A confluent hypergeometric system of holonomic rank 1 is necessarily atomic, and by [18, Theorem 4.1] its solution has the form $x^\alpha y^\beta e^{-x^p y^q - x^r y^s}$, where $\alpha,\beta \in \mathbb{C}$, $p,q,r,s \in \mathbb{Z}$, and $\left|\begin{smallmatrix} p & q \\ r & s\end{smallmatrix}\right| \neq 0$. This function has analytic complexity 3 for generic values of $\alpha,\beta\in\mathbb{C}$, since it is given by the product of three functions of analytic complexity 1.

The polygon of the Ore–Sato coefficient [16] which generates a nonconfluent hypergeometric system with holonomic rank 1 is either a triangle or a parallelogram. The analytic complexity of the generating solutions to either of these systems equals 3 by [18, Propositions 4.4, 4.7] as long as their parameters are generic. By assumption no other solutions are present. □

We remark that the canonical reduction of a hypergeometric system identifies polygons of the Ore–Sato coefficients that differ by integer shifts and allows nondegenerate monomial changes of variables. After such a reduction, the analytic complexity of a solution to a system with holonomic rank 1 is also equal to 1. The next statement is a consequence of the results in [17].

Lemma 4.3. *A bivariate hypergeometric system with a maximally reducible monodromy representation can be deformed into a system whose fundamental system of solutions consists of functions of finite analytic complexity.*

In the case when one of the operators in a hypergeometric system contains a right factor of the form $P(\theta_x)$ for a univariate polynomial P, the analytic complexity of its solutions can be estimated from above by means of the Palamodov–Malgrange–Ehrenpreis fundamental principle [13]. The next statement can be proved by means of the standard method of deriving differential consequences with constant coefficients of a given system of equations with polynomial coefficients.

Lemma 4.4. *Any hypergeometric function is differentially algebraic over* $\mathbb{C}$.

We remark that the holonomic property of a hypergeometric system of equations is in general not necessary for the analytic complexity of its solutions to be finite. For instance, the general solution to the nonholonomic system defined by the differential operators

$$\begin{cases} x(\theta_x+\theta_y+\alpha)-(\theta_x+\theta_y+\alpha), \\ y(\theta_x+\theta_y+\alpha)-(\theta_x+\theta_y+\alpha) \end{cases}$$

is given by the function $x^{-\alpha}c(y/x)$, which belongs to the second class of analytic complexity for any $\alpha\neq 0$. Here $c(\cdot)$ is an arbitrary univariate differentiable function.

The above statements together with numerous computer experiments suggest the following conjecture.

Conjecture 4.5. *Any solution to a bivariate hypergeometric system has finite analytic complexity.*

It appears to be important to be able to find the algebraic degree and the differential order of the optimal differential polynomial for which a given hypergeometric function is a root.

Example 4.6 (analytic complexity of the Mellin–Barnes integral associated with a Feynman diagram)**.** The Mellin–Barnes representation for the sunrise Feynman diagram studied in [11] has the following form:

$$J^{(L)}\big(M_1^2,\dots,M_{L+1}^2,\alpha_1,\dots,\alpha_{L+1},p^2\big)=(p^2)^{nL/2-\alpha}\big[i^{1-n}\pi^{n/2}\big]^L$$
$$\times\int\left\{\prod_{j=1}^{L+1}dt_j\,\frac{\Gamma(-t_j)\,\Gamma(n/2-\alpha_j-t_j)}{\Gamma(\alpha_j)}\left(-\frac{M_j^2}{p^2}\right)^{t_j}\right\}\frac{\Gamma(\alpha-nL/2+\vec{t}\,)}{\Gamma(n(L+1)/2-\alpha-\vec{t}\,)},\qquad(4.2)$$

where

$$\alpha=\sum_{j=1}^{L+1}\alpha_j,\qquad \vec{t}=\sum_{j=1}^{L+1}t_j,$$

α_j and L are positive integers, and M_j^2 and p^2 are some (in general, complex) parameters.

