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New Results on Sums and Products in $\mathbb{R}$

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Abstract—We improve previous sum–product estimates in $\mathbb{R}$; namely, we prove the inequality $\max\{|A+A|,|AA|\} \gg |A|^{4/3+c}$, where c is any number less than 5/9813. New lower bounds for sums of sets with small product set are found. We also obtain results on the additive and multiplicative energies; in particular, we improve a result of Balog and Wooley.

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1. INTRODUCTION

Let $A,B\subset\mathbb{R}$ be finite sets. We define the *sum set, product set,* and *quotient set* of A and B as

$$A+B:=\{a+b\colon\ a\in A,\ b\in B\},\qquad AB:=\{ab\colon\ a\in A,\ b\in B\},$$

and

$$A/B:=\{a/b\colon\ a\in A,\ b\in B,\ b\neq 0\},$$

respectively. The Erdős–Szemerédi conjecture [2] says that for any $\epsilon>0$ one has

$$\max\{|A+A|,|AA|\}\gg |A|^{2-\epsilon}.$$

Roughly speaking, it asserts that an arbitrary subset of real numbers (or integers) cannot have good additive and multiplicative structures simultaneously. Using some beautiful geometrical arguments, Solymosi [9] proved the following

Theorem 1. *Let $A\subset\mathbb{R}$ be a set. Then*

$$|A+A|^2|A/A|\ge \frac{|A|^4}{4\lceil\log|A|\rceil},\qquad |A+A|^2|AA|\ge \frac{|A|^4}{4\lceil\log|A|\rceil}. \tag{1.1}$$

In particular,

$$\max\{|A+A|,|AA|\}\gg \frac{|A|^{4/3}}{\log^{1/3}|A|}. \tag{1.2}$$

Here and below we suppose that $|A|\ge 2$.

It is easy to see that bound (1.1) is tight up to logarithmic factors if the size of $A+A$ is small relatively to A. We will write $a\lesssim b$ or $b\gtrsim a$ if $a=O(b\log^c|A|)$, $c>0$. The notation $a\sim b$ means that $a\lesssim b$ and $b\lesssim a$.

In [4] we improved bound (1.2).

Theorem 2. *Let $A\subset\mathbb{R}$ be a set. Then*

$$\max\{|A+A|,|AA|\}\gtrsim |A|^{4/3+c'},$$

where $c'=1/20\,598$. The same is true if one replaces AA by A/A.

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The main result of the article is the following.

Theorem 3. *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$\max\{|A+A|, |AA|\} \gtrsim |A|^{4/3+c},$$

where $c = 5/9813$. The same is true if one replaces AA by A/A.

In [4] the case of sets with small product/quotient sets was considered (sharper bounds for the *difference* of two sets with small multiplicative doubling can be found in [7]).

Theorem 4. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then*

$$|A+A| \gtrsim |A|^{19/12} K^{-5/6} \tag{1.3}$$

and

$$|A+A| \gtrsim |A|^{49/32} K^{-19/32}. \tag{1.4}$$

Inequality (1.4) is stronger than (1.3) for $K \gtrsim |A|^{5/23}$.

We improve Theorem 4 for some range of parameters in the case of a small quotient set.

Theorem 5. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|A/A| \leq K|A|$. Then*

$$|A+A| \gtrsim \max\bigl\{|A|^{19/12} K^{-5/6}, |A|^{1313/830} K^{-336/415}\bigr\}. \tag{1.5}$$

One can check that the lower bound (1.5) coincides with (1.3) for $K \lesssim |A|^{5/23}$ and is stronger than both estimates (1.3) and (1.4) for $|A|^{5/23} \lesssim K \lesssim |A|^{673/2867}$. If $K \gtrsim |A|^{673/2867}$, then (1.4) gives a better result.

Finally, in Section 4 we obtain sum–product results that involve the additive and multiplicative energies E^+ and $\mathsf{E}^\times$ of sets. Similar results in this direction were obtained in [1], where the following theorem was proved.

Theorem 6. *Let $A \subset \mathbb{R}$ be a finite set and $\delta = 2/33$. Then there are two disjoint subsets B and C of A such that $A = B \sqcup C$ and*

$$\max\{\mathsf{E}^+(B), \mathsf{E}^\times(C)\} \ll |A|^{3-\delta} (\log|A|)^{1-\delta},$$

$$\max\{\mathsf{E}^+(B,C), \mathsf{E}^\times(B,C)\} \ll |A|^{3-\delta/2} (\log|A|)^{(1-\delta)/2}.$$

It was also proved in [1] that one cannot take δ greater than $2/3$. Our method gives an improvement of Theorem 6.

