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A Triangular Finite Element with New Approximation Properties

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Abstract—A finite element with new properties of approximation of higher derivatives is constructed, and a method for the construction of a finite element space in the planar case is proposed. The method is based on Yu.N. Subbotin's earlier results as well as on the results obtained in this paper. The constructed piecewise polynomial function possesses the continuity property and new approximation properties.

Keywords: multidimensional interpolation, finite element method, maximum angle condition, splines on triangulations.

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let a function f be defined on Ω and belong to the class $W^{n+1}M$; i.e., f is continuous on Ω together with all its partial derivatives up to order $n+1$, and all its derivatives of order $n+1$ are bounded in absolute value by the constant M. Let Ω be triangulated, and let Δ be an arbitrary triangle from the triangulation. We consider the problem of interpolation of f on Δ by a polynomial of degree at most n for $n \geq 3$. Denote by a_i $(i = 1, 2, 3)$ the vertices of Δ; let τ_{ij} $(i, j = 1, 2, 3,\ i \neq j)$ be the unit vector directed from a_i to a_j; let n_{ij} $(i, j = 1, 2, 3,\ i \neq j)$ be the unit normal vector to the side $[a_i, a_j]$; let α, β, and θ be the angles at the vertices a_1, a_2, and a_3, respectively; and let H be the diameter of Δ. Let $\alpha \leq \beta \leq \theta$. Then, $H = \|a_2 - a_1\|$. Denote by $D^s_{\xi_1 \ldots \xi_s}$ the derivative of order s along arbitrary unit vectors $\xi_1, \ldots, \xi_s$.

For values φ_1 and φ_2, each of which, in general, may depend on the function f, the geometric characteristics H, α, β, and θ of the triangle Δ introduced above, and the point $u \in \Delta$, we will write $\varphi_1 \overset{(\gtrsim)}{\lesssim} \varphi_2$ if $\varphi_1 \overset{(\geq)}{\leq} C(n)\varphi_2$ for some number $C(n) > 0$, which can depend only on the degree n of the interpolation polynomial.

Let $\mathcal{P}_n$ be the set of polynomials of total degree at most n (i.e., the sum of degrees of the monomials does not exceed n) such that all coefficients of any polynomial $P \in \mathcal{P}_n$ are uniquely defined by the fact that it interpolates the values of the function f and, possibly, values of some of its derivatives at specified points of the triangle Δ. It is known that, for a sufficiently wide set $\mathcal{P}_n$ and any unit vectors $\xi_1, \ldots, \xi_s$, the following estimates hold (Ciarlet and Raviart [1]):

$$\|D^s_{\xi_1 \ldots \xi_s}(f - P)\|_{C(\Delta)} \lesssim MH^{n+1-s}\sin^{-s}\alpha, \quad 0 \leq s \leq n, \tag{0.1}$$

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where $P \in \mathcal{P}_n$ and $\|\cdot\|_{C(\Delta)}$ is the uniform norm on Δ.

Let us introduce the set of multi-indices $I = \{(i,j,k)|\ i,j,k \in \mathbb{Z}_+;\ i+j+k=n\}$ and consider the following set of points of the triangle Δ:

$$Q = \{u_{ijk}\}_{(i,j,k)\in I} = \{u_{ijk}|\ u_{ijk} = (i/n)a_1 + (j/n)a_2 + (k/n)a_3,\ (i,j,k) \in I\}.$$

Consider a polynomial $\widetilde{P} \in \mathcal{P}_n$ such that

$$\forall u \in Q \quad \widetilde{P}(u) = f(u). \tag{0.2}$$

Subbotin proved in [2, 3] that, for any unit vectors $\xi_1,\ldots,\xi_s$, the following estimates hold:

$$\|D^s_{\xi_1\ldots\xi_s}(f-\widetilde{P})\|_{C(\Delta)} \lesssim MH^{n+1-s}\sin^{-s}\theta, \quad 0 \le s \le n. \tag{0.3}$$

Note that, in [1, 3], estimates of type (0.1) and (0.3) were obtained not only in the planar case but also in multidimensional cases (in particular, when Δ is an m-simplex); however, we turn our attention to the case of a triangle. An advantage of (0.3) over (0.1) is that the use of estimates (0.1) requires imposing the "minimum angle condition" on the triangulation, which is a lower bound on the values of the minimum angles of the triangles, whereas estimates (0.3) enable us to consider a weaker condition of the separability of the maximum angles of the triangles from π. In the present paper, we propose to keep the conditions from (0.2) imposed at the points belonging to the sides of Δ and replace the remaining conditions by the interpolation of higher derivatives at the point a_2.

Consider the set $I_0 = \{(i,j,k) \in I|\ ijk = 0\}$ and the corresponding subset $Q_0 \subset Q$ (of points from Q belonging to the sides of the triangle):

$$Q_0 = \{u_{ijk}\}_{(i,j,k)\in I_0} = \{u_{ijk}|\ u_{ijk} = (i/n)a_1 + (j/n)a_2 + (k/n)a_3,\ (i,j,k) \in I_0\}.$$

Let the polynomial P be defined by the following conditions:

$$P(u) = f(u), \quad u \in Q_0; \tag{0.4}$$

$$\frac{\partial^{i+j}P(a_2)}{\partial\tau_{12}^i\partial n_{12}^j} = \frac{\partial^{i+j}f(a_2)}{\partial\tau_{12}^i\partial n_{12}^j}, \quad j=\overline{3,n}, \quad i=\overline{0,n-j}. \tag{0.5}$$

Denote by $\mathcal{P}_n[Q_0]$ the set of all polynomials of degree at most n satisfying condition (0.4). It is clear that $P, \widetilde{P} \in \mathcal{P}_n[Q_0] \subset \mathcal{P}_n$.

