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One-Sided Weighted Integral Approximation of Characteristic Functions of Intervals by Polynomials on a Closed Interval

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Abstract—We consider the problem of one-sided weighted integral approximation on the interval $[-1,1]$ to the characteristic functions of intervals $(a,1] \subset (-1,1]$ and $(a,b) \subset (-1,1)$ by algebraic polynomials. In the case of half-intervals, the problem is solved completely. We construct an example to illustrate the difficulties arising in the case of an open interval.

Keywords: one-sided approximation, characteristic function, polynomials.

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1. INTRODUCTION

One-sided integral approximations to step functions by algebraic polynomials in a weighted integral metric on the interval $[-1,1]$ were studied in the 1880s by Markov [4, Paper 1] and Stieltjes [13] (see [6, Sect. 3.411; 5, Ch. 1, Sect. 1.2, Lemmas 9, 9′]). In particular, these authors considered a polynomial of even degree $2m-2$ whose graph lies over the graph of the characteristic function $\mathbf{1}_{[-1,h]}$ of the interval $[-1,h]$, $h \in (-1,1)$. In the case when h coincides with one of the nodes of an m-point Gauss quadrature formula, this polynomial is extremal in the problem of the best weighted integral approximation to the function $\mathbf{1}_{[-1,h]}$ from above. Problems of one-sided weighted integral approximation to the characteristic function of an interval by algebraic or trigonometric polynomials arise in various areas of mathematics and have a rich history. Let us outline several exact results closely related to the present paper. The problem of one-sided approximation to the periodic extension of the characteristic function of an interval (a,b) by trigonometric polynomials in the integral metric with Jacobi weight on the period was studied in [11]. An exact solution was found in [11, Theorem 3] for some values of a and b satisfying specific equations. In the case of the unit weight, the problem for an arbitrary interval located on the period was solved in [1]. In [8], the problem of one-sided integral approximation to the characteristic function of an arbitrary half-open interval $(h,1] \subset (-1,1]$ by algebraic polynomials on $[-1,1]$ with the unit weight was solved, and the whole class of extremal polynomials was described.

In the present paper, we study similar problems on the interval $[-1,1]$ with a quite general weight. More exactly, in Section 3, we consider the problem of one-sided weighted integral approximation to the characteristic function of a half-open interval $(a,1] \subset (-1,1]$ by algebraic polynomials.

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To obtain lower bounds, we follow the known scheme of using quadrature formulas of the highest degree of precision with positive coefficients and several fixed nodes. There is a large number of papers devoted to such formulas (see [9, 10, 12] and references therein). These formulas are rich in applications. The most convenient for our purposes formulas were obtained in [9,10]. To find upper bounds, we use polynomials constructed according to methods going back to Markov and Stieltjes (see the beginning of this section); later, these methods were developed in [1, Lemma 1; 8, Sect. 5]. We also consider the problem of one-sided weighted integral approximation to the characteristic function of an interval $(a, b) \subset (-1, 1)$ by polynomials. In Section 3, we construct an example illustrating an essential difference of this case from the case of a half-open interval $(a, 1]$.

2. NOTATION AND AUXILIARY STATEMENTS

Let $\mathscr{P}_m$ be the set of algebraic polynomials of degree at most m with real coefficients. We call the number m the *degree* of the space $\mathscr{P}_m$. A polynomial with unit leading coefficient is called a *monic polynomial.*

Consider a nondecreasing function $\mu\colon [-1, 1] \to \mathbb{R}$ with infinite number of growth points. We call the distribution $d\mu(x)$ the *weight.* Let L be the space of real-valued μ-integrable on $[-1, 1]$ functions $f\colon [-1, 1] \to \mathbb{R}$ equipped with the norm $\|f\| = \displaystyle\int_{-1}^{1} |f(x)|\, d\mu(x)$.

The inequality $f \le g$ for functions f and g will mean that $f(x) \le g(x)$ for all $x \in [-1, 1]$.