Theorem 7. *Let $A \subset \mathbb{R}$ be a finite set and $\delta = 1/5$. Then there are two disjoint subsets B and C of A such that $A = B \sqcup C$ and*

$$\max\{\mathsf{E}^+(B), \mathsf{E}^\times(C)\} \lesssim |A|^{3-\delta}.$$

In the proof of our results we use a combination of the methods from [9, 6] and, of course, [4]. The main new idea is to introduce some more flexible quantity $d_*(A)$ instead of $d(A)$ (see the definitions below). It allows us to avoid using the Balog–Szemerédi–Gowers theorem [11] and to obtain stronger results. In addition, it allows us to prove a number of theorems on the additive and multiplicative energies (see Section 4). We hope that our new quantity $d_*(A)$ will help in other problems of sum–product type.

2. DEFINITIONS AND PRELIMINARY RESULTS

The *additive energy* $\mathsf{E}^+(A,B)$ between two sets A and B is the number of the solutions of the equation $a_1+b_1=a_2+b_2$, $a_1,a_2\in A$, $b_1,b_2\in B$ (see [11]):

$$\mathsf{E}^+(A,B)=\big|\big\{a_1+b_1=a_2+b_2\colon\ a_1,a_2\in A,\ b_1,b_2\in B\big\}\big|.$$

The *multiplicative energy* $\mathsf{E}^\times(A,B)$ between two sets A and B is the number of solutions of the equation $a_1b_1=a_2b_2$, $a_1,a_2\in A$, $b_1,b_2\in B$ (see [11]):

$$\mathsf{E}^\times(A,B)=\big|\big\{a_1b_1=a_2b_2\colon\ a_1,a_2\in A,\ b_1,b_2\in B\big\}\big|.$$

In the case of $A=B$ we write $\mathsf{E}^+(A)$ for $\mathsf{E}^+(A,A)$ and $\mathsf{E}^\times(A)$ for $\mathsf{E}^\times(A,A)$. Given $\lambda\in A/A$, we put $A_\lambda=A\cap\lambda A$. Clearly, if $0\notin A$, then

$$\mathsf{E}^\times(A)=\sum_{\lambda\in A/A}|A_\lambda|^2 \tag{2.1}$$

and similarly for the energy $\mathsf{E}^+(A)$. Next, the Cauchy–Schwarz inequality implies that

$$\mathsf{E}^\times(A_1,A_2)|A/A|\ge|A_1|^2|A_2|^2,\qquad \mathsf{E}^\times(A_1,A_2)|AA|\ge|A_1|^2|A_2|^2 \tag{2.2}$$

for $0\notin A$, $A_1\subset A$, and $A_2\subset A$. In particular,

$$\mathsf{E}^\times(A)|A/A|\ge|A|^4,\qquad \mathsf{E}^\times(A)|AA|\ge|A|^4. \tag{2.3}$$

Finally, we will use the following inequality.

Lemma 8. *Let $A_1,\dots,A_n$ be finite subsets of $\mathbb{R}$. Then*

$$\left(\mathsf{E}^+\left(\bigcup_{i=1}^n A_i\right)\right)^{1/4}\le\sum_{i=1}^n(\mathsf{E}^+(A_i))^{1/4}.$$

Similarly, if $A_1,\dots,A_n$ are finite subsets of $\mathbb{R}\setminus\{0\}$, then

$$\left(\mathsf{E}^\times\left(\bigcup_{i=1}^n A_i\right)\right)^{1/4}\le\sum_{i=1}^n(\mathsf{E}^\times(A_i))^{1/4}.$$

Proof. A similar result for subsets of finite abelian groups follows from [11, identity (4.18), Exercise 4.2.1]. Subsets of $\mathbb{R}$ can be reduced to subsets of finite groups by [11, Lemma 5.26]. $\square$

We need several auxiliary statements. The first one is the Szemerédi–Trotter theorem [10] (see also [11]). We call a set $\mathcal{L}$ of continuous plane curves a *pseudo-line system* if any two members of $\mathcal{L}$ share at most one point in common. Define the *incidence number* $\mathcal{I}(\mathcal{P},\mathcal{L})$ between points and pseudo-lines as $\mathcal{I}(\mathcal{P},\mathcal{L})=|\{(p,l)\in\mathcal{P}\times\mathcal{L}\colon p\in l\}|$.

