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On Some Properties of the Navier–Stokes System of Equations

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Abstract—We discuss the initial and boundary value problems for the system of dimensionless Navier–Stokes equations describing the dynamics of a viscous incompressible fluid using the method of characteristics and the geometric method developed by the authors. Some properties of the formulation of these problems are considered. We study the effect of the Reynolds number on the flow of a viscous fluid near the surface of a body.

Keywords: Navier–Stokes equations, initial value problem, boundary value problem, Reynolds number, turbulence.

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INTRODUCTION

Consider the system of dimensionless Navier–Stokes equations for a viscous incompressible fluid [1]

$$\begin{aligned}
&S\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+E\frac{\partial p}{\partial x}-\frac{1}{R}\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\Big)=0,\\
&S\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+E\frac{\partial p}{\partial y}-\frac{1}{R}\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\Big)=0,\\
&S\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+E\frac{\partial p}{\partial z}-\frac{1}{R}\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\Big)=\frac{1}{F},\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,
\end{aligned} \tag{0.1}$$

where S is the Strouhal number, E is the Euler number, R is the Reynolds number, F is the Froude number, $\{u, v, w\}$ are the components of the velocity vector, and p is pressure.

Various aspects in the study of the Navier–Stokes system of equations and its applications are addressed in the vast literature. Note only several lines of research: the study of symmetries, conservation laws, and group properties for these equations and equations connected with them (see, for example, the review in [2] and the references therein); the search for and application of exact and approximate solutions [3]; the analytic, approximate, and numerical solution or the study of various problems whose mathematical model is based on the application of the system of Navier–Stokes

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equations (see for example, [4]); and the asymptotic behavior of solutions [5,6]. A special place in this series is taken by the studies devoted to issues of solvability of problems (in particular and first of all, of the Cauchy problem [7] and of boundary and initial–boundary value problems) formulated for the system of Navier–Stokes equations [8] and to questions of blow-up solutions connected with these issues [9]. Much attention in the study of the system of Navier–Stokes equations is paid to the problems of appearance, development, and description of the turbulence phenomenon in a viscous incompressible fluid. The interest in the problem of initiation of a turbulent motion and in the reasons for the violation of a laminar flow has a long history and remains high at present [10,11]. Two main approaches to the theoretical study of this problem are known [12]. One of them imposes perturbations on a laminar flow and studies how these perturbation affect the flow [13,14]. The other approach attempts to reduce the description of the arising flow to dynamic systems and considers their peculiar features (bifurcation and strange attractors) [10].

In this paper, we examine the initial and boundary value problems for the system of Navier–Stokes equations (0.1). We study the effect of the Reynolds number on the flow of a viscous fluid near the surface of a streamlined body. In our studies, we use the method of characteristics and the geometric method developed by the authors. We restrict our consideration to the case of continuously differentiable functions.

We distinguish the properties of the model that can lead in "critical" cases to the breakdown of a laminar flow and to the appearance of certain phenomena such as scattering or turbulence (Assertion 2). It is shown how this can be avoided (Assertion 1).

1. SOME PROPERTIES OF THE PROBLEM FORMULATION WITH INITIAL DATA

The study of the initial–boundary value problem for system (0.1) features prominently in the literature. In some papers (see [8]), the problem is solved in the class of generalized solutions. Other papers consider strong solutions in addition to weak solutions; in the three-dimensional case, it is possible to construct either a local solution or a complete (for all $t > 0$) solution only for sufficiently small initial velocities [15]. In [16], it was noted that the velocities and pressure in the system of Navier–Stokes equations satisfy the continuity equation $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ and

$$\Delta p = g, \quad g = \Big(\frac{\partial u}{\partial x}\Big)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \Big(\frac{\partial v}{\partial y}\Big)^2 + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \Big(\frac{\partial w}{\partial z}\Big)^2. \tag{1.1}$$

It is clear that both relations must be fulfilled at any time, including $t = 0$. It is also clear that, if the initial velocities are assumed to be zero [8] and the pressure is considered as a complex function, then both relations written above turn into identities. We will consider the initial value problem in the class of sufficiently smooth functions in the range of real variables.

We pass in system (0.1) to new independent variables $\xi = z - \psi(x, y, t)$, $x = \eta$, $y = \zeta$, and $t = \tau$ and consider the solution of this system near the surface $\xi = 0$. The trace of this surface on the initial manifold is $\xi = z - \psi(x, y, 0) = 0$. For arbitrary sufficiently smooth functions $p(x, y, z, 0)$, $u(x, y, z, 0)$, $v(x, y, z, 0)$, and $w(x, y, z, 0)$, we write the continuity equation in the new variables:

$$\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}\psi_\eta + \frac{\partial v}{\partial \zeta} - \frac{\partial v}{\partial \xi}\psi_\zeta + \frac{\partial w}{\partial \xi} = 0.$$

Its differential consequences and relation (1.1) are written as

$$\begin{aligned}
&\frac{\partial u}{\partial \xi}\psi_{\eta\eta} + \frac{\partial v}{\partial \xi}\psi_{\eta\zeta} = G_1(\psi_\eta, \psi_\zeta, u_{\eta\eta}, u_{\eta\xi}, u_{\xi\xi}, v_{\eta\zeta}, v_{\zeta\xi}, v_{\eta\xi}, v_{\xi\xi}, w_{\eta\xi}, w_{\xi\xi}),\\
&\frac{\partial u}{\partial \xi}\psi_{\eta\zeta} + \frac{\partial v}{\partial \xi}\psi_{\zeta\zeta} = G_2(\psi_\eta, \psi_\zeta, u_{\eta\zeta}, u_{\eta\xi}, u_{\xi\xi}, v_{\zeta\zeta}, v_{\zeta\xi}, v_{\xi\xi}, w_{\zeta\xi}, w_{\xi\xi}),\\
&\frac{\partial p}{\partial \xi}(\psi_{\eta\eta} + \psi_{\zeta\zeta}) = F(\psi_\eta, \psi_\zeta, u_\eta, u_\zeta, u_\xi, v_\eta, v_\zeta, v_\xi, w_\eta, w_\zeta, w_\xi, p_{\eta\eta}, p_{\zeta\zeta}, p_{\xi\xi}, p_{\eta\xi}, p_{\zeta\xi}).
\end{aligned} \tag{1.2}$$

Determine $\psi_{\eta\eta}$, $\psi_{\eta\zeta}$, and $\psi_{\zeta\zeta}$ from the algebraic system of equations (1.2):

$$\psi_{\eta\eta} = \frac{1}{F_1}\Big[G_1\frac{\partial u}{\partial \xi}\frac{\partial p}{\partial \xi} - G_2\frac{\partial v}{\partial \xi}\frac{\partial p}{\partial \xi} + F\Big(\frac{\partial v}{\partial \xi}\Big)^2\Big], \quad \psi_{\eta\zeta} = \frac{1}{F_1}\Big[G_1\frac{\partial v}{\partial \xi}\frac{\partial p}{\partial \xi} + G_2\frac{\partial u}{\partial \xi}\frac{\partial p}{\partial \xi} - F\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \xi}\Big],$$

$$\psi_{\zeta\zeta} = \frac{1}{F_1}\Big[G_2\frac{\partial v}{\partial \xi}\frac{\partial p}{\partial \xi} - G_1\frac{\partial u}{\partial \xi}\frac{\partial p}{\partial \xi} + F\Big(\frac{\partial u}{\partial \xi}\Big)^2\Big], \quad F_1 = \frac{\partial p}{\partial \xi}\Big[\Big(\frac{\partial u}{\partial \xi}\Big)^2 + \Big(\frac{\partial v}{\partial \xi}\Big)^2\Big].$$