The values of the best approximations to a bounded function $f \in L$ from below and from above by the set $\mathscr{P}_m$ in the metric of the space L are defined by the formulas

$$E^-_m(f) = \inf_{p \le f,\ p \in \mathscr{P}_m} \|f - p\|, \qquad E^+_m(f) = \inf_{f \le p,\ p \in \mathscr{P}_m} \|f - p\|, \tag{2.1}$$

respectively. Polynomials at which the infima in (2.1) are attained are called polynomials of the best (weighted integral) approximation to the function f from below and from above, respectively.

The set $\mathscr{P}_{m-1}$ of algebraic polynomials of degree at most $m - 1$ is an m-dimensional linear subspace of L. The space $\mathscr{P}^*_{m-1}$ of all linear functionals on $\mathscr{P}_{m-1}$ also has dimension m. For a number $\xi \in \mathbb{R}$, we consider the Dirac linear functional δ_ξ acting on polynomials $p \in \mathscr{P}_{m-1}$ by the rule $\delta_\xi p = p(\xi)$. It is known that, if $x_1 < x_2 < \cdots < x_m$, then the family of functionals $\{\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_m}\}$ is a basis in $\mathscr{P}^*_{m-1}$. Therefore, every linear functional on $\mathscr{P}_{m-1}$ can be represented as a linear combination of these functionals. In particular, the linear functional I_μ acting on $\mathscr{P}_{m-1}$ by the rule $I_\mu p = \displaystyle\int_{-1}^{1} p(x)\, d\mu(x)$ can be written as a linear combination $I_\mu = \lambda_1\delta_{x_1} + \lambda_2\delta_{x_2} + \cdots + \lambda_m\delta_{x_m}$. The coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are defined by the family $x_1, x_2, \dots, x_m$ uniquely. In other words, the formula

$$\int_{-1}^{1} p(x)\, d\mu(x) = \sum_{k=1}^{m} \lambda_k p(x_k), \tag{2.2}$$

which is called an *m-point quadrature formula*, is valid on $\mathscr{P}_{m-1}$. The numbers $\lambda_1, \lambda_2, \dots, \lambda_m$ are called the *coefficients* of this quadrature formula, the points $x_1, x_2, \dots, x_m$ are its *nodes*, and the polynomial $w_m(x) = (x - x_1)(x - x_2)\cdots(x - x_m)$ is its *generating polynomial.*

To a node x_k, we assign the *Lagrange fundamental polynomial* $\ell_k(x) = \dfrac{w_m(x)}{(x - x_k)w'_m(x_k)}$, which is 1 at the node x_k and 0 at the remaining nodes. Substituting this polynomial into (2.2), we obtain

$$\lambda_k = \int_{-1}^{1} \ell_k(x)\, d\mu(x) = \frac{1}{w_m'(x_k)} \int_{-1}^{1} \frac{w_m(x)}{x - x_k}\, d\mu(x), \quad k \in \{1, 2, \ldots, m\}. \tag{2.3}$$

The quadrature formula (2.2), whose coefficients are calculated by formula (2.3), is called an *interpolation formula*. Thus, *the m-point quadrature formula* (2.2) *is an interpolation formula if and only if it holds on $\mathscr{P}_{m-1}$* (see [3, Ch. 6, Sect. 1, Theorem 1]).

Only such quadrature formulas will be considered below. We use the same indices for the coefficients of these quadrature formulas and for the corresponding nodes. To construct an interpolation quadrature formula, it is sufficient to know its nodes (or, equivalently, its generating polynomial), because, by (2.3), the coefficients are recovered from the nodes uniquely. That is why we will sometimes talk only about the nodes of such quadrature formula or about its generating polynomial.

It is known that, choosing the nodes in a special way, one can gain an increase in the degree of the space of polynomials where the quadrature formula (2.2) remains valid (and the coefficients of this formula are calculated by (2.3)). The maximum degree of the space of polynomials where a quadrature formula of form (2.2) is valid is called the *degree of precision* of this quadrature formula.