Theorem 9. *Let $\mathcal{P}$ be a set of points and $\mathcal{L}$ a pseudo-line system. Then*

$$\mathcal{I}(\mathcal{P},\mathcal{L})\ll|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3}+|\mathcal{P}|+|\mathcal{L}|.$$

We need a definition from [8].

Definition 10. A finite set $A\subset\mathbb{R}$ is said to be of *Szemerédi–Trotter type* (abbreviated as *SzT type*) with a parameter $D>0$ if the inequality

$$\big|\big\{s\in A-B\colon\ |A\cap(B+s)|\ge\tau\big\}\big|\le\frac{D|A|\cdot|B|^2}{\tau^3} \tag{2.4}$$

holds for every finite set $B\subset\mathbb{R}$ and every real number $\tau\ge1$.

The quantity $D(A)$ can be considered as the infimum of the numbers D such that (2.4) holds for all B and $\tau \geq 1$ but, of course, the definition is applicable only to sets A with small quantity $D(A)$.

Any SzT-type set contains only a small number of solutions to a wide class of linear equations (see, e.g., [4, Corollary 8], where nevertheless another quantity $D(A)$ was used, and [8, Lemmas 7, 8]).

Corollary 11. *Let $A_1, A_2, A_3 \subset \mathbb{R}$ be any finite sets and α_1, α_2, and α_3 be arbitrary nonzero numbers. Then the number*

$$\sigma(\alpha_1 A_1, \alpha_2 A_2, \alpha_3 A_3) := \big|\big\{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0\colon\ a_1 \in A_1,\ a_2 \in A_2,\ a_3 \in A_3\big\}\big| \tag{2.5}$$

does not exceed $O(D(A_1)^{1/3}|A_1|^{1/3}|A_2|^{2/3}|A_3|^{2/3})$. Moreover, $\mathsf{E}^+(A_1, A_2) \ll D(A_1)^{1/2}|A_1| \cdot |A_2|^{3/2}$.

We also need a result from [8] on the connection between sumsets and $D(A)$ for SzT-type sets A.

Theorem 12. *Let A be of SzT type. Then*

$$|A + A| \gtrsim |A|^{58/37} D(A)^{-21/37}. \tag{2.6}$$

Now we can introduce a new characteristic of a set $A \subset \mathbb{R}$. Put

$$\mathsf{Sym}^{\times}_t(Q, R) = \{x\colon\ |Q \cap xR^{-1}| \geq t\}$$

and

$$d_*(A) = \min_{t>0} \min_{\varnothing \neq Q, R \subset \mathbb{R}\setminus\{0\}} \frac{|Q|^2|R|^2}{|A|t^3}, \tag{2.7}$$

where the second minimum in (2.7) is taken over all Q and R such that $A \subseteq \mathsf{Sym}^{\times}_t(Q, R)$ and $\max\{|Q|, |R|\} \geq |A|$.

Lemma 13. *Let $A \subset \mathbb{R}$ be a finite set. Then A is of Szemerédi–Trotter type with a parameter $O(d_*(A))$.*

Proof. Let R and Q be two sets and $t > 0$ be a real number such that $A \subseteq \mathsf{Sym}^{\times}_t(Q, R)$. Without loss of generality assume that $|Q| = \max\{|Q|, |R|\} \geq |A|$. Let also

$$S_\tau := \big\{s \in A - B\colon\ |A \cap (B + s)| \geq \tau\big\}.$$

Our task is to estimate the size of S_τ. It is easy to see that the bound

$$|S_\tau| \ll \frac{|Q|^2|R|^2|B|^2}{t^3\tau^3} \tag{2.8}$$

is enough. We have

$$\tau|S_\tau| \leq \sum_{s \in S_\tau} |A \cap (B + s)| = \big|\big\{a - b = s\colon\ a \in A,\ b \in B,\ s \in S_\tau\big\}\big| := \sigma.$$

Because $A \subseteq \mathsf{Sym}^{\times}_t(Q, R)$, we obtain the following upper bound for the number σ:

$$\sigma \leq t^{-1}\big|\big\{qr - b = s\colon\ q \in Q,\ r \in R,\ b \in B,\ s \in S_\tau\big\}\big|. \tag{2.9}$$

