ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2016, Vol. 295, Suppl. 1, pp. S120–S130.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2015, Vol. 21, No. 3.

A PTAS for Min-*k*-SCCP in Euclidean Space of Arbitrary Fixed Dimension

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Received May 13, 2015

Abstract—We study the minimum weight k-size cycle cover problem (Min-k-SCCP), which consists in partitioning a complete weighted digraph into k vertex-disjoint cycles of minimum total weight. This problem is a generalization of the known traveling salesman problem and a special case of the classical vehicle routing problem. It is known that Min-k-SCCP is strongly NP-hard in the general case and preserves its intractability even in the geometric statement. For Euclidean Min-k-SCCP in $\mathbb{R}^d$ with $k = O(\log n)$, we construct a polynomial-time approximation scheme (PTAS), which generalizes the approach proposed earlier for planar Min-2-SCCP. For each fixed $c > 1$ the scheme finds a $(1 + 1/c)$-approximate solution in time $O(n^{O(d)}(\log n)^{(O(\sqrt{d}c))^{d-1}})$.

Keywords: cycle cover of size k, traveling salesman problem, NP-hard problem, polynomial-time approximation scheme.

DOI: 10.1134/S0081543816090133

INTRODUCTION

In the classical statement of the traveling salesman problem (TSP), it is required to find a minimum weight Hamiltonian cycle in a given complete edge-weighted graph. TSP plays a special role in combinatorial optimization and operations research and ranks among the typical NP-hard problems [1].

In the 1970s, a number of results were obtained that were related to studying the computational complexity of problems arising in graph theory, mathematical programming, combinatorial optimization, mathematical logic, and so on. Karp proved [2] the NP-completeness of a series of problems, including the Hamiltonian cycle problem, by reducing to them the Boolean satisfiability problem. The inapproximability of TSP in the classical statement was proved in [3].

The study of special cases of TSP seems to be important from the viewpoint of practical applications. Two subclasses are of special interest: metric TSP and Euclidean TSP. These subclasses are characterized by a special form of input data: the input of metric TSP is an undirected graph whose edge weights satisfy the triangle inequality, whereas in Euclidean TSP the vertices of the graph are points in $\mathbb{R}^d$ and edge weights are defined as pairwise distances between them.

TSP is NP-hard even in the Euclidean statement [4]. That is why optimal solutions of metric TSP and Euclidean TSP cannot be found in polynomial time unless P = NP. Efficient approximation algorithms were developed for these subclasses. For metric TSP, a 2-approximation

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algorithm [5] and a 3/2-approximation algorithm (Christofides, [6]) are known. Nevertheless, metric TSP has no polynomial-time approximation scheme (PTAS) unless P = NP [7], and the estimation of the effective approximability threshold in this problem remains one of the important open questions in the study of TSP. A PTAS for planar Euclidean TSP was developed by Mitchell in [8]. Euclidean TSP in the space of arbitrary fixed dimension has an asymptotically optimal algorithm [9] and a PTAS proposed by Arora in [10]. Note that Mitchell's and Arora's algorithms were developed almost simultaneously, but Mitchell's approach could not be generalized to the case of $\mathbb{R}^d$.

The object of study in the present paper is the minimum weight k-size cycle cover problem (Min-k-SCCP) [11, 12], which is natural generalization of TSP.

The input of Min-k-SCCP is a complete edge-weighted directed graph $G = (V, E, w)$ on n vertices and a weight function $w\colon E \to \mathbb{R}$ given by a matrix $W = (w_{ij})$ $(1 \le i, j \le n)$. It is required to find k vertex-disjoint cycles $\{C_1, \dots, C_k\}$ of minimum total weight that collectively visit all vertices of the graph. Min-k-SCCP is strongly NP-hard, and the metric and Euclidean statements of this problem have a similar status of computational complexity [11]. An efficient 2-approximation algorithm for metric Min-k-SCCP was proposed in [13]; its approximation ratio is asymptotically attainable. For planar Euclidean Min-k-SCCP with fixed parameter $k = 2$, a PTAS that extends Arora's approach [10] was constructed in [11].

In this paper, we consider Euclidean Min-k-SCCP. We assume that the graph G is undirected and its edge weights coincide with the distances between the corresponding vertices, which are points in $\mathbb{R}^d$. We give a generalization of the PTAS constructed for planar Euclidean Min-2-SCCP to the case of arbitrary fixed values of the parameter k and the dimension of the space.