Let us introduce a rectangular coordinate system Oxy such that the coordinates of the vertices of Δ have the form $a_1 = (a+b,0)$, $a_2 = (0,0)$, and $a_3 = (a,h)$ for some positive a, b, and h (see Fig. 1). Since $\alpha \le \beta \le \theta$, we have $a \le b$ and $H = a+b$ (it is also obvious that a, b, and h are some functions of H, α, β, and θ, which must be taken into account when one uses the relations "$\lesssim$" and "$\gtrsim$"). In this case, conditions (0.5) take the form

$$\frac{\partial^{i+j}P(a_2)}{\partial x^i\partial y^j} = \frac{\partial^{i+j}f(a_2)}{\partial x^i\partial y^j}, \quad j=\overline{3,n}, \quad i=\overline{0,n-j}.$$

Our aim is to prove the four theorems formulated below. In addition, in the last section, we describe a method for constructing a finite element space with new approximation properties. This space is constructed on the basis of Yu.N. Subbotin's results as well as on the results obtained in the present paper.

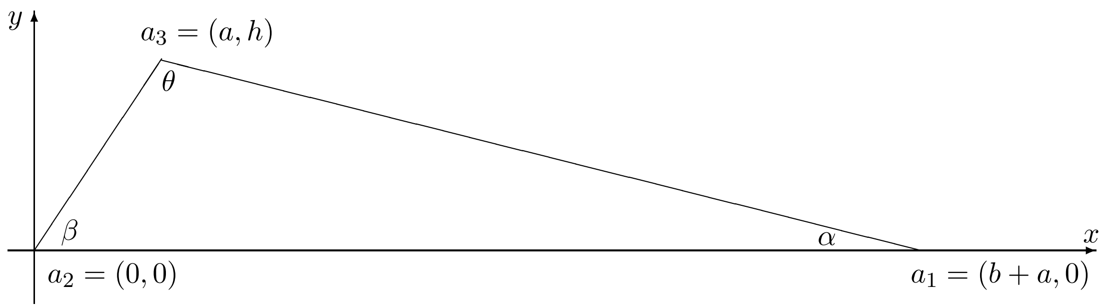

Fig. 1. The position of the triangle Δ in the coordinate system Oxy.

Theorem 1. *Let, for $f \in W^{n+1}M$, the polynomial $P \in \mathcal{P}_n$ be defined by interpolation conditions* (0.4) *and* (0.5). *Then, for any $\beta_0 < \pi/2$, any triangle Δ satisfying the condition $\beta \le \beta_0$, and any nonnegative integer s such that $0 \le s \le n$, the following estimate is valid:*

$$\Big\|\frac{\partial^{s-j}(f-P)}{\partial x^{s-j}\partial y^j}\Big\|_{C(\Delta)} \le \begin{cases} C(n,\beta_0)MH^{n+1-s}\sin^{-j}\theta, & j=\overline{0,\min\{2,s\}},\\ C(n,\beta_0)MH^{n+1-s}, & j=3,\ldots,s,\end{cases} \tag{0.6}$$

where $C(n,\beta_0)$ is a nonnegative value depending only on n and β_0.

Theorem 2. *Let, for $f \in W^{n+1}M$, the polynomial $P \in \mathcal{P}_n$ be defined by interpolation conditions* (0.4) *and* (0.5). *Then, for any $\beta_0 < \pi/2$, any triangle Δ satisfying the condition $\beta \le \beta_0$, any $s = 0,\ldots,n$, and arbitrary unit vectors $\xi_1,\ldots,\xi_s$, the following estimate is valid:*

$$\big\|D^s_{\xi_1\ldots\xi_s}(f-P)\big\|_{C(\Delta)} \le \begin{cases} C(n,\beta_0)MH^{n+1-s}\sin^{-s}\theta, & s=0,1,2,\\ C(n,\beta_0)MH^{n+1-s}\sin^{-2}\theta, & s=3,\ldots,n,\end{cases} \tag{0.7}$$

where $C(n,\beta_0)$ is a nonnegative value depending only on n and β_0.

Instead of conditions (0.5), we can use similar conditions for the maximum angle:

$$\frac{\partial^{i+j}P(a_3)}{\partial\tau_{31}^i\partial n_{31}^j} = \frac{\partial^{i+j}f(a_2)}{\partial\tau_{31}^i\partial n_{31}^j}, \qquad j=\overline{3,n},\quad i=\overline{0,n-j}. \tag{0.8}$$

In this case, we have the following theorems.

Theorem 3. *Let, for $f \in W^{n+1}M$, the polynomial $P \in \mathcal{P}_n$ be defined by interpolation conditions* (0.4) *and* (0.8). *Then, for any θ_0 such that $|\cos\theta_0| > 0$, any triangle Δ satisfying the condition $|\cos\theta| > |\cos\theta_0|$, and any nonnegative integer s such that $0 \le s \le n$, we have*

$$\Big\|\frac{\partial^{s-j}(f-P)}{\partial x^{s-j}\partial y^j}\Big\|_{C(\Delta)} \le \begin{cases} C(n,\theta_0)MH^{n+1-s}\sin^{-j}\theta, & j=\overline{0,\min\{2,s\}},\\ C(n,\theta_0)MH^{n+1-s}, & j=3,\ldots,s,\end{cases} \tag{0.9}$$

where $C(n,\theta_0)$ is a nonnegative value depending only on n and θ_0.

Theorem 4. *Let, for $f \in W^{n+1}M$, the polynomial $P \in \mathcal{P}_n$ be defined by interpolation conditions* (0.4) *and* (0.8). *Then, for any θ_0 such that $|\cos\theta_0| > 0$, any triangle Δ satisfying the condition $|\cos\theta| > |\cos\theta_0|$, any $s = 0,\ldots,n$, and arbitrary unit vectors $\xi_1,\ldots,\xi_s$, we have*

$$\big\|D^s_{\xi_1\ldots\xi_s}(f-P)\big\|_{C(\Delta)} \le \begin{cases} C(n,\theta_0)MH^{n+1-s}\sin^{-s}\theta, & s=0,1,2,\\ C(n,\theta_0)MH^{n+1-s}\sin^{-2}\theta, & s=3,\ldots,n,\end{cases} \tag{0.10}$$

where $C(n,\theta_0)$ is a nonnegative value depending only on n and θ_0.