It is also known that there exist m-point interpolation quadrature formulas of form (2.2) with positive coefficients $\lambda_k > 0$, $k = 1, 2 \ldots, m$. Such quadrature formulas are called *positive.* An important example of a positive m-point quadrature formula is a Gauss quadrature formula, whose generating polynomial coincides with the polynomial $p_m(x) = p_m(x, d\mu)$ of degree m orthogonal to $\mathscr{P}_{m-1}$ with weight $d\mu$, i.e., orthogonal to $\mathscr{P}_{m-1}$ with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x)$. The Gauss quadrature formula

$$\int_{-1}^{1} p(x) d\mu(x) = \sum_{j=1}^{m} \lambda_j^* p(x_j^*), \quad -1 < x_1^* < \cdots < x_m^* < 1, \quad p \in \mathscr{P}_{2m-1}, \tag{2.4}$$

has the highest degree of precision $2m - 1$, and all its nodes lie in the open interval $(-1, 1)$. The nodes and coefficients of a Gauss quadrature formula depend on m; to emphasize this dependence, we will sometimes use the notation $\lambda_{m,j}^*$ and $x_{m,j}^*$ instead of λ_j^* and x_j^*.

In what follows, we will need m-point quadrature formulas with several fixed nodes and maximum degree of precision. Such formulas for algebraic polynomials have been investigated for a long time starting with studies of R. Lobatto (1852), E.B. Christoffel (1858), and R. Radau (1880) related to cases of fixed nodes located either at the end points of an interval of integration or outside this interval.

Denote by $\mathfrak{u}$ the subset of nodes of a quadrature formula consisting of fixed nodes. For our purposes, it is sufficient to consider m-point quadrature formulas with maximum degree of precision and one, two, or three fixed nodes. More exactly, $\mathfrak{u}$ coincides with one of the sets $\{-1\}$, $\{1\}$, $\{-1, 1\}$, $\{\theta\}$, $\{-1, \theta\}$, $\{\theta, 1\}$, or $\{-1, \theta, 1\}$, where $\theta \in (-1, 1)$.

Let $Q_m^{\mathfrak{u}}$ denote an m-point quadrature formula with a fixed set of nodes $\mathfrak{u}$ and maximum degree of precision $\deg Q_m^{\mathfrak{u}} = 2m - 1 - |\mathfrak{u}|$:

$$\int_{-1}^{1} p(x) d\mu(x) = \sum_{j=1}^{m} \lambda_j^{\mathfrak{u}} p\big(x_j^{\mathfrak{u}}\big), \quad x_1^{\mathfrak{u}} < x_2^{\mathfrak{u}} < \cdots < x_m^{\mathfrak{u}}, \quad p \in \mathscr{P}_{2m-1-|\mathfrak{u}|}; \tag{2.5}$$

here, $|\mathfrak{u}|$ means the number of points contained in $\mathfrak{u}$. The nodes and coefficients of this formula depend on m; to emphasize this dependence, we will sometimes use the notation $\lambda^{\mathfrak{u}}_{m,j}$ and $x^{\mathfrak{u}}_{m,j}$ instead of $\lambda^{\mathfrak{u}}_j$ and $x^{\mathfrak{u}}_j$. Since $m \ge |\mathfrak{u}|$, we have $\deg Q^{\mathfrak{u}}_m = 2m-1-|\mathfrak{u}| \ge m-1$. Hence, (2.5) is an interpolation formula. In the cases $\mathfrak{u}=\{-1\}$ and $\mathfrak{u}=\{1\}$, formula (2.5) is a left and right Radau quadrature formula, respectively; if $\mathfrak{u}=\{-1,1\}$, (2.5) is a Lobatto quadrature formula. Such formulas are also called Markov quadrature formulas; it is known that they are positive. Define

$$\omega^{\{-1\}}(x)=1+x,\quad \omega^{\{1\}}(x)=1-x,\quad \omega^{\{-1,1\}}(x)=(1+x)(1-x). \tag{2.6}$$