System (1.2) as a partial differential system is consistent if the mixed derivatives are identical ($\psi_{\eta\eta\zeta} = \psi_{\eta\zeta\eta}$, $\psi_{\eta\zeta\zeta} = \psi_{\zeta\zeta\eta}$). Hence, substituting the above expressions for the second derivatives into the obtained relations $\psi_{\eta\eta\zeta} = \psi_{\eta\zeta\eta}$ and $\psi_{\eta\zeta\zeta} = \psi_{\zeta\zeta\eta}$, we come to the first-order equation with respect to the function $\psi(\eta, \zeta)$:

$$A(\psi_\eta, \psi_\zeta, p_\xi, p_\eta, p_\zeta, p_{\xi\eta}, p_{\xi\zeta}, p_{\eta\zeta}, u_\eta, u_\zeta, u_{\xi\eta}, u_{\xi\zeta}, u_{\eta\zeta}, v_\eta, v_\zeta, v_{\xi\eta}, v_{\xi\zeta}, v_{\eta\zeta}, \dots) = 0.$$

Thus, the trace $\xi = z - \psi(x, y, 0) = 0$ on the initial manifold is a smooth surface if the continuity equation and the first-order equation $A = 0$ give the derivatives ψ_η and ψ_ζ of the same function, which leads to the dependence between the initial data (this dependence is set in the continuity equation and relation (1.1)). If the dependence (the compatibility of the initial conditions) does not hold, then the flow near the trace will have some peculiarities. Actually, for arbitrary initial conditions, one can obtain two first-order partial differential equations and, as a consequence, two intersecting characteristics. One of these characteristic is given by the continuity equation and the other is given by the equation $A = 0$, which leads to the fact that the perturbations, which would spread over a scalar manifold in the case of matched initial conditions, spread in a two-dimensional domain between the two characteristics. It is known that a one-dimensional flow and a two-dimensional flow may differ considerably (see, for example, [17]). This causes difficulties in constructing a strong solution of the initial–boundary value problem for arbitrary initial data.

2. SOME PROPERTIES OF THE FORMULATION OF THE BOUNDARY VALUE PROBLEM

Consider for system (0.1) the problem formulation with the boundary conditions given in the form of the no-slip condition on a fixed streamlined body.

The last equation of the system implies that

$$\frac{\partial u}{\partial x} = -\Big(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big), \quad \frac{\partial v}{\partial y} = -\Big(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\Big), \quad \frac{\partial w}{\partial z} = -\Big(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\Big).$$

In view of these relations, we rewrite system (0.1) in the form

$$\begin{gathered}
S\frac{\partial u}{\partial t}-u\Big(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+E\frac{\partial p}{\partial x}-\frac{1}{R}\Big(-\frac{\partial^2 v}{\partial y\partial x}-\frac{\partial^2 w}{\partial z\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\Big)=0,\\
S\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}-v\Big(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\Big)+w\frac{\partial v}{\partial z}+E\frac{\partial p}{\partial y}-\frac{1}{R}\Big(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 w}{\partial z\partial y}+\frac{\partial^2 v}{\partial z^2}\Big)=0,\\
S\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}-w\Big(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\Big)+E\frac{\partial p}{\partial z}-\frac{1}{R}\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 v}{\partial y\partial z}-\frac{\partial^2 u}{\partial x\partial z}\Big)=\frac{1}{F}.
\end{gathered}\tag{2.1}$$

In system (2.1), we pass to the new variables $\psi(x,y)-z=\xi$, $x=\eta$, $y=\zeta$, and $t=\tau$. We obtain

$$\begin{gathered}
S\frac{\partial u}{\partial \tau}-u\Big(\frac{\partial v}{\partial \zeta}+\frac{\partial v}{\partial \xi}\psi_y-\frac{\partial w}{\partial \xi}\Big)+v\Big(\frac{\partial u}{\partial \zeta}+\frac{\partial u}{\partial \xi}\psi_y\Big)-w\frac{\partial u}{\partial \xi}+E\Big(\frac{\partial p}{\partial \eta}+\frac{\partial p}{\partial \xi}\psi_x\Big)\\
-\frac{1}{R}\Big(-\frac{\partial^2 v}{\partial \zeta\partial \eta}-\frac{\partial^2 v}{\partial \eta\partial \xi}\psi_y-\frac{\partial^2 v}{\partial \zeta\partial \xi}\psi_x-\frac{\partial^2 v}{\partial \xi^2}\psi_x\psi_y-\frac{\partial v}{\partial \xi}\psi_{xy}+\frac{\partial^2 w}{\partial \eta\partial \xi}+\frac{\partial^2 w}{\partial \xi^2}\psi_x\\
+\frac{\partial^2 u}{\partial \zeta^2}+2\frac{\partial^2 u}{\partial \zeta\partial \xi}\psi_y+\frac{\partial^2 u}{\partial \xi^2}\psi_y^2+\frac{\partial u}{\partial \xi}\psi_{yy}+\frac{\partial^2 u}{\partial \xi^2}\Big)=0,\\
S\frac{\partial v}{\partial \tau}+u\Big(\frac{\partial v}{\partial \eta}+\frac{\partial v}{\partial \xi}\psi_x\Big)-v\Big(\frac{\partial u}{\partial \eta}+\frac{\partial u}{\partial \xi}\psi_x-\frac{\partial w}{\partial \xi}\Big)-w\frac{\partial v}{\partial \xi}+E\Big(\frac{\partial p}{\partial \zeta}+\frac{\partial p}{\partial \xi}\psi_y\Big)\\
-\frac{1}{R}\Big(\frac{\partial^2 v}{\partial \eta^2}+2\frac{\partial^2 v}{\partial \eta\partial \xi}\psi_x+\frac{\partial^2 v}{\partial \xi^2}\psi_x^2+\frac{\partial v}{\partial \xi}\psi_{xx}-\frac{\partial^2 u}{\partial \eta\partial \zeta}-\frac{\partial^2 u}{\partial \eta\partial \xi}\psi_y\\
-\frac{\partial^2 u}{\partial \zeta\partial \xi}\psi_x-\frac{\partial^2 u}{\partial \xi^2}\psi_x\psi_y-\frac{\partial u}{\partial \xi}\psi_{xy}+\frac{\partial^2 w}{\partial \zeta\partial \xi}+\frac{\partial^2 w}{\partial \xi^2}\psi_y+\frac{\partial^2 v}{\partial \xi^2}\Big)=0,\\
S\frac{\partial w}{\partial \tau}+u\Big(\frac{\partial w}{\partial \eta}+\frac{\partial w}{\partial \xi}\psi_x\Big)+v\Big(\frac{\partial w}{\partial \zeta}+\frac{\partial w}{\partial \xi}\psi_y\Big)-w\Big(\frac{\partial u}{\partial \eta}+\frac{\partial u}{\partial \xi}\psi_x+\frac{\partial v}{\partial \zeta}+\frac{\partial v}{\partial \xi}\psi_y\Big)\\
-E\frac{\partial p}{\partial \xi}-\frac{1}{R}\Big(\frac{\partial^2 w}{\partial \eta^2}+2\frac{\partial^2 w}{\partial \eta\partial \xi}\psi_x+\frac{\partial^2 w}{\partial \xi^2}\psi_x^2+\frac{\partial w}{\partial \xi}\psi_{xx}\\
+\frac{\partial^2 w}{\partial \zeta^2}+2\frac{\partial^2 w}{\partial \zeta\partial \xi}\psi_y+\frac{\partial^2 w}{\partial \xi^2}\psi_y^2+\frac{\partial w}{\partial \xi}\psi_{yy}+\frac{\partial^2 u}{\partial \eta\partial \xi}+\frac{\partial^2 u}{\partial \xi^2}\psi_x+\frac{\partial^2 v}{\partial \zeta\partial \xi}+\frac{\partial^2 v}{\partial \xi^2}\psi_y\Big)=\frac{1}{F}.
\end{gathered}\tag{2.2}$$