1. ALGORITHM OF A POLYNOMIAL-TIME APPROXIMATION SCHEME FOR EUCLIDEAN MIN-k-SCCP IN $\mathbb{R}^d$

Definition 1. A *polynomial-time approximate scheme* (PTAS) for a combinatorial optimization problem is a family of algorithms that contains for each fixed $c > 1$ an approximation algorithm solving this problem with performance guarantee $(1 + 1/c)$ in a time upper bounded by some polynomial in the input size (the order and the coefficients of the polynomial may, in general, depend on c).

The general *scheme of the algorithm* develops the approach proposed in [11] for planar Min-2-SCCP and consists of five main stages.

1. The decomposition of the problem into m $(m \le k)$ independent subproblems on the cycle cover of a graph and the derivation of an upper bound for the sides of hypercubes enclosing the vertices of the graphs that define these subproblems using a function that expresses the linear dependence on the weight OPT of an optimal k-cycle cover.

2. The proof of the statement that, to an arbitrary instance of Euclidean Min-k-SCCP and any value of the parameter c, one can assign in polynomial time a rounded instance so that an arbitrary $(1 + 1/c)$-approximation algorithm for the rounded instance induces a $(1 + c_1/c)$-approximation algorithm for the original instance (for some independent value $c_1 > 1$).

3. The construction of a recursive dissection of the hypercube enclosing the vertices of the graph that defines the rounded instance of Euclidean Min-k-SCCP.

4. The proof of the theorem asserting that, with probability at least 1/2 within the chosen probabilistic model, there exists a $(1 + 1/c)$-optimal family of paths of a special form; this family

is called a cycle (m, r, k)-cover of the graph G. The notion of cycle (m, r, k)-cover of a graph was introduced in [11].

5. The construction of a $(1+1/c)$-optimal cycle (m, r, k)-cover by means of the dynamic programming method and the standard derandomization scheme.

Stages 1, 4, and 5 play the key role, whereas stage 2 is a simple corollary of the lemma from [11]. The method of implementing stage 3 coincides with the corresponding approach proposed in [10] for the solution of Euclidean TSP in $\mathbb{R}^d$.

2. A PTAS FOR EUCLIDEAN MIN-k-SCCP IN $\mathbb{R}^d$

2.1. The decomposition of the Euclidean cycle cover problem. It is known that the diameter D of a set and the radius R of the ball circumscribed about it in Euclidean space of dimension d are related by Young's inequality [14]

$$\frac{1}{2}D \leq R \leq \left(\frac{d}{2d+2}\right)^{1/2} D. \tag{1}$$

We construct a minimum spanning forest with k trees (a k-MSF) $T_1, T_2, \ldots, T_k$ using a simple modification [11] of the Borůvka–Kruskal algorithm [5].

Introduce the notation:

D_i is the diameter of the vertex set of a tree T_i, $i \in \mathbb{N}_k = \{1, \ldots, k\}$;

D is the maximum value of D_i $(i \in \mathbb{N}_k)$;

R_i is the radius of the sphere circumscribed about T_i;

R is the maximum value of R_i $(i \in \mathbb{N}_k)$.

Consider an auxiliary complete graph $G_T = (V_T, E_T)$ in which the vertex set V_T is the set of the trees of a k-MSF and the edge weights are defined by the function $\rho\colon V_T \times V_T \to \mathbb{R}$ such that the value $\rho(T_i, T_j)$ is the distance between the centers of the spheres circumscribed about the trees T_i and T_j for $\{i, j\} \subset \mathbb{N}_k$.

We construct a partition of the vertex set of G_T into m $(m \leq k)$ clusters by the nearest neighbor method [15, 16] with threshold value of the distance between clusters equal to $(2k+1)R$; the computational complexity of this procedure is $O(k^3)$. For each of the constructed clusters, we combine the vertex sets of all the trees of this cluster and denote the resulting sets by S_i; thus, we obtain the clustering $S_1, S_2, \ldots, S_m$ of the vertices of the original graph G. The diameter of a cluster is understood as the maximum distance between its vertices.

Assertion 1. *Any path from an optimal k-cycle cover of a graph G passes through vertices corresponding to only one of the clusters $S_1, S_2, \ldots, S_m$. In addition, the diameters of the clusters are upper bounded*:

$$\max_{i \in \mathbb{N}_m} D_{S_i} \leq \left(\frac{d}{2d+2}\right)^{1/2} (2k^2 - k + 1)\mathrm{OPT}. \tag{2}$$

Proof. Assume by contradiction that one of the k paths of the minimum weight cycle cover contains vertices from S_i and from S_j, where $\{i, j\} \subset \mathbb{N}_m$.

Denote this path by P. By the assumption, P contains at least two edges connecting vertices from different clusters, and their total length is greater than $2(2k-1)R$.