Remark 1. Theorems 2 and 4 are trivial corollaries of Theorems 1 and 3, respectively, and are given here for the only reason that form (0.7) and (0.10) of writing estimates is more traditional in comparison with (0.6) and (0.9).

Remark 2. Since $\sin\theta \lesssim \sin\beta \lesssim \sin\theta$, we can write $\sin\beta$ instead of $\sin\theta$ in estimates (0.3), (0.6), (0.7), (0.9), and (0.10), which has no affect on the accuracy of the estimates.

Remark 3. The question about the optimality of the found interpolation conditions and estimates on the class $W^{n+1}M$ remains open. In [4], lower estimates for a function from $W^{n+1}M$ were obtained for a wider class of finite elements providing a specified smoothness or continuity of a resulting spline on the triangulation. In the case of continuity, the denominators of the lower estimates contain the sine of the maximum (or middle) angle to the first power. The denominators of the upper estimates obtained in the present paper for the approximation of higher order derivatives can contain the squared sine of the maximum (or middle) angle, whereas the interpolation condition under consideration provide the continuity of the spline on Ω.

Reviews related to the "maximum angle condition" in the finite elements method can be found, for example, in [4, 5].

1. ON SUBBOTIN'S ESTIMATES IN MORE DETAIL

Along with estimates (0.3), the following fact for the polynomial $\widetilde{P}$ was proved in [3, Lemma 4]. Under one of the conditions

$$\xi_k \in \{\tau_{21}, \tau_{23}\} \quad \text{for all} \quad k = \overline{1, s}; \tag{1.1}$$

$$\xi_k \in \{\tau_{31}, \tau_{32}\} \quad \text{for all} \quad k = \overline{1, s}, \tag{1.2}$$

the following relation holds for any $s = 0, \dots, n$:

$$\big\|D^s_{\xi_1\dots\xi_s}(f - \widetilde{P})\big\|_{C(\Delta)} \lesssim MH^{n+1-s}. \tag{1.3}$$

The method for choosing either condition (1.1) or (1.2) is given in [3]; however, it does not matter for us. Let us show that this, in particular, means that, for $i = 0, \dots, s$ and $s = 0, \dots, n$,

$$\Big\|\frac{\partial^s(f - \widetilde{P})}{\partial x^{s-i}\partial y^i}\Big\|_{C(\Delta)} \lesssim MH^{n+1-s}\sin^{-i}\beta. \tag{1.4}$$

Consider first the situation when (1.3) holds under condition (1.1). Let $s \in \{0, \dots, n\}$. For $i = 0$, estimate (1.4) coincides with (1.3) for $\xi_1 = \dots = \xi_s = \tau_{21}$. Now, let $1 \le j \le s$, and let (1.4) hold for $i = 0, \dots, j-1$. Then, we take $\xi_1 = \dots = \xi_{s-j+1} = \tau_{21}$ and $\xi_{s-j+1} = \dots = \xi_s = \tau_{23}$ in (1.3) and represent the sth derivative along τ_{21} and τ_{23} as the sum of partial derivatives

$$D^s_{\xi_1\dots\xi_s}(f - \widetilde{P}) = \sum_{k=0}^{j} C^k_j \frac{\partial^s(f - \widetilde{P})}{\partial x^{s-k}\partial y^k}\cos^{j-k}\beta\sin^k\beta,$$

which implies the estimate

$$\Big\|\frac{\partial^s(f - \widetilde{P})}{\partial x^{s-j}\partial y^j}\sin^j\beta\Big\|_{C(\Delta)} = \Big\|D^s_{\xi_1\dots\xi_s}(f - \widetilde{P}) - \sum_{k=0}^{j-1} C^k_j \frac{\partial^s(f - \widetilde{P})}{\partial x^{s-k}\partial y^k}\cos^{j-k}\beta\sin^k\beta\Big\|_{C(\Delta)}$$

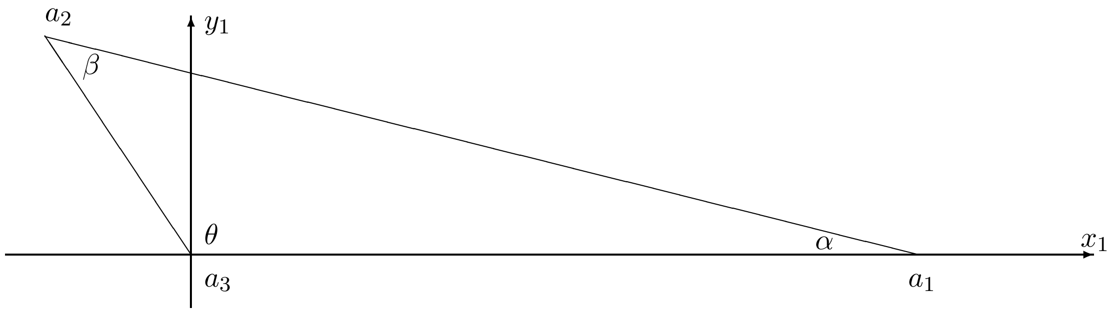

Fig. 2. The position of the triangle Δ in the coordinate system Ox_1y_1.

$$\lesssim \big\|D^s_{\xi_1\ldots\xi_s}(f-\widetilde{P})\big\|_{C(\Delta)} + \sum_{k=0}^{j-1} C_j^k \Big\|\frac{\partial^s(f-\widetilde{P})}{\partial x^{s-k}\partial y^k}\cos^{j-k}\beta\sin^k\beta\Big\|_{C(\Delta)}$$

$$\lesssim MH^{n+1-s} + \sum_{k=0}^{j-1} C_j^k MH^{n+1-s}\sin^{-k}\beta\cos^{j-k}\beta\sin^k\beta \lesssim MH^{n+1-s};$$

i.e., (1.4) is proved.