The system of equations (2.2) can be written in the form $AU=B$, where

$$A=\frac{1}{R}\begin{pmatrix}\psi_y^2+1 & -\psi_x\psi_y & \psi_x\\ -\psi_x\psi_y & \psi_x^2+1 & \psi_y\\ \psi_x & \psi_y & \psi_x^2+\psi_y^2\end{pmatrix},\quad U=\begin{pmatrix}\dfrac{\partial^2 u}{\partial \xi^2}\\ \dfrac{\partial^2 v}{\partial \xi^2}\\ \dfrac{\partial^2 w}{\partial \xi^2}\end{pmatrix},$$

$$B=\begin{pmatrix}S\frac{\partial u}{\partial \tau}-u\Big(\frac{\partial v}{\partial \zeta}+\frac{\partial v}{\partial \xi}\psi_y-\frac{\partial w}{\partial \xi}\Big)+v\Big(\frac{\partial u}{\partial \zeta}+\frac{\partial u}{\partial \xi}\psi_y\Big)-w\frac{\partial u}{\partial \xi}+E\Big(\frac{\partial p}{\partial \eta}+\frac{\partial p}{\partial \xi}\psi_x\Big)-\frac{M}{R}\\
S\frac{\partial v}{\partial \tau}+u\Big(\frac{\partial v}{\partial \eta}+\frac{\partial v}{\partial \xi}\psi_x\Big)-v\Big(\frac{\partial u}{\partial \eta}+\frac{\partial u}{\partial \xi}\psi_x-\frac{\partial w}{\partial \xi}\Big)-w\frac{\partial v}{\partial \xi}+E\Big(\frac{\partial p}{\partial \zeta}+\frac{\partial p}{\partial \xi}\psi_y\Big)-\frac{N}{R}\\
S\frac{\partial w}{\partial \tau}+u\Big(\frac{\partial w}{\partial \eta}+\frac{\partial w}{\partial \xi}\psi_x\Big)+v\Big(\frac{\partial w}{\partial \zeta}+\frac{\partial w}{\partial \xi}\psi_y\Big)-wQ-E\frac{\partial p}{\partial \xi}-\frac{H}{R}-\frac{1}{F}\end{pmatrix},$$

$$M=-\frac{\partial^2 v}{\partial\zeta\partial\eta}-\frac{\partial^2 v}{\partial\eta\partial\xi}\psi_y-\frac{\partial^2 v}{\partial\zeta\partial\xi}\psi_x-\frac{\partial v}{\partial\xi}\psi_{xy}+\frac{\partial^2 w}{\partial\eta\partial\xi}+\frac{\partial^2 u}{\partial\zeta^2}+2\frac{\partial^2 u}{\partial\zeta\partial\xi}\psi_y+\frac{\partial u}{\partial\xi}\psi_{yy},$$

$$N=\frac{\partial^2 v}{\partial\eta^2}+2\frac{\partial^2 v}{\partial\eta\partial\xi}\psi_x+\frac{\partial v}{\partial\xi}\psi_{xx}-\frac{\partial^2 u}{\partial\eta\partial\zeta}-\frac{\partial^2 u}{\partial\eta\partial\xi}\psi_y-\frac{\partial^2 u}{\partial\zeta\partial\xi}\psi_x-\frac{\partial u}{\partial\xi}\psi_{xy}+\frac{\partial^2 w}{\partial\zeta\partial\xi},$$

$$Q=\frac{\partial u}{\partial\eta}+\frac{\partial u}{\partial\xi}\psi_x+\frac{\partial v}{\partial\zeta}+\frac{\partial v}{\partial\xi}\psi_y,$$

$$H=\frac{\partial^2 w}{\partial\eta^2}+2\frac{\partial^2 w}{\partial\eta\partial\xi}\psi_x+\frac{\partial w}{\partial\xi}\psi_{xx}+\frac{\partial^2 w}{\partial\zeta^2}+2\frac{\partial^2 w}{\partial\zeta\partial\xi}\psi_y+\frac{\partial w}{\partial\xi}\psi_{yy}+\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{\partial^2 v}{\partial\zeta\partial\xi}.$$

From the last equation of system (0.1), we obtain

$$\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\zeta}+\frac{\partial u}{\partial\xi}\psi_x+\frac{\partial v}{\partial\xi}\psi_y-\frac{\partial w}{\partial\xi}=0. \tag{2.3}$$

Let us also write the differential consequences of expression (2.3):

$$\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial^2 v}{\partial\zeta\partial\eta}+\frac{\partial^2 u}{\partial\xi\partial\eta}\psi_x+\frac{\partial^2 v}{\partial\xi\partial\eta}\psi_y-\frac{\partial^2 w}{\partial\xi\partial\eta}+\frac{\partial u}{\partial\xi}\psi_{xx}+\frac{\partial v}{\partial\xi}\psi_{xy}=0,$$

$$\frac{\partial^2 u}{\partial\eta\partial\zeta}+\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi\partial\zeta}\psi_x+\frac{\partial^2 v}{\partial\xi\partial\zeta}\psi_y-\frac{\partial^2 w}{\partial\xi\partial\zeta}+\frac{\partial u}{\partial\xi}\psi_{xy}+\frac{\partial v}{\partial\xi}\psi_{yy}=0,$$

$$\frac{\partial^2 u}{\partial\eta\partial\tau}+\frac{\partial^2 v}{\partial\zeta\partial\tau}+\frac{\partial^2 u}{\partial\xi\partial\tau}\psi_x+\frac{\partial^2 v}{\partial\xi\partial\tau}\psi_y-\frac{\partial^2 w}{\partial\xi\partial\tau}=0,$$

$$\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{\partial^2 v}{\partial\zeta\partial\xi}+\frac{\partial^2 u}{\partial\xi^2}\psi_x+\frac{\partial^2 v}{\partial\xi^2}\psi_y-\frac{\partial^2 w}{\partial\xi^2}=0.$$