The path P may contain vertices from different spheres (Fig. 1). Let $\{u, v\}$ be an arbitrary edge from P such that $u \in S_i$ and $v \in S_j$. Fixing the order of visiting the vertices of the path $P\colon u \to u_{i_1} \to \ldots \to v$, we construct a partition of P into fragments so that each sphere corresponds

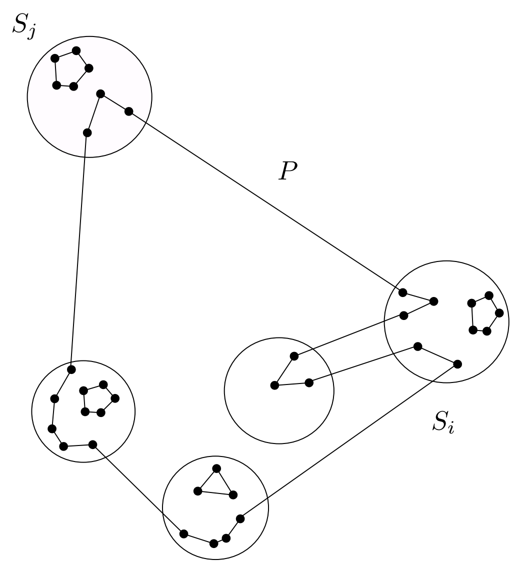

Fig. 1. Vertices from different spheres alternate in P, $k = 5$.

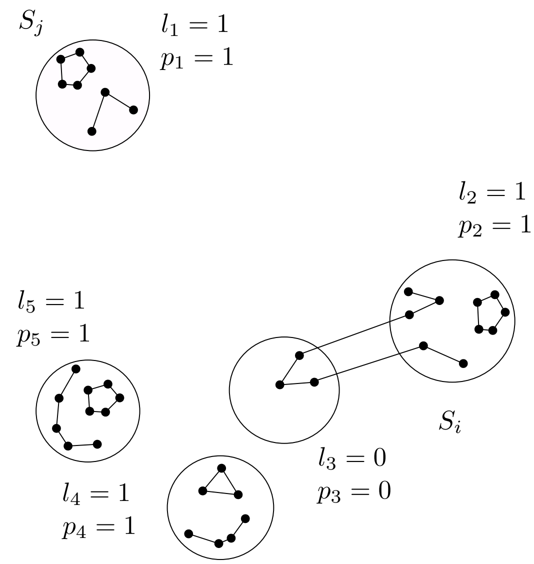

Fig. 2. A partition of P into fragments.

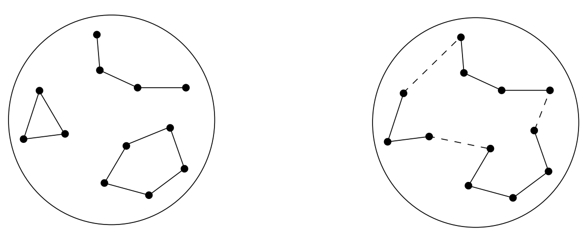

Fig. 3. The closure of a fragment of P and two cycles into one path ($l_i = 2$, $p_i = 1$).

to the fragment of P between the first and the last vertices visited inside this sphere, and we remove the edges of P that connect vertices from different fragments (Fig. 2).

To each sphere circumscribed about a tree T_i, we assign the number l_i ($0 \le l_i \le k-1$) of cycles and the number p_i ($0 \le p_i \le 1$) of fragments of the path P corresponding to this sphere (Fig. 2). If a cycle connects vertices from several trees and intersects the corresponding circumscribed spheres, then we assign it to only one of these spheres chosen arbitrarily. By construction, we have

$$\sum_{i=1}^{k} l_i = k - 1 \quad \text{and} \quad 2 \le \sum_{i=1}^{k} p_i \le k. \tag{3}$$

Let q be the number of spheres for which $l_i + p_i = 0$. Consider two cases: $q = 0$ and $q \ge 1$. In the first case, we construct a k-cycle cover as follows. For the ith sphere, we combine into one cycle the l_i closed paths and the fragment of P corresponding to this sphere by adding at most $(l_i + p_i)$ new edges; note that the weight of each new edge does not exceed $2R$ (Fig. 3). We perform this transformation for each sphere, increasing the total weight of the cycle cover by at most

$$2R\sum_{i=1}^{k}(l_i+p_i)\le 2(2k-1)R$$

in view of relations (3), whereas the total weight of the removed edges exceeds this value. Thus, under the assumption $q=0$, we have proved the existence of a k-cycle cover of the graph G such that its weight is strictly less than the weight of the original solution.