Consider the polynomials

$$w^{\{-1\}}_m(x)=\omega^{\{-1\}}(x)p_{m-1}\big(x,\omega^{\{-1\}}d\mu\big),\quad w^{\{1\}}_m(x)=-\omega^{\{1\}}(x)p_{m-1}\big(x,\omega^{\{1\}}d\mu\big), \tag{2.7}$$

$$w^{\{-1,1\}}_m(x)=-\omega^{\{-1,1\}}(x)p_{m-2}\big(x,\omega^{\{-1,1\}}d\mu\big), \tag{2.8}$$

where $p_k(x,d\sigma)$ is the monic orthogonal polynomial of degree k with weight $d\sigma$.

Polynomials (2.7) are generating polynomials of left and right Radau quadrature formulas, respectively, and polynomial (2.8) is the generating polynomial of a Lobatto quadrature formula. The roots of these polynomials and of the generating polynomial of the Gauss quadrature formula (2.4) satisfy the inequalities (see [10, (3)–(6)])

$$x^{\{-1\}}_{m,j}<x^{*}_{m,j}<x^{\{1\}}_{m,j},\quad j=1,2,\dots,m;\quad x^{*}_{m-1,j}<x^{\{-1\}}_{m,j+1}<x^{\{-1,1\}}_{m,j+1},\quad j=1,2,\dots,m-1;$$

$$x^{\{-1,1\}}_{m,j}<x^{\{1\}}_{m,j}<x^{*}_{m-1,j},\quad j=1,2,\dots,m-1;\quad x^{\{1\}}_{m-1,j}<x^{\{-1,1\}}_{m,j+1}<x^{\{-1\}}_{m-1,j+1},\quad j=1,2,\dots,m-2.$$

In what follows, we will need the main result of [10]. To formulate this result, introduce the sets

$$\mathbf{G}^{*}_m=\bigcup_{j=1}^{m}\big(x^{\{-1\}}_{m,j},x^{\{1\}}_{m,j}\big)\setminus\big\{x^{*}_{m,j}\big\},\quad m\ge 1;$$

$$\mathbf{G}^{\{-1\}}_m=\bigcup_{j=1}^{m-1}\big(x^{*}_{m-1,j},x^{\{-1,1\}}_{m,j+1}\big)\setminus\big\{x^{\{-1\}}_{m,j+1}\big\},\quad \mathbf{G}^{\{1\}}_m=\bigcup_{j=1}^{m-1}\big(x^{\{-1,1\}}_{m,j},x^{*}_{m-1,j}\big)\setminus\big\{x^{\{1\}}_{m,j}\big\},\quad m\ge 2;$$

$$\mathbf{G}^{\{-1,1\}}_m=\bigcup_{j=1}^{m-2}\big(x^{\{1\}}_{m-1,j},x^{\{-1\}}_{m-1,j+1}\big)\setminus\big\{x^{\{-1,1\}}_{m,j+1}\big\},\quad m\ge 3.$$

Note (see [10, Sect. 1]) that

$$\overline{\mathbf{G}^{\{1\}}_m\bigcup\mathbf{G}^{\{-1\}}_{m-1}}=\overline{\mathbf{G}^{*}_{m-1}\bigcup\mathbf{G}^{\{-1,1\}}_m}=[-1,1]. \tag{2.9}$$

To a set $G_m\in\{\mathbf{G}^{\{-1\}}_m,\mathbf{G}^{\{1\}}_m,\mathbf{G}^{\{-1,1\}}_m\}$, we assign the polynomial $\omega(x)=\omega(x,G_m)$ coinciding with the corresponding polynomial from (2.6), and to the set $\mathbf{G}^{*}_m$, we assign the polynomial $\omega(x)=\omega(x,\mathbf{G}^{*}_m)\equiv 1$. For a pair (G_m,θ), where $G_m\in\{\mathbf{G}^{*}_m,\mathbf{G}^{\{-1\}}_m,\mathbf{G}^{\{1\}}_m,\mathbf{G}^{\{-1,1\}}_m\}$, $\theta\in G_m$, we define

$$W^{\{\theta\}}_m(x,G_m)=\omega(x)\Big(p_\nu(x,\omega\,d\mu)-\frac{p_\nu(\theta,\omega\,d\mu)}{p_{\nu-1}(\theta,\omega\,d\mu)}p_{\nu-1}(x,\omega\,d\mu)\Big), \tag{2.10}$$

where $\omega(x)=\omega(x,G_m)$ and $\nu=m-\deg\omega$.