Let the boundary conditions $u(t,\psi(x,y),x,y)=0$, $v(t,\psi(x,y),x,y)=0$, $w(t,\psi(x,y),x,y)=0$, $\partial u/\partial\xi=u_1(\eta,\zeta)$, and $\partial v/\partial\xi=v_1(\eta,\zeta)$ be given on the surface $\xi=0$. Then, equation (2.3) implies $w_1(\eta,\zeta)=\partial w/\partial\xi=u_1\psi_x+v_1\psi_y$. In addition, define $p(t,\psi(x,y),x,y)=p_0(\eta,\zeta)$ and $\partial p/\partial\xi=p_1(\eta,\zeta)$ on this surface.

Note that the determinant of the matrix A is zero. Multiply the right- and the left-hand sides of the system $AU=B$ by the left null vector of the matrix A (see [18]):

$$L=\{\psi_x^2(\psi_x^2+\psi_y^2+1),\psi_x\psi_y(\psi_x^2+\psi_y^2+1)-\psi_x(\psi_x^2+\psi_y^2+1)\}.$$

We obtain the existence condition for a solution of system (2.2) on the surface $\xi=0$, assuming that $\psi_x\neq 0$, $\psi_x^2+\psi_y^2+1\neq 0$ and taking into account relation (2.3) and its differential consequences as well as the given boundary conditions

$$\frac{1}{R}\Big[\Big(\frac{\partial u_1}{\partial\eta}+\frac{\partial v_1}{\partial\zeta}\Big)(\psi_x^2+\psi_y^2+1)+2(\psi_x\psi_{xx}+\psi_y\psi_{xy})u_1+2(\psi_x\psi_{xy}+\psi_y\psi_{yy})v_1\Big]$$
$$+E\Big[\frac{\partial p_0}{\partial\eta}\psi_x+\frac{\partial p_0}{\partial\zeta}\psi_y+p_1(\psi_x^2+\psi_y^2+1)\Big]+\frac{1}{F}=0. \tag{2.4}$$

Thus, for the given first derivatives (boundary conditions), we have two relations (2.3) and (2.4), which must be satisfied on the streamlined surface in order that system (2.2) have a solution.

Let us transform relation (2.4) to a more convenient form. For this, we rewrite the differential consequences of equation (2.3) and relation (2.4) as follows:

$$\begin{aligned}
&u_1\psi_{\eta\eta} + v_1\psi_{\eta\zeta} = -\Big(\frac{\partial u_1}{\partial\eta}\psi_\eta + \frac{\partial v_1}{\partial\eta}\psi_\zeta - \frac{\partial w_1}{\partial\eta}\Big),\\
&u_1\psi_{\eta\zeta} + v_1\psi_{\zeta\zeta} = -\Big(\frac{\partial u_1}{\partial\zeta}\psi_\eta + \frac{\partial v_1}{\partial\zeta}\psi_\zeta - \frac{\partial w_1}{\partial\zeta}\Big),\\
&(\psi_\eta\psi_{\eta\eta} + \psi_\zeta\psi_{\eta\zeta})u_1 + (\psi_\eta\psi_{\eta\zeta} + \psi_\zeta\psi_{\zeta\zeta})v_1 = -0.5ER\Big[\frac{\partial p_0}{\partial\eta}\psi_\eta + \frac{\partial p_0}{\partial\zeta}\psi_\zeta + p_1 n\Big]\\
&-0.5\Big(\frac{\partial u_1}{\partial\eta} + \frac{\partial v_1}{\partial\zeta}\Big)n - 0.5\frac{R}{F}, \qquad \text{where}\quad n = \psi_\eta^2 + \psi_\zeta^2 + 1.
\end{aligned} \tag{2.5}$$

Multiply the first equation of (2.5) by ψ_η and the second equation by ψ_ζ and add them together. Subtracting the third equation of (2.5) from the obtained sum, we reduce (2.4) to the relation

$$\begin{aligned}
0.5n\frac{\partial u_1}{\partial\eta} + 0.5n\frac{\partial v_1}{\partial\zeta} + \psi_\eta\frac{\partial w_1}{\partial\eta} + \psi_\zeta\frac{\partial w_1}{\partial\zeta} - \psi_\eta\psi_\zeta\Big(\frac{\partial u_1}{\partial\zeta} + \frac{\partial v_1}{\partial\eta}\Big)\\
+ 0.5ER\Big[\frac{\partial p_0}{\partial\eta}\psi_\eta + \frac{\partial p_0}{\partial\zeta}\psi_\zeta + p_1 n\Big] + 0.5\frac{R}{F} = 0.
\end{aligned} \tag{2.6}$$

Relation (2.3) under the given boundary conditions takes the form

$$w_1 = u_1\psi_\eta + v_1\psi_\zeta. \tag{2.7}$$

Then, for the given outer derivatives $u_1 = \partial u/\partial\xi$, $v_1 = \partial v/\partial\xi$, and $w_1 = \partial w/\partial\xi$, the surface $\xi = 0$ must satisfy system (2.6), (2.7) of two first-order partial differential equations if the pressure p_0 and its outer derivative $p_1 = \partial p/\partial\xi$ are known on the body.

The condition of existence of a sufficiently smooth solution of system (2.2) near the streamlined surface reduces to obtaining a consistency condition for system (2.6), (2.7).

Assertion 1. *The system of first-order differential equations* (2.6), (2.7) *is consistent if and only if the following equality holds identically*:

$$\{[-b \pm \sqrt{b^2 - 4ac}]/(2a)\}_\eta = \{w_1/u_1 - v_1[-b \pm \sqrt{b^2 - 4ac}]/(2au_1)\}_\zeta, \tag{2.8}$$

where the lower indices η and ζ denote the variables with respect to which the derivatives of the corresponding expressions are taken,

$$a = 0.5\frac{u_1^2 - v_1^2}{u_1^2}\frac{\partial u_1}{\partial\eta} + \frac{v_1}{u_1}\frac{\partial u_1}{\partial\zeta} + \frac{v_1}{u_1}\frac{\partial v_1}{\partial\eta} + 0.5\frac{v_1^2 - u_1^2}{u_1^2}\frac{\partial v_1}{\partial\zeta} + 0.5ERp_1\frac{u_1^2 + v_1^2}{u_1^2},$$