In the second case, $q\ge 1$. Consequently, the $(k-q)$ spheres for which $l_i+p_i>0$ correspond to $(k-1)$ cycles and at lest two fragments of P. Note that one sphere cannot correspond to more than one fragment of P. Then at least $(q+1)$ cycles are distributed among spheres that contain a cycle or a fragment of P. We exclude from consideration q cycles so that the number of spheres for which $l_i+p_i=0$ remain the same. Then $(k-q)$ spheres contain $(k-q-1)$ cycles and fragments of P. Therefore, we have the relations $l_i+p_i>0,\ i=1,\dots,k-q$,

$$\sum_{i=1}^{k-q} l_i = k-q-1, \quad \text{and} \quad 2\le\sum_{i=1}^{k-q}p_i\le k-q. \tag{4}$$

For these $(k-q)$ spheres, we implement a transformation similarly to the transformation in first case. For each sphere we combine into one cycle the corresponding closed paths and a fragment of P by adding at most (l_i+p_i) new edges inside this sphere (Fig. 3). Thus, we construct a cycle cover of size $(k-q)$ increasing the weight of the cycle cover by at most

$$2R\sum_{i=1}^{k-q}(l_i+p_i)\le 2(2k-2q-1)R$$

in view of (4). Adding to this cycle cover q cycles that were excluded earlier, we obtain a k-cycle cover whose weight is also strictly less than the weight of the original cover.

Thus, we have proved the existence of a k-cycle cover whose weight is strictly less than the weight of the original solution, which contradicts the assumption about the optimality of the latter.

Let us estimate the diameters of the obtained clusters. The diameter of a cluster in the space $\mathbb{R}^d$ is greatest if the centers of the spheres circumscribed about the trees of the k-MSF that form this cluster lie on the same straight line. Any cluster from a partition of the vertex set V of the graph G contains at most k spheres; hence,

$$\max_{i\in\mathbb{N}_m} D_{S_i}\le (k-1)(2k+1)R+2R.$$

Applying inequality (1) and using the obvious two-sided bound $D\le \text{MSF}\le \text{OPT}$ for the weight of the minimum spanning forest $\{T_1,T_2,\dots,T_k\}$, we obtain

$$R\le\Big(\frac{d}{2d+2}\Big)^{1/2}D\le\Big(\frac{d}{2d+2}\Big)^{1/2}\text{OPT}.$$

Hence,

$$\max_{i\in\mathbb{N}_m} D_{S_i}\le\Big(\frac{d}{2d+2}\Big)^{1/2}(2k^2-k+1)\text{OPT}.$$

The assertion is proved.

Thus, if $m = k$, then Euclidean Min-k-SCCP is decomposed into k independent subproblems equivalent to TSP. In particular, a PTAS for Euclidean Min-k-SCCP can be constructed as a combination of PTASs for these subproblems.

However, if the number of the constructed clusters is strictly less than k, then it is necessary to consider all possible variants of the distribution of k cycles among m clusters, solve in each case m independent cycle cover subproblems, and choose an optimal solution. The number of such cases coincides with the number of compositions of k of length m and equals the binomial coefficient of $(k-1)$ and $(m-1)$ [17].

Let us consider a special case of Euclidean Min-k-SCCP in which $m = 1$, since this case is the worst from the point of view of computational complexity.

2.2. A rounded instance. For completeness of presentation, we give a definition of a rounded instance of Euclidean Min-k-SCCP introduced in [10, 11].

Definition 2. A *rounded* instance of Euclidean Min-k-SCCP is an instance satisfying the following constraints: all vertices of the graph have integer coordinates and the weight of any edge e_{ij} is greater than or equal to 4.

To obtain a rounded instance of the problem, it is necessary and sufficient to perform the following transformation.

1. Define

$$L = \Big(\frac{d}{2d+2}\Big)^{1/2}(2k^2 - k + 1)\mathrm{MSF} \le \Big(\frac{d}{2d+2}\Big)^{1/2}(2k^2 - k + 1)\mathrm{OPT}$$

and construct a hypercube $\mathcal{S}$ with side L enclosing the vertices of the graph G. This construction is possible since the original instance of Euclidean Min-k-SCCP satisfies inequality (2).

2. We construct inside $\mathcal{S}$ an orthogonal grid with distance $L/8cn\sqrt{d}$ between its hyperplanes and shift the vertices of the graph to the nearest nodes of the grid. Since the distance between any two vertices will increase by at most $L/4cn$, the weight of the k-cycle will change by at most $L/4c$.

3. Change the distance between the hyperplanes of the grid multiplying all the coordinates by $32cn\sqrt{d}/L$. Then the minimum distance between the vertices will be 4, and the side of the enclosing hypercube will be $O(cn\sqrt{d})$.