In the situation when (1.3) holds under condition (1.2), we introduce an auxiliary coordinate system Ox_1y_1 such that the point a_3 coincides with the origin, the vector τ_{31} is codirectional with the axis Ox_1, and the triangle Δ is for definiteness in the upper half-plane (see Fig. 2). Then, similarly to the considered case, we obtain the estimates

$$\Big\|\frac{\partial^s(f-\widetilde{P})}{\partial x^{s-i}\partial y^i}\Big\|_{C(\Delta)} \lesssim MH^{n+1-s}\sin^{-i}\theta \lesssim MH^{n+1-s}\sin^{-i}\beta$$

for $i=0,\ldots,s$ and $s=0,\ldots,n$. Since $\dfrac{\partial}{\partial\tau_{21}} = \dfrac{\partial}{\partial x_1}\cos\alpha - \dfrac{\partial}{\partial y_1}\sin\alpha$, $\dfrac{\partial}{\partial\tau_{23}} = -\dfrac{\partial}{\partial x_1}\cos\theta - \dfrac{\partial}{\partial y_1}\sin\theta$, and $\sin\beta \lesssim \sin\theta \lesssim \sin\beta$, we can assert that (1.3) also holds under (1.1). Thus, we are in the situation of the first case.

Remark 4. It is also easy to see that, if relation (1.3) holds under one of conditions (1.1) or (1.2), then it holds under the other condition (we have shown that the validity of (1.3) under (1.2) implies the validity of (1.3) under (1.1); the converse statement is proved similarly).

2. PROOFS OF THEOREMS 1 AND 2

Consider a polynomial $R \in \mathcal{P}_n$. Define

$$e[R](x,y) = f(x,y) - R(x,y).$$

Using Taylor's formula with the Cauchy integral remainder term, we obtain the expansion

$$\frac{\partial^s e[R](x,y)}{\partial x^{s-j}\partial y^j} = \sum_{i=j}^{n-s+j}\frac{1}{(i-j)!}y^{i-j}\sum_{k=0}^{n-s+j-i}\frac{\partial^{s-j+i+k}e[R](0,0)}{\partial x^{s-j+k}\partial y^i}\,\frac{x^k}{k!}$$

$$+\sum_{i=j}^{n-s+j}\frac{1}{(i-j)!}y^{i-j}\int\limits_0^x \frac{(x-v)^{n-s+j-i}}{(n-s+j-i)!}\,\frac{\partial^{n+1}e[R](v,0)}{\partial v^{n+1-i}\partial y^i}\,dv$$

$$+\int_0^y \frac{(y-t)^{n-s}}{(n-s)!}\,\frac{\partial^{n+1} f(x,t)}{\partial x^{s-j}\partial t^{n+1-s+j}}\,dt, \qquad (x,y)\in\Delta. \tag{2.1}$$

To prove (0.6), it is sufficient to estimate for the polynomial $R = P$ defined by conditions (0.4) and (0.5) the value $\partial^s e[P](0,0)/\left(\partial x^{s-j}\partial y^j\right)$ for $0 \le s \le n$ and $0 \le j \le s$.

Lemma. *Under the conditions of Theorem 1, there exists a nonnegative value $K(n,\beta_0)$ depending only on n and β_0 such that, for any $s = 1,\ldots,n$, the following relations hold*:

$$\left|\frac{\partial^s e[P](0,0)}{\partial x^{s-j}\partial y^j}\right| \le K(n,\beta_0) M H^{n+1-s}\sin^{-j}\beta \quad \textit{if} \quad j = \overline{0,\min\{2,s\}}, \tag{2.2}$$

$$\frac{\partial^s e[P](0,0)}{\partial x^{s-j}\partial y^j} = 0 \quad \textit{if} \quad s\ge 3 \quad \textit{and} \quad j = 3,\ldots,s. \tag{2.3}$$

Proof. Equalities (2.3) follow from (0.5). It remains to prove (2.2). Let an integer s be such that $1 \le s \le n$. Recall that $a_2 = (0,0)$. Considering $e[P](x,0)$ on the interval $[a_2,a_1]$ and using formulas for the estimate of the interpolation error in the scalar case (see, e.g., [6]), we obtain

$$\frac{\partial^s e[P](a_2)}{\partial x^s} = C_1(s)\frac{\partial^{n+1} f(\zeta_{21}^s)}{\partial x^{n+1}} H^{n+1-s}, \tag{2.4}$$

where $C_1(s)$ depends only on s and can be upper bounded by a value depending on n; ζ_{21}^s is a point between a_2 and a_1. Thus,

$$\left|\frac{\partial^s e[P](a_2)}{\partial x^s}\right| \lesssim M H^{n+1-s};$$

i.e., (2.2) is proved for $j = 0$ for any $s = 1,\ldots,n$. If $s \ge 2$, then, to estimate the remaining two derivatives (for $j = 1,2$), we write a system of equations by applying mathematical induction.

Assume that (2.2) is proved for $s = r+1,\ldots,n$, where $r \in \{2,\ldots,n-1\}$. Then, taking into account (2.1), we can assert that formula (0.6) from Theorem 1 holds for all $s = r+1,\ldots,n$. Consider an arbitrary $s = r \in \{2,\ldots,n-1\}$ (which corresponds to the induction step) or $s = r = n$ (the induction base). Since the induction step and the induction base are proved almost similarly, we will consider these cases simultaneously, only sometimes focusing on the case $s = r = n$ if necessary. Consider the restriction of the function $e[P]$ to the interval $[a_2,a_3]$. On the one hand, applying known results for estimates of interpolation errors in the scalar case, we obtain the equality

$$\frac{\partial^s e[P](a_2)}{\partial \tau_{23}^s} = C_2(s)\frac{\partial^{n+1} f(\zeta_{23}^s)}{\partial \tau_{23}^{n+1}}\left(a^2+h^2\right)^{(n+1-s)/2}, \tag{2.5}$$