$$\begin{aligned}
b = \frac{v_1 w_1}{u_1^2}\frac{\partial u_1}{\partial\eta} - \frac{w_1}{u_1}\frac{\partial u_1}{\partial\zeta} - \frac{w_1}{v_1}\frac{\partial v_1}{\partial\eta} - \frac{w_1 v_1}{u_1^2}\frac{\partial v_1}{\partial\zeta} - \frac{v_1}{u_1}\frac{\partial w_1}{\partial\eta} + \frac{\partial w_1}{\partial\zeta}\\
- 0.5ER\frac{v_1}{u_1}\frac{\partial p_0}{\partial\eta} + 0.5ER\frac{\partial p_0}{\partial\zeta} - ERp_1\frac{v_1 w_1}{u_1^2},
\end{aligned}$$

$$c = 0.5\frac{u_1^2 - w_1^2}{u_1^2}\frac{\partial u_1}{\partial\eta} + 0.5\frac{u_1^2 + w_1^2}{u_1^2}\frac{\partial v_1}{\partial\zeta} + \frac{w_1}{u_1}\frac{\partial w_1}{\partial\eta} + 0.5ER\frac{w_1}{u_1}\frac{\partial p_0}{\partial\eta} + 0.5ERp_1\frac{u_1^2 + w_1^2}{u_1^2} + 0.5\frac{R}{F}.$$

Proof. The Jacobian system obtained from the system (2.6), (2.7) of first-order differential equations is completely integrable if and only if the mixed derivatives are identically equal.

Substituting ψ_η from relation (2.7) into equation (2.6), we get

$$0.5\frac{\partial u_1}{\partial \eta}\Big(\frac{u_1^2-v_1^2}{u_1^2}\psi_\zeta^2+2\psi_\zeta\frac{v_1w_1}{u_1^2}+\frac{u_1^2-w_1^2}{u_1^2}\Big)-\frac{\partial u_1}{\partial \zeta}\Big(\frac{w_1}{u_1}\psi_\zeta-\frac{v_1}{u_1}\psi_\zeta^2\Big)$$
$$-\frac{\partial v_1}{\partial \eta}\Big(\frac{w_1}{u_1}\psi_\zeta-\frac{v_1}{u_1}\psi_\zeta^2\Big)+0.5\frac{\partial v_1}{\partial \zeta}\Big(\frac{v_1^2-u_1^2}{u_1^2}\psi_\zeta^2-2\psi_\zeta\frac{v_1w_1}{u_1^2}+\frac{u_1^2+w_1^2}{u_1^2}\Big)$$
$$+\frac{\partial w_1}{\partial \eta}\Big(\frac{w_1}{u_1}-\frac{v_1}{u_1}\psi_\zeta\Big)+\frac{\partial w_1}{\partial \zeta}\psi_\zeta+0.5ER\frac{\partial p_0}{\partial \eta}\Big(\frac{w_1}{u_1}-\psi_\zeta\frac{v_1}{u_1}\Big)+0.5ER\psi_\zeta\frac{\partial p_0}{\partial \zeta}$$
$$+0.5ERp_1\Big(\frac{u_1^2+v_1^2}{u_1^2}\psi_\zeta^2-2\psi_\zeta\frac{v_1w_1}{u_1^2}+\frac{u_1^2+w_1^2}{u_1^2}\Big)+0.5\frac{R}{F}=0. \tag{2.9}$$

Expressing ψ_ζ from (2.9) and then ψ_η from (2.7), we obtain the Jacobian system

$$\psi_\zeta=c_1/(2a),\quad \psi_\eta=w_1/u_1-v_1c_1/(2au_1),\quad \text{where}\quad c_1=-b\pm\sqrt{b^2-4ac}.$$

The requirement of the equality of the mixed derivatives leads to relation (2.8), connecting the velocity and the pressure, which must be identically satisfied on the streamlined body.

The assertion is proved. □

If relation (2.8) does not become an identity, then the mixed derivatives are identical only on a certain line $\varphi(\eta,\zeta)=0$.

Consider relation (2.7). In this relation,

$$u_1=\lim_{\xi\to 0}\frac{u(\eta,\zeta,\xi)-u(\eta,\zeta,0)}{\xi}=\lim_{\xi\to 0}\frac{u(\eta,\zeta,\xi)}{\xi},\quad v_1=\lim_{\xi\to 0}\frac{v(\eta,\zeta,\xi)-v(\eta,\zeta,0)}{\xi}=\lim_{\xi\to 0}\frac{v(\eta,\zeta,\xi)}{\xi},$$
$$w_1=\lim_{\xi\to 0}\frac{w(\eta,\zeta,\xi)-w(\eta,\zeta,0)}{\xi}=\lim_{\xi\to 0}\frac{w(\eta,\zeta,\xi)}{\xi};$$

consequently, $u(\eta,\zeta,\xi)\psi_\eta+v(\eta,\zeta,\xi)\psi_\zeta-w(\eta,\zeta,\xi)=o(\xi)$.

Thus, by relation (2.7), the flow near the surface of the streamlined body "repeats" the form of its surface.

As shown earlier, one condition (2.7) is insufficient for the existence of a sufficiently smooth solution of system (2.2). To construct the solution in the class of smooth functions, it is necessary that conditions (2.6)–(2.8) hold. If condition (2.8) fails, then relations (2.7) and (2.6) specify two cones of normals and two different surfaces intersecting along the line $\varphi(\eta,\zeta)=0$, i.e., the surface of the streamlined body and the surface described by equation (2.6). Since we have the no-slip condition on the surface of the streamlined body, we can consider this surface as a characteristic surface of weak discontinuity. The situation with two different surfaces that have a common part is similar to the situation in geometrical optics when the direction of an incident ray of light coincides with an optic axis of a crystal. Such a situation is mathematically described by two characteristic surfaces having one common normal. In the case of geometrical optics, this leads to light scattering called the conical refraction [19]. As follows from observations [14], zones of mixing and scattering often emerge in flows past smooth surfaces (the problem under consideration). The reason for such phenomena is that the perturbation from the body on the line of intersection of two surfaces $\varphi(\eta,\zeta)=0$ is scattered similarly to the scattering of a ray under conical refraction. A rather complicated picture of the flow between surfaces (2.6) and (2.7) may arise (strong and weak discontinuities, a blow-up, and turbulent mixing).