4. Let the origin coincide with a corner of $\mathcal{S}$, and let the coordinate axes be directed so that the edges of the hypercube belong to the axes and all vertices of the graph have nonnegative coordinates.

The sufficiency of constructing a PTAS for rounded Euclidean Min-k-SCCP follows immediately from the lemma proved in [11]. An equivalent formulation of this lemma is given below.

Lemma. *Let a rounded instance of Euclidean Min-k-SCCP be obtained from an original instance of Euclidean Min-k-SCCP by transformations that increase the weight of an arbitrary k-cycle cover by at most $O(\mathrm{OPT}/c)$. Then a PTAS for rounded Euclidean Min-k-SCCP induces a PTAS for general Euclidean Min-k-SCCP.*

In what follows, we describe the construction of a PTAS for rounded Euclidean Min-k-SCCP.

2.3. A recursive dissection of the enclosing hypercube $\mathcal{S}$. We construct a geometric decomposition of the problem using the data structure of a 2^d-tree, which is similar to the 4-tree used in the planar case [10, 11].

We take the side L of the enclosing hypercube $\mathcal{S}$ equal to the smallest appropriate power of two. We organize the construction of a 2^d-tree as follows. Let $\mathcal{S}$ be the root of the tree, and we dissect

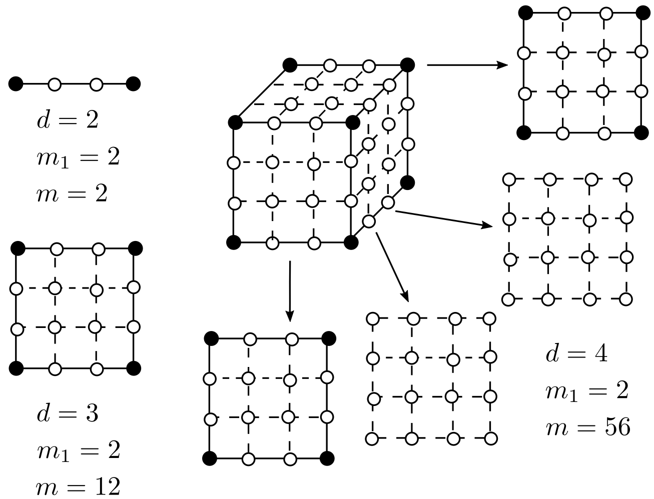

Fig. 4. An example of constructing an orthogonal grid of portals for $d = 2$, 3, and 4.

each hypercube, including the root, into 2^d equal child hypercubes. We repeat this procedure recursively until we obtain hypercubes containing at most one vertex of the original problem. Let us agree that $\mathcal{S}$ belongs to level 0, its 2^d child hypercubes belong to level 1, and so on. Note that the constructed tree contains $O(2^d n)$ leaves and $O(\log L) = O(\log(cn\sqrt{d}))$ levels. Thus, the number of nodes of the 2^d-tree can be estimated as $O(2^d n \log(cn\sqrt{d}))$.

Fix a value of the parameter $m \in \mathbb{N}$ and assign to each $(d-1)$-dimensional face of a node an orthogonal grid consisting of $m + 2^{d-1}$ portals. Note that, in contrast to the planar case, not every positive integer is an admissible value of the parameter m.

Assertion 2. *The distance between nearest portals on a face of a node of the ith level in a 2^d-tree is $O(L/2^i m^{1/(d-1)})$.*

Proof. The construction of an orthogonal grid on a face of a d-dimensional cube involves $(m_1+2)^{d-1} - 2^{d-1}$ portals, where $m_1 \in \mathbb{N}$ is the number of portals on an edge of the hypercube (see Fig. 4).

The side of a hypercube of the ith level is $L/2^i$ by construction, whence the distance between neighboring portals on the face of this node is $L/2^i(m_1+1)$.

Let us show that

$$\frac{L}{2^i(m_1+1)} \le \frac{3}{2}\,\frac{L}{2^i m^{1/(d-1)}}.$$

This inequality is equivalent to $m_1 + 1 \ge \dfrac{2}{3}((m_1+2)^{d-1} - 2^{d-1})^{1/(d-1)}$. As d grows infinitely large, the value of the expression $((m_1+2)^{d-1} - 2^{d-1})^{1/(d-1)}$ tends asymptotically from below to the value $m_1 + 2$; hence,

$$\frac{m_1+1}{((m_1+2)^{d-1} - 2^{d-1})^{1/(d-1)}} \ge \frac{m_1+1}{m_1+2} \ge \frac{2}{3}.$$

Thus, in a hypercube of level i in a 2^d-tree, the distance between nearest portals can be estimated as $O(L/2^i m^{1/(d-1)})$.

The assertion is proved.