Following [10, Sect. 1], we call an m-point quadrature formula a *quasi* Gauss quadrature formula if it has one fixed node $\theta\in(-1,1)\setminus\{x^{*}_{m,j}\}_{j=1}^{m}$ and the highest degree of precision (equal to $2m-2$).

Similarly, we define m-point *quasi* left and right Radau quadrature formulas of the highest degree of precision (equal to $2m-3$) and an m-point *quasi* Lobatto quadrature formula of the highest degree of precision (equal to $2m-4$); i.e., in addition to the fixed nodes of the corresponding formulas, we consider one more fixed node $\theta \in (-1,1)$ different from the nodes of the corresponding classical m-point formula. Let us formulate the main result of [10] using the introduced terms.

Theorem A [10, Theorem 1.1]. *There exist a positive quasi Gauss quadrature formula, quasi left and right Radau formulas, and a quasi Lobatto formula with fixed node $\theta \in (-1,1)$ if and only if θ belongs to the set $\mathbf{G}_m^*$; $m \ge 1$, $\mathbf{G}_m^{\{-1\}}$; $m \ge 2$, $\mathbf{G}_m^{\{1\}}$; $m \ge 2$, and $\mathbf{G}_m^{\{-1,1\}}$; $m \ge 3$, respectively. The generating polynomials for these quasi quadrature formulas are given by equality* (2.10)*, in which G_m coincides with $\mathbf{G}_m^*$, $\mathbf{G}_m^{\{-1\}}$, $\mathbf{G}_m^{\{1\}}$, and $\mathbf{G}_m^{\{-1,1\}}$, respectively. Consequently, these quasi quadrature formulas are unique.*

Positive quadrature formulas are used (see Theorem B) for the lower estimation of values (2.1) of the best one-sided (weighted integral) approximation to a bounded function $f \in L$ by polynomials. In the present paper, as an approximated function, we consider the characteristic function

$$\mathbf{1}_J(x) = \begin{cases} 1, & x \in J, \\ 0, & x \in [-1,1] \setminus J, \end{cases}$$

of a subset $J \subset [-1,1]$. In the case $J = (a,1]$, for the upper estimation of the values of the best one-sided approximation to the function $\mathbf{1}_{(a,1]}$, we use Hermite interpolation polynomials that interpolate the function $\mathbf{1}_{(a,1]}$ at nodes of positive quadrature formulas.

Consider conditions under which polynomials interpolate the function $\mathbf{1}_{(a,1]}$ on $[-1,1]$. Let

$$s, r \in \{0,1\}, \quad \ell, k \in \mathbb{Z}_+ = \{0,1,2,\dots\}, \quad s + \ell \ge 1.$$

To an ordered quadruple (s,ℓ,k,r), we assign a type $T(s,\ell,k,r)$ of conditions under which a polynomial p of minimum degree interpolates the function $\mathbf{1}_{(a,1]}$. This type characterizes the arrangement and multiplicity of interpolation nodes.

Let us describe the values of the parameters s, ℓ, k, and r. If $s = 0$, then -1 is not an interpolation node; if $s = 1$, then -1 is an interpolation node, i.e., $p(-1) = 0$. The number ℓ is the number of interpolation nodes $x_1 < x_2 < \dots < x_\ell$ located in the interval $(-1,a)$, and each of them has double multiplicity: $p(x_j) = 0$, $p'(x_j) = 0$, $j = 1,2,\dots,\ell$; the point $x_{\ell+1} = a$ always is a (simple) interpolation node, i.e., $p(a) = 0$; if $\ell = 0$, then there is no interpolation node in $(-1,a)$. The number k is the number of interpolation nodes $x_{\ell+2} < x_{\ell+3} < \dots < x_{\ell+k+1}$ located in the interval $(a,1)$, and each of them has double multiplicity: $p(x_j) = 1$, $p'(x_j) = 0$, $j = \ell+2, \ell+3, \dots, \ell+k+1$; if $k = 0$, then there are no interpolation nodes in $(a,1)$. If $r = 0$, then 1 is not an interpolation node; if $r = 1$, then 1 is an interpolation node: $p(1) = 1$.