where $C_2(s)$ depends only on s and can be upper bounded by a value depending on n; ζ_{23}^s is a point between a_2 and a_3. On the other hand, using equalities (2.3), we can represent the derivative along τ_{23} as the sum of partial derivatives:

$$\frac{\partial^s e[P](a_2)}{\partial \tau_{23}^s} = \sum_{k=0}^{s} C_s^k \frac{\partial^s e[P](a_2)}{\partial x^{s-k}\partial y^k}\cos^{s-k}\beta\sin^k\beta$$

$$= \frac{\partial^s e[P](a_2)}{\partial x^s}\cos^s\beta + s\frac{\partial^s e[P](a_2)}{\partial x^{s-1}\partial y}\cos^{s-1}\beta\sin\beta + \frac{s(s-1)}{2}\frac{\partial^s e[P](a_2)}{\partial x^{s-2}\partial y^2}\cos^{s-2}\beta\sin^2\beta. \tag{2.6}$$

Combining (2.4)–(2.6), we come to the equality

$$s\frac{\partial^s e[P](a_2)}{\partial x^{s-1}\partial y}\cos^{s-1}\beta\sin\beta+\frac{s(s-1)}{2}\frac{\partial^s e[P](a_2)}{\partial x^{s-2}\partial y^2}\cos^{s-2}\beta\sin^2\beta=m_1(s), \tag{2.7}$$

where

$$|m_1(s)|=\Big|C_2(s)\frac{\partial^{n+1}f(\zeta_{23}^s)}{\partial\tau_{23}^{n+1}}\left(a^2+h^2\right)^{(n+1-s)/2}-C_1(s)\frac{\partial^{n+1}f(\zeta_{21}^s)}{\partial x^{n+1}}H^{n+1-s}\cos^s\beta\Big|$$
$$\lesssim M\left(a^2+h^2\right)^{(n+1-s)/2}+MH^{n+1-s}\cos^s\beta.$$

To obtain the second equation of the system, consider an arbitrary polynomial $R\in\mathcal{P}_n[Q_0]$, i.e., a polynomial satisfying conditions (0.4). In particular, this can be $\widetilde{P}$ or P. Consider the restriction of the function $\partial^s e[R]/(\partial\tau_{31}^s)$ to the interval $[a_2,a_3]$ and expand the value of this function at the point a_2 by Taylor's formula at the point a_3. As a result, we obtain the equality

$$\frac{\partial^s e[R](a_2)}{\partial\tau_{31}^s}=\frac{\partial^s e[R](a_3)}{\partial\tau_{31}^s}+\sum_{k=1}^{n-s}\frac{(-1)^k}{k!}\frac{\partial^{s+k}e[R](a_3)}{\partial\tau_{31}^s\partial\tau_{23}^k}\left(a^2+h^2\right)^{k/2}$$
$$+\frac{(-1)^{n+1-s}}{(n+1-s)!}\frac{\partial^{n+1}f\left(\eta_{23}^s\right)}{\partial\tau_{31}^s\partial\tau_{23}^{n+1-s}}\left(a^2+h^2\right)^{(n+1-s)/2} \tag{2.8}$$

(if $s=n$, the right-hand side of this equality contains only the first and the last terms; i.e., the sum $\sum_{k=1}^0$ is zero). On the other hand, we can represent the derivative along τ_{31} as the sum of partial derivatives:

$$\frac{\partial^s e[R](a_2)}{\partial\tau_{31}^s}=\sum_{k=0}^{s}C_s^k\frac{\partial^s e[R](a_2)}{\partial x^{s-k}\partial y^k}\cos^{s-k}\alpha(-\sin\alpha)^k. \tag{2.9}$$

From (2.8) and (2.9), we come to the equality

$$-s\frac{\partial^s e[R](a_2)}{\partial x^{s-1}\partial y}\cos^{s-1}\alpha\sin\alpha+\frac{s(s-1)}{2}\frac{\partial^s e[R](a_2)}{\partial x^{s-2}\partial y^2}\cos^{s-2}\alpha\sin^2\alpha=\mu_1^s[R]+\mu_2^s[R], \tag{2.10}$$

where

$$\mu_1^s[R]=\sum_{k=1}^{n-s}\frac{(-1)^k}{k!}\frac{\partial^{s+k}e[R](a_3)}{\partial\tau_{31}^s\partial\tau_{23}^k}\left(a^2+h^2\right)^{k/2}+\frac{(-1)^{n+1-s}}{(n+1-s)!}\frac{\partial^{n+1}f\left(\eta_{23}^s\right)}{\partial\tau_{31}^s\partial\tau_{23}^{n+1-s}}\left(a^2+h^2\right)^{(n+1-s)/2}$$
$$-\sum_{k=3}^{s}C_s^k\frac{\partial^s e[R](a_2)}{\partial x^{s-k}\partial y^k}\cos^{s-k}\alpha(-\sin\alpha)^k,\qquad \mu_2^s[R]=\frac{\partial^s e[R](a_3)}{\partial\tau_{31}^s}-\frac{\partial^s e[R](a_2)}{\partial x^s}\cos^s\alpha.$$