The notions of central point and tree with cyclic shift, which were defined for planar Euclidean Min-2-SCCP, are easily extended to the case of an arbitrary fixed dimension of the space.

A d-dimensional point is called *central* if each of its coordinates equals $L/2$.

Definition 3. Let $a_1, a_2, \ldots, a_d \in \mathbb{N}_L$. A 2^d-tree for which the point $((L/2 + a_1) \bmod L, (L/2 + a_2) \bmod L, \ldots, (L/2 + a_d) \bmod L)$ is central is called the 2^d-*tree* $T(a_1, a_2, \ldots, a_d)$.

Hypercubes belonging to an arbitrary level $i \geq 1$ of a tree $T(a_1, a_2, \ldots, a_d)$, as well as its central point, are subject to a cyclic shift in each of the d coordinate axes. Under the shift, the position of $\mathcal{S}$ and the coordinates of the vertices of G remain unchanged.

By analogy with the case $d = 2$, the union of dissections by portals of $(d-1)$-dimensional faces of all nodes of a 2^d-tree $T(a_1, a_2, \ldots, a_d)$ (except for $\mathcal{S}$) is called an *m-regular set of portals* and is denoted by $P(a_1, a_2, \ldots, a_d; m)$.

2.4. The existence theorem. In this section we prove the existence of a k-cycle cover with a number of properties for a given graph G.

Denote by $V(C) \subseteq V$ the vertex set of an arbitrary cycle C in the graph G. We assign to C an (m, r)-approximation, which is a closed polygonal line $l(C)$ satisfying the following conditions:

(i) the vertex set of $l(C)$ is a subset of $V(C) \cup P(a_1, a_2, \ldots, a_d; m)$;

(ii) $l(C)$ visits the vertices $V(C)$ in the order corresponding to the path C;

(iii) $l(C)$ intersects each face of an arbitrary node of the tree $T(a_1, a_2, \ldots, a_d)$ at most r times $(r \in \mathbb{N})$ and only at points of the set $P(a_1, a_2, \ldots, a_d; m)$.

The following statement is equivalent to Theorem 5 from [10].

Theorem 1 (the structure theorem for Euclidean TSP in $\mathbb{R}^d$)**.** *Suppose that a rounded instance of TSP in $\mathbb{R}^d$ is given by a complete Euclidean graph G, the side of the enclosing hypercube $\mathcal{S}$ for this graph is L, and a positive constant c is given. Let discrete random values $a_1, a_2, \ldots, a_d$ be distributed uniformly and independently on the set $\mathbb{N}_L$.*

Then for any $\eta \in (0, 1)$ there exist $D_1, D_2 > 0$ such that, for $r = \left\lceil (D_1 \sqrt{d} c)^{d-1} \right\rceil$ and $m = \left\lceil (D_2 d c \log L)^{d-1} \right\rceil$, for an arbitrary simple cycle C of weight $W(C)$ in the graph G, with probability at least $1 - \eta$ there exists an (m, r)-approximation $l(C)$ whose weight does not exceed $(1 + 1/c)W(C)$.

Definition 4 [11]. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be an arbitrary k-cycle cover of a graph G, and let $l(C_i)$ be an (m, r)-approximation of a cycle C_i. The set $\mathcal{L}(\mathcal{C}) = \{l(C_1), \ldots, l(C_k)\}$ is called a *cycle (m, r, k)-cover* of the graph G.

Let us extend the result of Theorem 3 from [11] to the case of Euclidean Min-k-SCCP in the space of arbitrary fixed dimension. We prove the following theorem.

Theorem 2. *Suppose that $c > 0$ is an arbitrary constant, L is the size of the enclosing hypercube $\mathcal{S}$ for a rounded statement of Euclidean Min-k-SCCP in $\mathbb{R}^d$, and discrete random values $a_1, a_2, \ldots, a_d$ are distributed uniformly and independently on the set $\mathbb{N}_L$. Then for the parameters $m = (O(dc \log L))^{d-1}$ and $r = (O(\sqrt{d}c))^{d-1}$ with probability at least $1/2$ there exists a cycle (m, r, k)-cover of cost not exceeding $(1 + 1/c)\mathrm{OPT}$.*

Proof. Consider a minimum weight k-cycle cover $\mathcal{C}^* = \{C_1^*, C_2^*, \ldots, C_k^*\}$, which is a solution the rounded instance of Euclidean Min-k-SCCP. As usual, we denote its weight by OPT; i.e., $\sum_{i=1}^{k} W(C_i^*) = \mathrm{OPT}$.