A polynomial p (of minimum degree) implementing a type $T(s,\ell,k,r)$ of the Hermite interpolation conditions for the function $\mathbf{1}_{(a,1]}$ will be called a *polynomial of type* $T(s,\ell,k,r)$ for brevity. The degree of this polynomial is the number of interpolation conditions minus one; i.e., $\deg p = s + r + 2(\ell + k)$. It is known (see [2, Ch. 2, Sect. 11]) that, under more general conditions, there exists a unique Hermite interpolation polynomial (of minimum degree).

To find the value of the best one-sided approximation to the function $\mathbf{1}_{(a,1]}$ by polynomials of given degree and to construct the corresponding extremal polynomial, we will need the following lemma, which, for a polynomial p of type $T(0,\ell,k,0)$, is the result of Markov and Stieltjes mentioned at the beginning of the Introduction.

Lemma. *Assume that $s, r \in \{0,1\}$, $\ell, k \in \mathbb{Z}_+$, $s+\ell \geq 1$, $a \in (-1,1)$, and p is an interpolation polynomial (of minimum degree) implementing the type $T(s,\ell,k,r)$ of interpolation conditions for the function $\mathbf{1}_{(a,1]}$. Then, $p(x) \leq \mathbf{1}_{(a,1]}(x)$ for all $x \in [-1,1]$.*

Proof. The proof follows from [1, Lemma 1] after the cosine change. □

Theorem B (Bojanic, DeVore [7]). *Assume that the positive quadrature formula*

$$\int_{-1}^{1} p(x)\,d\mu(x) = \sum_{k=1}^{m} \lambda_k p(x_k), \quad -1 \leq x_1 < x_2 < \cdots < x_m \leq 1, \tag{2.11}$$

holds on $\mathscr{P}_n$. Then, for every bounded function $f \in L$, we have the inequalities

$$E_n^-(f) \geq \int_{-1}^{1} f(x)\,d\mu(x) - \sum_{k=1}^{m} \lambda_k f(x_k); \qquad E_n^+(f) \geq \sum_{k=1}^{m} \lambda_k f(x_k) - \int_{-1}^{1} f(x)\,d\mu(x). \tag{2.12}$$

Substituting the characteristic function of a subset $J \subset [-1,1]$ into (2.12), we arrive at the following statement.

Proposition 1. *For $f = \mathbf{1}_J$, where $J \subset [-1,1]$, we have $E_n^-(\mathbf{1}_J) \geq \int_J d\mu(x) - \sum \lambda_k$ and $E_n^+(\mathbf{1}_J) \geq \sum \lambda_k - \int_J d\mu(x)$, where the sums are taken over indices k such that $x_k \in J$. If, in addition, J does not contain any nodes of quadrature formula (2.11), then $E_n^-(\mathbf{1}_J) = \int_J d\mu(x)$ and $p \equiv 0$ is a polynomial of the best approximation from below. In particular, this statement holds in the case when J is an open interval $(a,b) \subset [-1,1]$ under the condition $x_k \leq a < b \leq x_{k+1}$, where x_k and x_{k+1} are arbitrary neighboring nodes of quadrature formula (2.11). If the right end point of the interval $[-1,1]$ is not a node of quadrature formula (2.11), then we can take a half-open interval $(a,1]$, where a is not less than the maximum node of quadrature formula (2.11), as the set J.*

It is not hard to understand that, if a polynomial $p^- \in \mathscr{P}_n$ implements the infimum in the problem on calculating the value $E_n^-(\mathbf{1}_J)$, then the polynomial $p^+ = 1 - p^-$ implements the infimum in the problem on calculating $E_n^+(\mathbf{1}_{[-1,1]\setminus J})$. Moreover, $E_n^+(\mathbf{1}_{[-1,1]\setminus J}) = E_m^-(\mathbf{1}_J)$. In view of this fact, in what follows, we will consider only the problem of the one-sided approximation to the characteristic function of a subset J of the interval $[-1,1]$ from below.