Note that $\mu_2^s[R]$ depends only on the interpolation conditions on the sides $[a_3,a_1]$ and $[a_2,a_1]$, i.e., on a part of conditions (0.4). Thus, $\mu_2^s[R]=\mu_2^s[\widetilde{P}]=\mu_2^s[P]$. Consequently, to estimate this value, we can use Subbotin's result (1.3), (1.4) and equality (2.10):

$$\left|\mu_2^s[R]\right|=\left|\mu_2^s[\widetilde{P}]\right|=\Big|-s\frac{\partial^s e[\widetilde{P}](a_2)}{\partial x^{s-1}\partial y}\cos^{s-1}\alpha\sin\alpha+\frac{s(s-1)}{2}\frac{\partial^s e[\widetilde{P}](a_2)}{\partial x^{s-2}\partial y^2}\cos^{s-2}\alpha\sin^2\alpha-\mu_1^s[\widetilde{P}]\Big|$$

$$=\Big|\sum_{k=1}^{s}C_s^k\frac{\partial^s e[\widetilde{P}](a_2)}{\partial x^{s-k}\partial y^k}\cos^{s-k}\alpha(-\sin\alpha)^k-\sum_{k=1}^{n-s}\frac{(-1)^k}{k!}\frac{\partial^{s+k}e[\widetilde{P}](a_3)}{\partial\tau_{31}^s\partial\tau_{23}^k}\left(a^2+h^2\right)^{k/2}$$

$$- \frac{(-1)^{n+1-s}}{(n+1-s)!} \frac{\partial^{n+1} f\left(\eta_{23}^s\right)}{\partial \tau_{31}^s \partial \tau_{23}^{n+1-s}} \left(a^2+h^2\right)^{(n+1-s)/2} \Big| \lesssim \sum_{k=1}^{s} C_s^k M H^{n+1-s} \frac{\cos^{s-k}\alpha \sin^k \alpha}{\sin^k \beta}$$

$$+ \sum_{k=1}^{n-s} M H^{n+1-s-k} \left(a^2+h^2\right)^{k/2} + M\left(a^2+h^2\right)^{(n+1-s)/2} \lesssim M H^{n-s} \left(a^2+h^2\right)^{1/2}.$$

Substituting $R = P$ into (2.10) and taking into account conditions (0.5), we obtain

$$-s \frac{\partial^s e[P](a_2)}{\partial x^{s-1} \partial y} \cos^{s-1}\alpha \sin \alpha + \frac{s(s-1)}{2} \frac{\partial^s e[P](a_2)}{\partial x^{s-2} \partial y^2} \cos^{s-2}\alpha \sin^2 \alpha = \mu_1^s[P] + \mu_2^s[P], \tag{2.11}$$

$$\text{where} \quad |\mu_2^s[P]| = |\mu_2^s[R]| \lesssim M H^{n-s} \left(a^2+h^2\right)^{1/2},$$

$$\mu_1^s[P] = \sum_{k=1}^{n-s} \frac{(-1)^k}{k!} \frac{\partial^{s+k} e[P](a_3)}{\partial \tau_{31}^s \partial \tau_{23}^k} \left(a^2+h^2\right)^{k/2} + \frac{(-1)^{n+1-s}}{(n+1-s)!} \frac{\partial^{n+1} f\left(\eta_{23}^s\right)}{\partial \tau_{31}^s \partial \tau_{23}^{n+1-s}} \left(a^2+h^2\right)^{(n+1-s)/2}$$

$$- \sum_{k=3}^{s} C_s^k \frac{\partial^s e[P](a_2)}{\partial x^{s-k} \partial y^k} \cos^{s-k}\alpha (-\sin \alpha)^k$$

$$= \sum_{k=1}^{n-s} \frac{(-1)^k}{k!} \frac{\partial^{s+k} e[P](a_3)}{\partial \tau_{31}^s \partial \tau_{23}^k} \left(a^2+h^2\right)^{k/2} + \frac{(-1)^{n+1-s}}{(n+1-s)!} \frac{\partial^{n+1} f\left(\eta_{23}^s\right)}{\partial \tau_{31}^s \partial \tau_{23}^{n+1-s}} \left(a^2+h^2\right)^{(n+1-s)/2}. \tag{2.12}$$

If $s = n$, then the right-hand side of equality (2.12) consists of one term; in this case,

$$|\mu_1^s[P]|_{s=n}| = |\mu_1^n[P]| = \Big| - \frac{\partial^{n+1} f\left(\eta_{23}^n\right)}{\partial \tau_{31}^n \partial \tau_{23}} \left(a^2+h^2\right)^{1/2} \Big| \lesssim M\left(a^2+h^2\right)^{1/2}.$$

If $s = r \in \{2, \ldots, n-1\}$, then the right-hand side of (2.12) also contains a sum over k. By the induction assumption, Theorem 1 holds for all $s = r+1, \ldots, n$. Since

$$\Big|\frac{\partial^{s+k} e[P](a_3)}{\partial \tau_{31}^s \partial \tau_{23}^k}\Big| \lesssim \Big|\frac{\partial^{s+k} e[P](a_3)}{\partial x^{s_1+k_1} \partial y^{s_2+k_2}} \cos^{s_1}\alpha \cos^{k_1}\beta (-\sin\alpha)^{s_2} \sin^{k_2}\beta\Big|,$$

where $s_1 + s_2 = s$ and $k_1 + k_2 = k$, we obtain the estimate

$$\left|\frac{\partial^{s+k} e[P](a_3)}{\partial \tau_{31}^s \partial \tau_{23}^k}\right| \lesssim M H^{n+1-s-k}.$$

Thus, $|\mu_1^s[P]| \lesssim M H^{n-s} \left(a^2+h^2\right)^{1/2}$.