We apply Theorem 1: for $\eta = 1/2k$ and each cycle C_i^* with probability at least $1 - 1/2k$, there exists an (m, r)-approximation $l(C_i^*)$ with weight

$$W(l(C_i^*)) \leq (1 + 1/c)W(C_i^*). \tag{5}$$

The values of the parameters m and r satisfy the bounds $(O(dc\log L))^{d-1}$ and $(O(\sqrt{d}c))^{d-1}$, respectively.

Since the values $a_1, a_2, \ldots, a_d$ are distributed uniformly, the probability of the union of the complementary events, which consist in the absence of an (m,r)-approximation $l(C_i^*)$ with weight satisfying (5) for a cycle C_i^*, is upper bounded by 1/2. Thus, with probability at least 1/2, there exists a cycle (m,r,k)-cover $\{l(C_1^*), l(C_2^*), \ldots, l(C_k^*)\}$ such that

$$\sum_{i=1}^{k} W(l(C_i^*)) \le \Big(1+\frac{1}{c}\Big)\sum_{i=1}^{k} W(C_i^*) = \Big(1+\frac{1}{c}\Big)\mathrm{OPT}.$$

The theorem is proved.

2.5. A remark on the dynamic programming procedure. The procedure of finding a minimum weight cycle (m,r,k)-cover $\{l_1, l_2, \ldots, l_k\}$ in Euclidean Min-k-SCCP is based on the dynamic programming method and develops the approach proposed in [10, 11]. The running time of the algorithm is $O(n(\log n)^{(O(\sqrt{d}c))^{d-1}})$.

The interior subproblem for a node of the 2^d-tree S consists in finding a minimum cost part of the cycle (m,r,k)-cover that lies entirely inside S and visits all the vertices of the graph that belong to this node.

In the beginning of the dynamic programming procedure, we consider the leaves of the 2^d-tree $T(a_1, a_2, \ldots, a_d)$. Let S be an arbitrary leaf of the tree. By construction S contains at most one vertex of the graph G, and the corresponding subproblem can be solved by direct search in $O(2dr)$ operations.

Consider the case when S is not a leaf of $T(a_1, a_2, \ldots, a_d)$. Denote by $S^1, \ldots, S^{2^d}$ its child hypercubes, for which the interior subproblems are assumed to be solved. We construct a solution for S recursively, assuming that the answer consists of intervals of (m,r)-approximations $\{l_1, l_2, \ldots, l_k\}$.

To estimate the complexity of the dynamic programming procedure, we preserve the notation introduced in [11]: $\mathfrak{P}$ is the family of all possible multisets P consisting of at most $2dr$ portals located on the faces of child hypercubes $S^1, \ldots, S^{2^d}$ that are inner with respect to S. By construction, on each of these faces, there are $m + 2^{d-1}$ portals and, by the constraints imposed on a cycle cover, a face can be intersected at most r times. We find that $|\mathfrak{P}| = O((m+2^{d-1})^{2dr})$.

For an arbitrary multiset $P \in \mathfrak{P}$, there exist $O((dr)^{2dr})$ ways to assign to each portal a polygonal line corresponding to one of the k (m,r)-approximations, and there are $O((2dr)!)$ variants of partitioning the multiset into ordered pair.

The complexity of the subproblem corresponding to S can be estimated as

$$O\big((m+2^{d-1})^{2dr}(dr)^{2dr}(2dr)!\big).$$

The problem of finding a minimum weight cycle (m,r,k)-cover is equivalent to solving the subproblem for the enclosing hypercube $\mathcal{S}$.

Let us find an upper bound for the number of subproblems in order to derive the total complexity of the dynamic programming procedure. Note that, for each node of an 2^d-tree S, there are $O((m+2^{d-1})^{2dr})$ ways to choose the multiset P of portals on $(d-1)$-dimensional faces of S.

For each of these multisets, there are $O((2dr)!)$ ways of partitioning it into ordered pairs; for each of these partitions, there are $O(k^{dr})$ ways to distribute these pairs among the paths $\{l_1, l_2, \ldots, l_k\}$. Since the total number of nodes of $T(a_1, a_2, \ldots, a_d)$ is $O(2^d n\log(cn\sqrt{d}))$, we obtain the desired bound for the complexity of the procedure for an arbitrary fixed central point $(a_1, a_2, \ldots, a_d)$:

$$O\big(2^d n \log(cn\sqrt{d}) \times (m + 2^{d-1})^{4dr}((2dr)!)^2 (dr)^{2dr} \times k^{dr}\big). \quad (6)$$

The variables d and c are not a part of the input of Euclidean Min-k-SCCP. For the values $m = (O(dc \log(cn\sqrt{d})))^{d-1}$ and $r = (O(\sqrt{d}c))^{d-1}$, since the number of the ways to distribute k cycles among m clusters is $O(2^k)$, bound (6) is equivalent to the bound

$$O\big(n(k \log n)^{(O(\sqrt{d}c))^{d-1}} 2^k\big).$$

Remark. The computational complexity of the standard derandomization scheme that involves the exhaustive search of shifts of the 2^d-tree $T(a_1, a_2, \ldots, a_d)$ is $O(n^d)$.