3. APPROXIMATION TO THE CHARACTERISTIC FUNCTION OF A HALF-OPEN INTERVAL FROM BELOW. AN EXAMPLE IN THE CASE OF AN OPEN INTERVAL

Let us pass to a more detailed study of the problem of approximation of the characteristic function $\mathbf{1}_{(a,1]}$ of a half-open interval $(a,1] \subset [-1,1]$ from below.

Proposition 2. *For any $m \in \mathbb{N}$, the following relations hold:*

$$E_n^-(\mathbf{1}_{(a,1]}) = \int_{(a,1]} d\mu(x) \quad \text{for} \quad \begin{cases} n = 2m-1, & x_{m,m}^* \leq a < 1; \\ n = 2m-2, & x_{m,m}^{\{-1\}} \leq a < 1, \end{cases}$$

and $p \equiv 0$ is a polynomial of the best approximation to the function $\mathbf{1}_{(a,1]}$ from below.

Proof. To prove this statement, it is sufficient to use Proposition 1, taking into account the positivity of the m-point Gauss quadrature formulas (2.4) and of a left Radau formula (see (2.5)).

Theorem 1. *Assume that $m \in \mathbb{N}$, $m \geq 2$, and the number a coincides with some of the nodes of the m-point Gauss quadrature formula* (2.4) *different from the maximum node, i.e., $a = x^*_{v,m}$, where $v \in \{1, 2, \dots, m-1\}$. Then, $E^-_{2m-1}(\mathbf{1}_{(a,1]}) = \int_{(a,1]} d\mu(x) - \sum_{j=v+1}^{m} \lambda^*_{m,j}$, and a polynomial of the best approximation from below is a polynomial p of type $T(0,\ell,k,0)$ and degree $2m-2$ that interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes of Gauss quadrature formula* (2.4). *In addition, $E^-_{2m-2}(\mathbf{1}_{(a,1]}) = E^-_{2m-1}(\mathbf{1}_{(a,1]})$.*

Proof. It follows from the Markov–Stieltjes result (see [5, Ch. 1, Sect. 1.2, Lemma 9′]) that $p \leq \mathbf{1}_{(a,1]}$. This polynomial provides an upper bound for the required value $E^-_{2m-1}(\mathbf{1}_{(a,1]})$. The same lower bound follows from Proposition 1. The last assertion of the theorem is valid since the degree of extremal polynomial is $2m-2$. □

Theorem 2. *Assume that $m \in \mathbb{N}$, $m \geq 3$, and the number a coincides with some node of an m-point left Radau quadrature formula* (*see* (2.5)) *different from the minimum and maximum nodes, i.e., $a = x^{\{-1\}}_{v,m}$, where $v \in \{2, 3, \dots, m-1\}$. Then, $E^-_{2m-2}(\mathbf{1}_{(a,1]}) = \int_{(a,1]} d\mu(x) - \sum_{j=v+1}^{m} \lambda^{\{-1\}}_{m,j}$ and a polynomial of the best approximation from below is a polynomial p of type $T(1,\ell,k,0)$ and degree $2(\ell+k)+1 = 2m-3$ that interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes of this Radau quadrature formula. In addition, $E^-_{2m-3}(\mathbf{1}_{(a,1]}) = E^-_{2m-2}(\mathbf{1}_{(a,1]})$.*