Combining (2.7) and (2.11), we obtain the system of equations

$$\begin{cases} s \dfrac{\partial^s e[P](a_2)}{\partial x^{s-1} \partial y} \cos^{s-1}\beta \sin \beta + \dfrac{s(s-1)}{2} \dfrac{\partial^s e[P](a_2)}{\partial x^{s-2} \partial y^2} \cos^{s-2}\beta \sin^2 \beta = m_1(s), \\ -s \dfrac{\partial^s e[P](a_2)}{\partial x^{s-1} \partial y} \cos^{s-1}\alpha \sin \alpha + \dfrac{s(s-1)}{2} \dfrac{\partial^s e[P](a_2)}{\partial x^{s-2} \partial y^2} \cos^{s-2}\alpha \sin^2 \alpha = m_2(s), \end{cases}$$

$$|m_1(s)| \lesssim M\left(a^2+h^2\right)^{(n+1-s)/2} + M H^{n+1-s} \cos^s \beta,$$

$$|m_2(s)| = |\mu_1^s[P] + \mu_2^s[P]| \lesssim M H^{n-s} \left(a^2+h^2\right)^{1/2}.$$

To solve this system, we use Cramer's rule. Denote by A the main matrix of the system and calculate its determinant:

$$\det A = \frac{s^2(s-1)}{2}\cos^{s-2}\beta\sin\beta\cos^{s-2}\alpha\sin\alpha\begin{vmatrix} \cos\beta & \sin\beta \\ -\cos\alpha & \sin\alpha \end{vmatrix}$$

$$= \frac{s^2(s-1)}{2}\cos^{s-2}\beta\sin\beta\cos^{s-2}\alpha\sin\alpha\sin(\alpha+\beta).$$

Then, taking into account that $\cos\alpha \gtrsim 1$, $\sin\beta = h/\left(a^2+h^2\right)^{1/2}$, $h/H \lesssim \sin\alpha = h/b \lesssim h/H$, and $\sin\beta \lesssim \sin(\alpha+\beta) = \sin\theta \lesssim \sin\beta$, we obtain

$$\left|\frac{\partial^s e[P](a_2)}{\partial x^{s-1}\partial y}\right| \lesssim \frac{|m_1(s)|\cos^{s-2}\alpha\sin^2\alpha + |m_2(s)|\cos^{s-2}\beta\sin^2\beta}{\cos^{s-2}\beta\sin\beta\cos^{s-2}\alpha\sin\alpha\sin(\alpha+\beta)}$$

$$= \frac{|m_1(s)|\sin\alpha}{\cos^{s-2}\beta\sin\beta\sin(\alpha+\beta)} + \frac{|m_2(s)|\sin\beta}{\cos^{s-2}\alpha\sin\alpha\sin(\alpha+\beta)} \lesssim \frac{|m_1(s)|\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{|m_2(s)|}{\cos^{s-2}\alpha\sin\alpha}$$

$$\lesssim \frac{M\left(a^2+h^2\right)^{(n+1-s)/2}\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n+1-s}\cos^s\beta\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n-s}\left(a^2+h^2\right)^{1/2}}{\cos^{s-2}\alpha\sin\alpha}$$

$$\lesssim MH^{n+1-s}\frac{\left(a^2+h^2\right)^{(n+1-s)/2}}{H^{n+1-s}}\frac{\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n+1-s}}{\sin\beta}\cos^2\beta\frac{\sin\alpha}{\sin\beta} + \frac{MH^{n-s}\left(a^2+h^2\right)^{1/2}}{\sin\alpha}$$

$$\lesssim MH^{n+1-s}\left(\frac{\sin\alpha}{\sin\beta}\right)^{n+1-s}\frac{\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n+1-s}}{\sin\beta} + \frac{MH^{n+1-s}\left(a^2+h^2\right)^{1/2}}{h}$$

$$\lesssim MH^{n+1-s}\frac{\sin\alpha}{\sin\beta}\frac{\sin\alpha}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n+1-s}}{\sin\beta} + \frac{MH^{n+1-s}}{\sin\beta}$$

$$\lesssim \frac{MH^{n+1-s}}{\cos^{s-2}\beta\sin\beta}\frac{\sin^2\alpha}{\sin^2\beta} + \frac{MH^{n+1-s}}{\sin\beta}. \tag{2.13}$$

Since $\sin\alpha \le \sin(2\beta)$ (which follows from the fact that $\sin\alpha - \sin(2\beta) = \sin(\theta+\beta) - \sin(2\beta) = 2\sin\left((\theta-\beta)/2\right)\cos((\theta+\beta)/2+\beta) = 2\sin((\theta-\beta)/2)\cos((\pi-\alpha)/2+\beta) = -2\sin((\theta-\beta)/2)\sin(\beta-\alpha/2) \le 0$) and $\beta \le \beta_0$, we conclude that (2.13) leads to the estimate

$$\left|\frac{\partial^s e[P](a_2)}{\partial x^{s-1}\partial y}\right| \lesssim \frac{MH^{n+1-s}}{\cos^{s-4}\beta\sin\beta}\frac{\sin^2\alpha}{\sin^2(2\beta)} + \frac{MH^{n+1-s}}{\sin\beta} \lesssim \frac{MH^{n+1-s}}{\cos^{n-4}\beta\sin\beta}$$

$$\le \left(\frac{1}{\cos\beta_0}\right)^{n-4}\frac{MH^{n+1-s}}{\sin\beta}. \tag{2.14}$$

The remaining derivative is estimated similarly:

$$\left|\frac{\partial^s e[P](a_2)}{\partial x^{s-2}\partial y^2}\right| \lesssim \frac{|m_1(s)|\cos^{s-1}\alpha\sin\alpha + |m_2(s)|\cos^{s-1}\beta\sin\beta}{\cos^{s-2}\beta\sin\beta\cos^{s-2}\alpha\sin\alpha\sin(\alpha+\beta)}$$

$$= \frac{|m_1(s)|\cos\alpha}{\cos^{s-2}\beta\sin\beta\sin(\alpha+\beta)} + \frac{|m_2(s)|\cos\beta}{\cos^{s-2}\alpha\sin\alpha\sin(\alpha+\beta)} \lesssim \frac{|m_1(s)|}{\cos^{s-2}\beta\sin^2\beta} + \frac{|m_2(s)|\cos\beta}{\sin\alpha\sin\beta}$$

$$\lesssim \frac{M\left(a^2+h^2\right)^{(n+1-s)/2}}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n+1-s}\cos^s\beta}{\cos^{s-2}\beta\sin^2\beta} + \frac{MH^{n-s}\left(a^2+h^2\right)^{1/2}\cos\beta}{\sin\alpha\sin\beta}$$

$$\lesssim MH^{n+1-s}\Big(\frac{\sin\alpha}{\sin\beta}\Big)^{n+1-s}\frac{1}{\cos^{s-2}\beta\sin^2\beta}+MH^{n+1-s}\frac{\cos^2\beta}{\sin^2\beta}+MH^{n+1-s}\frac{\sin\alpha}{\sin\beta}\frac{\cos\beta}{\sin\alpha\sin\beta}$$

$$\lesssim\frac{MH^{n+1-s}}{\cos^{s-2}\beta\sin^2\beta}\frac{\sin\alpha}{\sin\beta}+\frac{MH^{n+1-s}}{\sin^2\beta}+\frac{MH^{n+1-s}}{\sin^2\beta}. \tag{2.15}$$