Thus, the following theorem is proved.

Theorem 3. *Euclidean Min-k-SCCP in $\mathbb{R}^d$ has a PTAS with complexity*

$$O\big(n^{d+1}(k \log n)^{(O(\sqrt{d}c))^{d-1}} 2^k\big). \quad (7)$$

For $d = 2$, the computational complexity of the PTAS constructed for Euclidean Min-k-SCCP coincides with the complexity of the PTAS constructed in [11] for Euclidean Min-2-SCCP and differs from it, as follows from relation (7), by a constant factor $2^k k^{O(c)}$.

Corollary. *Euclidean Min-k-SCCP in $\mathbb{R}^d$ with the condition that k is a part of the input has a PTAS with complexity $O(n^{O(d)}(\log n)^{(O(\sqrt{d}c))^{d-1}})$ for $k = O(\log n)$.*

CONCLUSIONS

In this paper, we have extended the result obtained for planar Euclidean Min-2-SCCP and validated a polynomial-time approximation scheme for arbitrary fixed values of the parameter k and of the dimension of space d. The proposed algorithm is also a PTAS in the case if the parameter k is a part of the input of Euclidean Min-k-SCCP in $\mathbb{R}^d$ for $k = O(\log n)$.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation (project no. 14-11-00109).

REFERENCES

1. M. R. Garey and D. S. Johnson, *Computers and Intractability* (Freeman, San Fransisco, 1979; Mir, Moscow, 1982).
2. R. Karp, "Reducibility among combinatorial problems," in *Complexity of Computer Computations: Symposium Procedings*, Ed. by. R. E. Miller and J. W. Thatcher (Plenum, New York, 1972), pp. 85–103.
3. S. Sahni and T. Gonzales, "P-complete approximation problems," J. Assoc. Comput. Mach. **23** (3), 555–565 (1976).
4. C. Papadimitriou, "Euclidean TSP is NP-complete," Theoret. Comput. Sci. **4** (3), 237–244 (1977).
5. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, (MIT Press, Cambridge, MA, 1990).
6. N. Christofides, "Worst-case analysis of a new heuristic for the traveling salesman problem," in *Algorithms and Complexity: New Directions and Recent Results* (Academic, New York, 1976), p. 441.

7. S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy, "Proof verification and intractability of approximation problems," in *Proceedings of the 33rd Annual IEEE Symposium on Foundations of Computer Science, Pittsburgh, Pennsylvania, 1992*, pp. 13–22.
8. J. Mitchell, "Guillotine subdivisions approximate polygonal subdivisions: A simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems," SIAM J. Comp. **28** (4), 1298–1309 (1999).
9. E. Kh. Gimadi and V. A. Perepelitsa, "An asymptotically optimal approach to solving the traveling salesman problem," in *Control Systems*: *Collection of Papers* (Inst. Mat. SO AN SSSR, Novosibirsk, 1974), issue 12, pp. 35–45 [in Russian].
10. S. Arora, "Polynomial-time approximation schemes for Euclidean traveling salesman and other geometric problems," J. ACM. **45** (5), 753–782 (1998).
11. M. Yu. Khachai and E. D. Neznakhina, "A polynomial-time approximation scheme for the Euclidean problem on a cycle cover of a graph," Tr. Inst. Mat. Mekh. **20** (4), 297–311 (2014).
12. M. Yu. Khachai and E. D. Neznakhina, "Approximability of the problem about a minimum-weight cycle cover of a graph," Dokl. Math. **91** (2), 240–245 (2015).
13. M. Khachay and K. Neznakhina, "Approximation of Euclidean k-size cycle cover problem," Croat. Oper. Res. Rev. **5** (2), 177–188 (2014).
14. H. Jung, "Über die kleinste Kugel, die eine räumliche Figur einschliesst," J. Reine Angew. Math. **123**, 241–257 (1901).
15. P. Sneath, "Computers in taxonomy," J. Gen. Microbiol. **17**, 201–226 (1957).
16. J. Gower and G. Ross, "Minimum spanning trees and single linkage cluster analysis," Appl. Statist. **18**, 54–64 (1969).
17. G. Andrews, *The Theory of Partitions* (Addison-Wesley, London, 1976; Nauka, Moscow, 1982).

Translated by E. Vasil'eva