Proof. By the lemma, the polynomial p from the statement of the theorem is admissible; more exactly, $p \leq \mathbf{1}_{(a,1]}$ and $p \in \mathscr{P}_{2m-3} \subset \mathscr{P}_{2m-2}$. Therefore, it provides the required upper bound for $E^-_{2m-2}(\mathbf{1}_{(a,1]})$. The same lower bound follows from Proposition 1, since an m-point left Radau quadrature formula is positive and holds on $\mathscr{P}_{2m-2}$. The last assertion of the theorem is valid since the degree of the extremal polynomial is $2m-3$. □

Applying similar arguments based on Proposition 1, the lemma, and the positivity of a right Radau quadrature formula and a Lobatto quadrature formula, we find the values of the best approximation to the function $\mathbf{1}_{(a,1]}$ from below in the case when the number a coincides with one of the nodes of these formulas different from the end points of the interval $[-1,1]$. Polynomials of types $T(0,\ell,k,1)$ and $T(1,\ell,k,1)$ interpolating the function $\mathbf{1}_{(a,1]}$ at the nodes of the corresponding formulas are extremal and their degrees are one less than the degrees of precision of the corresponding quadrature formulas.

To solve the problem of the best approximation to the function $\mathbf{1}_{(a,1]}$ from below by polynomials in the remaining cases of position of the point a, we must argue similarly based on the positivity of the quasi-quadrature formulas from Theorem A and taking into account (2.9). The degree of the extremal polynomials coincides with the degree of precision of the corresponding quasi-formula. For example, in the case when the point θ belongs to the set $\mathbf{G}^*_m$, the following theorem is valid, which is similar to Theorem 1. Here, we denote by $x^{*,\theta}_{j,m}$, $j = 1, 2, \dots, m$, the roots of the generating polynomial $W^{\{\theta\}}_m(x, \mathbf{G}^*_m)$ (see (2.10)) of an m-point Gauss quadrature formula; $\lambda^{*,\theta}_{j,m}$, $j = 1, 2, \dots, m$, are the coefficients of this quadrature formula.

Theorem 3. *Assume that $m \in \mathbb{N}$, $m \geq 2$, $\theta \in \mathbf{G}^*_m$, the number a coincides with some root of the generating polynomial $W^{\{\theta\}}_m(x, \mathbf{G}^*_m)$ of an m-point quasi Gauss quadrature formula, and the number a is not its maximum root, i.e., $a = x^{*,\theta}_{v,m}$, where $v \in \{1, 2, \dots, m-1\}$. Then,*

$E^{-}_{2m-2}(\mathbf{1}_{(a,1]}) = \int_{(a,1]} d\mu(x) - \sum_{j=v+1}^{m} \lambda^{*,\theta}_{m,j}$. *A polynomial of the best approximation from below is a polynomial p of type $T(0,\ell,k,0)$ and degree $2m-2$ that interpolates the function $\mathbf{1}_{(a,1]}$ at the nodes of this quasi-formula.*

In conclusion, we give an example showing an essential difference between the problems of approximation from below to the characteristic functions of an open interval and a half-open interval.

Example. Consider a three-point Gauss quadrature formula on $[-1,1]$ with the weight $d\mu(x) = dx$. The degree of precision of this formula is five and the points $x_1 = -\sqrt{3/5}$, $x_2 = 0$, and $x_3 = \sqrt{3/5}$ are its nodes. Define $b = 2/5$, $A = -5(5b+\sqrt{15})/(9b^2)$, and $c = -\sqrt{15}\,b/(5b+\sqrt{15})$. A simple verification shows that the fifth-degree polynomial $p(x) = A(x-x_1)(x-b)(x+x_1)^2(x-c)$ interpolates the function $\mathbf{1}_{(x_1,b)}$ as follows: $p(x_1) = p(b) = p(x_3) = 0$, $p(x_2) = 1$, and $p'(x_2) = p'(x_3) = 0$. It is easy to see that, in fact, this polynomial is positive on $[-1, x_1)$; consequently, its graph does not lie below the graph of the characteristic function $\mathbf{1}_{(x_1,b)}$ on this half-open interval.

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