Corollary. *If* $u_1 = v_1 = w_1 = 0$ *on the surface of the streamlined body, then the consistency condition* (2.4) *takes the form*

$$E\Big[\frac{\partial p_0}{\partial \eta}\psi_x + \frac{\partial p_0}{\partial \zeta}\psi_y + p_1(\psi_x^2 + \psi_y^2 + 1)\Big] + \frac{1}{F} = 0.$$

3. DEPENDENCE OF OUTER DERIVATIVES ON THE REYNOLDS NUMBER

If we pass to the new variables in system (0.1) without regard to the continuity equation, then we obtain the system $GU = T_0$, where

$$G = \frac{1}{R}\begin{pmatrix} \psi_x^2 + \psi_y^2 + 1 & 0 & 0 \\ 0 & \psi_x^2 + \psi_y^2 + 1 & 0 \\ 0 & 0 & \psi_x^2 + \psi_y^2 + 1 \end{pmatrix}.$$

For the vector T_0, we have

$$T_0 = \begin{pmatrix} S\dfrac{\partial u}{\partial \tau} + u\Big(\dfrac{\partial u}{\partial \eta} + \dfrac{\partial u}{\partial \xi}\psi_x\Big) + v\Big(\dfrac{\partial u}{\partial \zeta} + \dfrac{\partial u}{\partial \xi}\psi_y\Big) - w\dfrac{\partial u}{\partial \xi} + E\Big(\dfrac{\partial p}{\partial \eta} + \dfrac{\partial p}{\partial \xi}\psi_x\Big) - \dfrac{M_1}{R} \\ S\dfrac{\partial v}{\partial \tau} + u\Big(\dfrac{\partial v}{\partial \eta} + \dfrac{\partial v}{\partial \xi}\psi_x\Big) + v\Big(\dfrac{\partial v}{\partial \zeta} + \dfrac{\partial v}{\partial \xi}\psi_y\Big) - w\dfrac{\partial v}{\partial \xi} + E\Big(\dfrac{\partial p}{\partial \zeta} + \dfrac{\partial p}{\partial \xi}\psi_y\Big) - \dfrac{N_1}{R} \\ S\dfrac{\partial w}{\partial \tau} + u\Big(\dfrac{\partial w}{\partial \eta} + \dfrac{\partial w}{\partial \xi}\psi_x\Big) + v\Big(\dfrac{\partial w}{\partial \zeta} + \dfrac{\partial w}{\partial \xi}\psi_y\Big) - w\dfrac{\partial w}{\partial \xi} - E\dfrac{\partial p}{\partial \xi} - \dfrac{H_1}{R} - \dfrac{1}{F} \end{pmatrix},$$

where $M_1 = \dfrac{\partial^2 u}{\partial \eta^2} + 2\dfrac{\partial^2 u}{\partial \eta \partial \xi}\psi_x + \dfrac{\partial^2 u}{\partial \zeta^2} + 2\dfrac{\partial^2 u}{\partial \zeta \partial \xi}\psi_y$, $N_1 = \dfrac{\partial^2 v}{\partial \eta^2} + 2\dfrac{\partial^2 v}{\partial \eta \partial \xi}\psi_x + \dfrac{\partial^2 v}{\partial \zeta^2} + 2\dfrac{\partial^2 v}{\partial \zeta \partial \xi}\psi_y$,

$$H_1 = \frac{\partial^2 w}{\partial \eta^2} + 2\frac{\partial^2 w}{\partial \eta \partial \xi}\psi_x + \frac{\partial^2 w}{\partial \zeta^2} + 2\frac{\partial^2 w}{\partial \zeta \partial \xi}\psi_y.$$

Hence, we find that, on the surface $\xi = 0$,

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} &= \frac{R}{\psi_x^2 + \psi_y^2 + 1}\Big[E\Big(\frac{\partial p_0}{\partial \eta} + p_1\psi_x\Big) - \frac{M_2}{R}\Big], \\ \frac{\partial^2 v}{\partial \xi^2} &= \frac{R}{\psi_x^2 + \psi_y^2 + 1}\Big[E\Big(\frac{\partial p_0}{\partial \zeta} + p_1\psi_y\Big) - \frac{N_2}{R}\Big], \\ \frac{\partial^2 w}{\partial \xi^2} &= -\frac{R}{\psi_x^2 + \psi_y^2 + 1}\Big(Ep_1 + \frac{H_2}{R} + \frac{1}{F}\Big), \end{aligned} \tag{3.1}$$

where $M_2 = 2\Big(\dfrac{\partial u_1}{\partial \eta}\psi_x + \dfrac{\partial u_1}{\partial \zeta}\psi_y\Big)$, $N_2 = 2\Big(\dfrac{\partial v_1}{\partial \eta}\psi_x + \dfrac{\partial v_1}{\partial \zeta}\psi_y\Big)$, $H_2 = 2\Big(\dfrac{\partial w_1}{\partial \eta}\psi_x + \dfrac{\partial w_1}{\partial \zeta}\psi_y\Big)$,

and $w_1 = u_1\psi_x + v_1\psi_y$. Substituting quantities (3.1) into differential consequences (2.3), we obtain relation (2.4) once again. Consequently, expressions (3.1) specify the second derivatives of the flow past the streamlined body if the boundary conditions given on the body comply with relation (2.4).

As follows from (3.1), the second derivatives of the velocity components also tend to infinity as $R \to \infty$.

Note that, as follows from relation (3.1), if the Reynolds number is such that

$$R < \min\Big\{M_2/\Big[E\Big(\frac{\partial p_0}{\partial \eta} + p_1\psi_x\Big)\Big],\ N_2/\Big[E\Big(\frac{\partial p_0}{\partial \zeta} + p_1\psi_y\Big)\Big],\ -H_2/\Big(Ep_1 + \frac{1}{F}\Big)\Big\},$$

then the values of the second derivatives are determined by the terms independent of the Reynolds number. Similarly, if the relations

$$\frac{\partial p_0}{\partial \eta} + p_1\psi_x = 0, \quad \frac{\partial p_0}{\partial \zeta} + p_1\psi_y = 0, \quad Ep_1 + \frac{1}{F} = 0$$

hold on the body, then the Reynolds number does not influence the second derivatives.

Assertion 2. *Near the streamlined plane $l = at + x + by + cz$, we have*

$$\lim_{R\to\infty} K = \infty, \quad \lim_{R\to\infty} T = \infty,$$

where K is the streamline curvature, T is the streamline torsion, and R is the Reynolds number.

Proof. Consider the influence of the Reynolds number on the flow near the streamlined body for the exact solution of the system of Navier–Stokes equations [1]. Assume that its solution is $u = u(\psi(x, y, z, t))$, $v = v(\psi(x, y, z, t))$, $w = w(\psi(x, y, z, t))$, and $p = p(\psi(x, y, z, t))$. Then, $\psi(x, y, z, t) = \text{const}$ is a level surface for u, v, w, and p, and system (0.1) can be written as

$$\begin{aligned}
&Su'\psi_t + uu'\psi_x + vu'\psi_y + wu'\psi_z - \frac{1}{R}(u''m + u'\Delta) = 0,\\
&Sv'\psi_t + uv'\psi_x + vv'\psi_y + wv'\psi_z - \frac{1}{R}(v''m + v'\Delta) = 0,\\
&Sw'\psi_t + uw'\psi_x + vw'\psi_y + ww'\psi_z - \frac{1}{R}(w''m + w'\Delta) = \frac{1}{F},\\
&u'\psi_x + v'\psi_y + w'\psi_z = 0, \quad m = \psi_x^2 + \psi_y^2 + \psi_z^2, \quad \Delta = \psi_{xx} + \psi_{yy} + \psi_{zz}.
\end{aligned} \tag{3.2}$$

In system (3.2), the prime ($'$) denotes differentiation with respect to ψ and lower indices denote differentiation of the function ψ with respect to the corresponding variables.