Let us introduce variables $z_j=M_j^2/p^2$, $j=1,2,\dots,L+1$, and define functions Φ_J as

$$\Phi_J=\frac{\prod_{j=1}^{L+1}\Gamma(\alpha_j)}{\big[i^{1-n}\pi^{n/2}\big]^L\,(p^2)^{nL/2-\alpha}}\,J^{(L)}\big(M_1^2,\dots,M_{L+1}^2,\alpha_1,\dots,\alpha_{L+1},p^2\big).$$

That is, Φ_J is defined through the Mellin–Barnes integral representation

$$\Phi_J=\int_C\left\{\prod_{j=1}^{L+1}dt_j\,\Gamma(-t_j)\Gamma\Big(\frac{n}{2}-\alpha_j-t_j\Big)(-z_j)^{t_j}\right\}\frac{\Gamma(\alpha-nL/2+\vec{t}\,)}{\Gamma(n(L+1)/2-\alpha-\vec{t}\,)}.\qquad(4.3)$$

The contour of integration C is to be chosen to be compatible with the singular divisors of the integrand in (4.3) (see [14]). In general there exist several ways to choose such a contour of integration and different choices typically lead to linearly independent integrals [18].

For any choice of the contour of integration C that is compatible with the singularities of the integrand in the sense of [14], the Mellin–Barnes integral (4.3) is a solution to the following system of partial differential equations:

$$\left(\Big(\vec{\theta}+\alpha-\frac{Ln}{2}\Big)\Big(\vec{\theta}+\alpha-\frac{n}{2}(L+1)+1\Big)-\theta_j\Big(\theta_j+\alpha_j-\frac{n}{2}\Big)\right)\Phi_J=0,\qquad j=1,\dots,L+1.\quad(4.4)$$

Here $\theta_j=z_j\frac{\partial}{\partial z_j}$ and $\vec{\theta}=\sum_{j=1}^{L+1}\theta_j$.

By [15], the differential operators in (4.4) commute and the holonomic rank of (4.4) equals 2^{L+1}. The initial exponents of a basis in the space of analytic solutions to (4.4) at the origin are given for generic parameters by $(\beta_1, \ldots, \beta_{L+1})$, where β_j equals either α_j or 0.

The special case of (4.4) with $n = 2$ is interesting due to the resonance of the singular divisors in (4.3). The corresponding hypergeometric system has the form

$$\big((\vec{\theta} + \alpha - L)^2 - \theta_j(\theta_j + \alpha_j - 1)\big)\Phi_J = 0, \qquad j = 1, \ldots, L+1. \tag{4.5}$$

For generic values of α_j, a basis in the solution space of (4.5) is given in [15, Theorem 3.1]. We remark that a very similar nonconfluent system of hypergeometric partial differential equations defined by a commutative family of operators was treated in [8, 9].

The physically less interesting case when $\alpha_j < 0$ leads to a system of equations with an appealing structure of the monodromy group. For example, taking $L = 1$ and $\alpha_1 = \alpha_2 = -4$, we obtain the system of equations defined by the differential operators

$$\begin{cases} x(\theta_x + \theta_y - 9)^2 - \theta_x(\theta_x - 5), \\ y(\theta_x + \theta_y - 9)^2 - \theta_y(\theta_y - 5). \end{cases}$$

One of the irreducible polynomial solutions to this system has the form

$$y^5\big(126 - 504x + 756x^2 - 504x^3 + 126x^4 + 336y - 756xy + 504x^2y - 84x^3y + 216y^2 - 216xy^2 + 36x^2y^2 + 36y^3 - 9xy^3 + y^4\big).$$

The analytic complexity of the essential irreducible factor of this polynomial does not exceed 4, that is, the holonomic rank of the defining system of equations. The analytic complexity of the irreducible polynomial solutions to (4.4) can be estimated for any values of L, n, and α_j by means of similar arguments.

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