Finally, since $\sin\alpha\le\sin(2\beta)$ and $\beta\le\beta_0$, inequality (2.15) gives the estimate

$$\Big|\frac{\partial^s e[P](a_2)}{\partial x^{s-2}\partial y^2}\Big|\lesssim\frac{MH^{n+1-s}}{\cos^{s-3}\beta\sin^2\beta}\frac{\sin\alpha}{\sin(2\beta)}+\frac{MH^{n+1-s}}{\sin^2\beta}\lesssim\Big(\frac{1}{\cos\beta_0}\Big)^{n-3}\frac{MH^{n+1-s}}{\sin^2\beta}. \tag{2.16}$$

To complete the proof of the lemma, it remains to prove (2.2) for $s=1$.

In the case $s=1$, there is no need to consider a system of equations. It is sufficient to consider the restriction of the function $e[P]$ to the side $[a_2,a_3]$ and use estimates for the error of approximation of the derivative of the function by the derivative of the interpolation polynomial, on the one hand, and the representation of a directional derivative as the sum of partial derivatives, on the other hand. Thus,

$$\frac{\partial e[P](a_2)}{\partial\tau_{23}}=C_2(1)\frac{\partial^{n+1}f(\zeta_{23}^1)}{\partial\tau_{23}^{n+1}}\left(a^2+h^2\right)^{n/2}=\frac{\partial e[P](a_2)}{\partial x}\cos\beta+\frac{\partial e[P](a_2)}{\partial y}\sin\beta,$$

where $C_2(1)$ can be upper bounded by a value depending on n and ζ_{23}^1 is a point between a_2 and a_3. Further, taking into account that (2.2) has been proved for all s for $j=0$, we obtain

$$\Big|\frac{\partial e[P](a_2)}{\partial y}\Big|\lesssim\frac{MH^n}{\sin\beta}.$$

The lemma is proved. □

To complete the proof of Theorem 1, we apply expansion (2.1) and the lemma.

Theorem 2 is a trivial corollary of Theorem 1 and follows from the representation of a derivative of any order along any direction as the sum of partial derivatives of the same order with coefficients whose absolute values can be upper bounded by values depending only on n.

Corollary. *If $n=3$, then the constraint on the angle β and the dependence of the value $C(n,\beta_0)$ on β_0 can be eliminated from the statements of Theorems* 1 *and* 2*.*

Proof. The proof follows from (2.14), (2.16), and (2.1).

3. PROOF OF THEOREMS 3 AND 4

Theorem 3 is proved similarly to Theorem 1. We introduce a rectangular coordinate system Ox_1y_1 such that the point a_3 coincides with the origin, the point a_1 belongs to the axis Ox_1, and the point a_2 is in the upper half-plane (see Fig. 2). Replacing the angle β by θ and the variables x and y by x_1 and y_1, respectively, we repeat almost completely the proof of the lemma. The exception is that estimates of form (2.14) and (2.16) are not proved. The proof ends at the estimates of form (2.13) and (2.15), which imply the following inequalities for any $s=2,\dots,n$:

$$\Big|\frac{\partial^s e[P](a_3)}{\partial x_1^{s-1}\partial y_1}\Big|\lesssim\Big(\frac{1}{\cos\beta_0}\Big)^{n-2}\frac{MH}{\sin\beta},\qquad\Big|\frac{\partial^s e[P](a_2)}{\partial x_1^{s-2}\partial y_1^2}\Big|\lesssim\Big(\frac{1}{\cos\beta_0}\Big)^{n-2}\frac{MH}{\sin^2\beta}.$$

The remaining part of the proof of the lemma is the same. □

Theorem 4 is a corollary of Theorem 3.

4. CONSTRUCTION OF A FINITE ELEMENT SPACE

Let Δ be an arbitrary triangle from the triangulation of the domain Ω, and let β be the middle angle of this triangle. Obviously, $0 \le \beta \le \pi/2$. Let us represent the interval $[0, \pi/2]$ in the form of the union of two intervals. For example, $[0, \pi/2] = [0, \pi/3] \cup [\pi/3, \pi/2]$.

If $\beta < \pi/3$, then, constructing a finite element, we take the polynomial P given by conditions (0.4) and (0.5), for which estimates (0.6) and (0.7) hold. In particular, for any unit vectors $\xi_1, \dots, \xi_s$ and $0 \le s \le n$, we have

$$\left\| D^s_{\xi_1 \dots \xi_s}(f - P) \right\|_{C(\Delta)} \lesssim M H^{n+1-s} (\sin\theta)^{-\min\{s,2\}}.$$

If $\beta \ge \pi/3$, we take the polynomial $\widetilde{P}$ given by conditions (0.2). In view of (0.3) and the constraint on the angle β, for any unit vectors $\xi_1, \dots, \xi_s$ and $0 \le s \le n$, we have

$$\left\| D^s_{\xi_1 \dots \xi_s}(f - \widetilde{P}) \right\|_{C(\Delta)} \lesssim M H^{n+1-s} \sin^{-s}\theta \lesssim M H^{n+1-s} \sin^{-s}\beta \lesssim M H^{n+1-s}.$$

Since the definitions of the polynomials P and $\widetilde{P}$ involve condition (0.4), the resulting spline on Ω is a continuous function.

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