Let $\psi_x \neq 0$. Define $\psi_t/\psi_x = f_1(\psi)$, $\psi_y/\psi_x = f_2(\psi)$, $\psi_z/\psi_x = f_3(\psi)$, and $(\psi_x^2 + \psi_y^2 + \psi_z^2)/\psi_x = f_4(\psi)$, where $f_i(\psi)$ are arbitrary functions ($i = 1, 2, 3, 4$). Then, $\psi_x = f_4/(1 + f_2^2 + f_3^2) = g(\psi)$. The equality of the mixed derivatives implies that $\psi_t = ag(\psi)$, $\psi_y = bg(\psi)$, $\psi_z = cg(\psi)$, $a = \text{const}$, $b = \text{const}$, and $c = \text{const}$ and, consequently, $\psi = \psi(l)$ and $l = x + at + by + cz$. Then, we find that $u = u(l)$, $v = v(l)$, $w = w(l)$, and $p = p(l)$ and the system of equations (0.1) reduces to the following system of ordinary differential equations:

$$\begin{aligned}
&(Sa + u + bv + cw)u_l + Ep_l - \frac{1}{R}qu_{ll} = 0, \quad (Sa + u + bv + cw)v_l + Ebp_l - \frac{1}{R}qv_{ll} = 0,\\
&(Sa + u + bv + cw)w_l + Ecp_l - \frac{1}{R}qw_{ll} = \frac{1}{F}, \quad u_l + bv_l + cw_l = 0, \quad \text{where} \quad q = 1 + b^2 + c^2.
\end{aligned} \tag{3.3}$$

The differential consequences of the last equation in system (3.3) yield the relations $u_{ll} + bv_{ll} + cw_{ll} = 0$, $u + bv + cw = A$, and $A = \text{const}$. If we multiply the second equation of system (3.3) by b and the third equation by c and add them to the first equation of the system, we obtain $p_l = c/(FEq)$. Then the first three equations of system (3.3) take the form

$$\begin{aligned}
&\frac{q}{R}u_{ll} - (Sa + A)u_l - \frac{c}{Fq} = 0, \quad \frac{q}{R}v_{ll} - (Sa + A)v_l - \frac{cb}{Fq} = 0,\\
&\frac{q}{R}w_{ll} - (Sa + A)w_l + \frac{1 + b^2}{Fq} = 0.
\end{aligned} \tag{3.4}$$

Let us write a general solution of linear system (3.4) with constant coefficients.

If $A \neq (-Sa)$, then

$$u = \frac{U_0}{\alpha}\exp(\alpha l) - \frac{\beta}{\alpha}l + U_1, \quad \alpha = \frac{R(Sa+A)}{q}, \quad \beta = -\frac{cR}{Fq^2}, \quad U_0 = \text{const}, \quad U_1 = \text{const};$$

$$\begin{aligned} v &= \frac{V_0}{\alpha}\exp(\alpha l) - \frac{b\beta}{\alpha}l + V_1, \quad V_0 = \text{const}, \quad V_1 = \text{const}; \\ w &= -\frac{U_0 + bV_0}{c\alpha}\exp(\alpha l) + \frac{\beta(1+b^2)}{c\alpha}l + \frac{A - U_1 - bV_1}{c}; \\ p &= \frac{c}{FEq}l + p_0, \quad p_0 = \text{const}, \quad l = at + x + by + cz. \end{aligned} \tag{3.5}$$

We use (3.5) to study the properties of the flow near the surface $\xi = 0$ assuming that $\xi = l$ is a movable streamlined plane. On the plane $l = 0$, we set the no-slip conditions $u(0) = 0$, $v(0) = 0$, and $w(0) = 0$. Then, on $l = 0$, we have $A = 0$, $U_1 = -U_0/\alpha$, and $V_1 = -V_0/\alpha$; consequently,

$$u_1 = U_0 - \frac{c}{qF(Sa+A)}, \quad v_1 = V_0 - \frac{bc}{qF(Sa+A)}, \quad w_1 = -\frac{U_0+bV_0}{c} + \frac{1+b^2}{qF(Sa+A)}, \quad p_1 = \frac{c}{qEF}.$$

Note that the first derivatives with respect to ξ are independent of the Reynolds number and, as in the general case (see the corollary), the flow near the streamlined surface repeats the form of the surface at any time: $u_1 + bv_1 + cw_1 = 0$; consequently, $u(l) + bv(l) + cw(l) = o(l)$.

Let us also write the second derivatives on the surface $\xi = l = 0$. We obtain

$$\frac{d^2u}{d\xi^2} = \alpha U_0, \quad \frac{d^2v}{d\xi^2} = \alpha V_0, \quad \frac{d^2w}{d\xi^2} = -\frac{(U_0+bV_0)\alpha}{c}, \quad \frac{d^2p}{d\xi^2} = 0.$$

Here, we observe the dependence on the Reynolds number (see α in (3.5)). Similarly, we have for the third derivatives

$$\frac{d^3u}{d\xi^3} = \alpha^2 U_0, \quad \frac{d^3v}{d\xi^3} = \alpha^2 V_0, \quad \frac{d^3w}{d\xi^3} = -\frac{(U_0+bV_0)\alpha^2}{c}, \quad \frac{d^3p}{d\xi^3} = 0.$$

Substituting the obtained values of the derivatives into the formulas for the curvature K and the torsion T of the line $u = u(\xi)$, $v = v(\xi)$, and $w = w(\xi)$, we get

$$K = \frac{R(Sa+A)}{qT_1}\sqrt{T_1T_2 - \frac{qT_3}{R(Sa+A)}}, \quad T = \Big(\frac{R(Sa+A)}{q}\Big)^3 \frac{T_4}{T_1^3K^2}, \tag{3.6}$$

where $T_1 = \Big(U_0 - \frac{c}{qF(Sa+A)}\Big)^2 + \Big(V_0 - \frac{bc}{qF(Sa+A)}\Big)^2 + \Big(-\frac{U_0+bV_0}{c} + \frac{1+b^2}{qF(Sa+A)}\Big)^2,$

$$T_2 = U_0^2 + V_0^2 + \Big[\frac{U_0+bV_0}{c}\Big]^2,$$

$$T_3 = U_0\Big(U_0 - \frac{c}{qF(Sa+A)}\Big) + V_0\Big(V_0 - \frac{bc}{qF(Sa+A)}\Big) + \frac{U_0+bV_0}{c}\Big(-\frac{U_0+bV_0}{c} + \frac{1+b^2}{qF(Sa+A)}\Big),$$

$$T_4 = V_0\Big(U_0 - \frac{c}{qF(Sa+A)}\Big)\Big(\frac{U_0+bV_0}{c}\Big) + U_0V_0\Big(-\frac{U_0+bV_0}{c} + \frac{1+b^2}{qF(Sa+A)}\Big)$$

$$+ U_0\Big(\frac{U_0+bV_0}{c}\Big)\Big(V_0-\frac{bc}{qF(Sa+A)}\Big)-U_0V_0\Big(-\frac{U_0+bV_0}{c}+\frac{1+b^2}{qF(Sa+A)}\Big)$$

$$- V_0\Big(\frac{U_0+bV_0}{c}\Big)\Big(U_0-\frac{c}{qF(Sa+A)}\Big)-U_0\Big(\frac{U_0+bV_0}{c}\Big)\Big(V_0-\frac{bc}{qF(Sa+A)}\Big).$$

Relations (3.6) imply that $\lim\limits_{R\to\infty} K=\infty$ and $\lim\limits_{R\to\infty} T=\infty$.

The assertion is proved. □

4. CONCLUSIONS

As shown above, when posing both the initial and boundary value problems, we have to deal with an overdetermined system of partial differential equations (see the continuity equation and equation (1.1) in the new independent variables or system (2.6), (2.7)). For a given pressure, the consistency requirement for such systems leads to certain requirements on the velocity of motion of the medium (see (2.8)). If these requirements are not fulfilled, then each equation of the system in the domain of its solution has its own cone of normals and (as the extended system of characteristic equations shows [20,21]) its own highest-order derivatives in terms of which the streamline curvature and torsion can be expressed (see (3.6)).

It is well-known in optics that, if the direction of an incident ray coincides with the direction of an optic axis of a crystal, then a bright spot is observed on a screen instead of a point. This effect of ray scattering (conical refraction) is explained by the presence of two different first-order partial differential equations. Their cones of normals have one common normal, but they do not coincide. A similar reason leads to the phenomenon of scattering for a nonstationary flow of an ideal plasma, when the vector of magnetic intensity is parallel to the normal of a characteristic surface and the velocity of sound is equal to the van Alphen velocity [19].

In this connection, recall the solution of system (2.6), (2.7) in the case when conditions (2.8) do not hold. We observe the resemblance to the mathematical description of conical refraction. Here, the line $\varphi(\eta,\zeta)=0$ plays the role of an optic axis of a crystal. An analogy of the behavior of a flow past a body (scattering and turbulent mixing) to the effect of scattering in optics comes to mind.

In the case of an initial value problem describing the motion of a viscous fluid at the initial time, the flow or the motion may mismatch the ambient atmospheric pressure, and relation (1.1) is not satisfied. There appear zones of additional perturbations, which, during their propagation, give rise to mixing, scattering, etc.

Thus, we can assume that the primary cause generating the breakdown of a laminar flow lies in the absence of matching between the ambient pressure and the velocity of motion of the fluid. Hence, additional perturbations [13], scattering [14], and bifurcation [10] may occur. As the above example of a boundary value problem shows, certain conditions for the elimination of "conflict" may exist (see, for example, (2.8)).

The transition to a turbulent motion in continuous media occurs with the growth of the Reynolds number. It is also known that the streamline curvature has a significant effect on turbulence, including its intensification, and is an important issue that requires a close examination in the study of turbulence and in the development of models of turbulence [22, 23]. Hence, it is interesting that, for the considered mathematical model (0.1), relations (3.1) and (3.6) imply that, if $A \neq (-Sa)$ and as the Reynolds number grows, the streamline curvature and torsion increase. Otherwise (for $A=-Sa$), the growth of the Reynolds number has no detectable effect.

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REFERENCES

1. L. G. Loitsyanskii, *Mechanics of Liquids and Gases* (Drofa, Moscow, 2003; Pergamon, Oxford, 1966).
2. V. V. Pukhnachev, "Symmetry in Navier–Stokes equations," Uspekhi Mekh., No. 6, 3–76 (2006).
3. L. Brandolese, "Fine properties of self-similar solutions of the Navier–Stokes equation," Arch. Ration. Mech. Anal. **192** (3), 375–401 (2009).
4. G. P. Galdi, "On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications," in *Handbook of Mathematical Fluid Dynamics* (North Holland, Amsterdam, 2002), Vol. 1, pp. 653–791.
5. M. E. Schonbek, "Asymptotic behavior of solutions to three-dimensional Navier–Stokes equations," Indiana Univ. Math. J. **41** (3), 809–823 (1992).
6. T. Miyakawa, "On space-time decay properties of nonstationary incompressible Navier–Stokes flows in $\mathbb{R}^n$," Funkcial. Ekvac. **43** (3), 541–557 (2000).
7. C. Fefferman, "Existence and smoothness of the Navier–Stokes equation," in *The Millennium Prize Problems*, (Clay Math. Inst., Cambridge, MA, 2006), pp. 57–67.
8. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Nauka, Moscow, 1970; Gordon and Breach, New York, 1968).
9. V. A. Galaktionov, "On a spectrum of blow-up patterns for a higher-order semilinear parabolic equations," Proc. Royal Soc. A, Ser. Math. Phys. Eng. Sci. **457**, 1623–1643 (2001).
10. J. Marsden and J. McCracken, *The Hopf Bifurcation and Its Applications* (Springer, New York, 1976), Ser. Applied Mathematical Sciences **19**.
11. Yu. N. Kochetkov, "Turbulence on a rough surface and new fundamental equations of boundary layer," Dvigatel', No. 3, 38–41 (2015).
12. N. A. Slezkin, *Dynamics of a Viscous Incompressible Fluid* (GITTL, Moscow, 1955).
13. A. J. Chorin, *Lectures on Turbulence Theory*, (Publish or Perish, Boston, 1976).
14. A. N. Kolmogorov, "Energy dissipation in local isotropic turbulence," Dokl. Akad. Nauk **32** (1), 19–21 (1941).
15. Y. Giga, "Weak and strong solutions of the Navier–Stokes initial value problem," Publ. Res. Inst. Math. Sci. **19** (3), 887–910 (1983).
16. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. 1: *Linear Steady Problems*, Vol. 2: *Nonlinear Steady Problems* (Springer, New York, 1994).
17. E. V. Ermolin, L. I. Rubina, and A. F. Sidorov, "On the problem of two pistons," J. Appl. Math. Mech. **32** (5), 967–976 (1968).
18. R. Courant, *Partial Differential Equations* (Mir, Moscow, 1964) [in Russian].
19. L. I. Rubina, "Propagation of weak discontinuities for quasi-linear systems," J. Appl. Math. Mech. **36** (3), 409–417 (1972).
20. L. I. Rubina and O. N. Ul'yanov, "On solving the potential equation," Proc. Steklov Inst. Math. **261** (Suppl. 1), S183–S200 (2008).
21. L. I. Rubina and O. N. Ul'yanov, "Solution of nonlinear partial differential equations by the geometric method," Trudy Inst. Mat. Mekh. UrO RAN **18** (2), 209–225 (2012).
22. P. Bradshaw, *Effects of Streamwise Curvature on Turbulent Flow* (London, 1973), Ser. AGARDograph **169**.
23. M. Pathak, A. Dewan, and A. K. Dass, "Effect of streamline curvature on flow field of a turbulent plane jet in cross-flow," Mech. Res. Comm. **34**, 241–248 (2007).

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