First of all, let us prove a trivial estimate for the size of S_τ. Namely, dropping the condition $s \in S_\tau$ in (2.9), we get

$$\tau|S_\tau|t \leq |Q| \cdot |R| \cdot |B|,$$

and hence inequality (2.8) should only be checked in the range

$$t^2\tau^2 \gg |Q| \cdot |R| \cdot |B|, \tag{2.10}$$

because otherwise

$$|S_\tau| \le \frac{|Q|\cdot|R|\cdot|B|}{t\tau} \ll \frac{|Q|^2|R|^2|B|^2}{t^3\tau^3}.$$

Further, consider the family $\mathcal{L}$ of $|R|\cdot|S_\tau|$ lines $l_{r,s} = \{(x,y)\colon ry - x = s\}$, $r \in R$, $s \in S_\tau$ and the family of points $\mathcal{P} = Q \times B$. Applying Theorem 9 to the pair $(\mathcal{P},\mathcal{L})$, we get

$$\sigma \le t^{-1}\mathcal{I}(\mathcal{P},\mathcal{L}) \ll t^{-1}\big((|\mathcal{P}|\cdot|\mathcal{L}|)^{2/3} + |\mathcal{P}| + |\mathcal{L}|\big). \tag{2.11}$$

If the first term in (2.11) dominates, then we obtain (2.8). Now suppose that the required bound (2.8) does not hold. Then, if the second term in (2.11) is the largest one, we obtain

$$\frac{|Q|^2|R|^2|B|^2}{t^2\tau^2} \ll t\tau|S_\tau| \ll |\mathcal{P}| = |Q|\cdot|B|.$$

But, clearly, $t \le \min\{|Q|,|R|\} = |R|$ and $\tau \le \min\{|A|,|B|\}$; thus we arrive at a contradiction in view of the assumption $|Q| \ge |A|$. Finally, we need to consider the case when the third term in (2.11) dominates. In the situation

$$t\tau|S_\tau| \ll |S_\tau|\cdot|R|$$

and hence in view of (2.10)

$$|R|\cdot|Q|\cdot|B| \ll |R|^2.$$

But this is a contradiction, because $|Q| \ge |R|$ and B is large enough. This completes the proof of the lemma. □

It is easy to see from the definition that $1 \le d_*(A) \le |A|$. The second inequality can be obtained by putting $Q = A$, $R = \{1\}$, and $t = 1$.

Remark 14. In [5, Lemma 7] (see also [6, Lemma 27]) the same result was obtained for the quantity

$$d(A) := \min_{C \neq \varnothing} \frac{|AC|^2}{|A|\cdot|C|}.$$

Clearly, $d_*(A) \le d(A)$. Indeed, just take $t = |C|$, $Q = AC$, and $R = C^{-1}$.

Remark 15. Let A be a set and $\Pi = AA$ or A/A. By the Katz–Koester inclusion [3], that is, $|\Pi \cap \lambda\Pi| \ge |A|$ for any $\lambda \in A/A$, one has $d_*(\Pi) \le |\Pi|^3/|A|^3$. The last estimate is usually better than the ordinary $|\Pi\Pi|^2/|\Pi|^2$ even if one applies the Plünnecke–Ruzsa inequality [11] (even for large subsets of A).

One can easily obtain an analog of Lemma 13 in a dual form. In this case, for any sets Q and R and a real number $t > 0$ we put

$$\mathsf{Sym}_t^+(Q,R) := \{x\colon\ |Q \cap (x - R)| \ge t\}$$

and consider the quantity

$$d_+(A) := \min_{t>0} \min_{\varnothing \neq Q,R \subset \mathbb{R}\setminus\{0\}} \frac{|Q|^2|R|^2}{|A|t^3}, \tag{2.12}$$

where the second minimum in (2.12) is taken over all Q and R such that $A \subseteq \mathsf{Sym}_t^+(Q,R)$ and $\max\{|Q|,|R|\} \ge |A|$. After that, repeating the proof of Lemma 13, we need to estimate the cardinality of the set

$$S_\tau := \{s \in AB^{-1}\colon\ |A \cap sB| \ge \tau\}.$$

So, we have arrived at the equation $ab^{-1} = s$, $s \in S_\tau$, $a \in A$, $b \in B$, and further to the equation $q + r = sb$, $s \in S_\tau$, $b \in B$, $q \in Q$, $r \in R$. It corresponds to the lines $l_{r,s} = \{(x, y)\colon\ y + r = sx\}$, and Theorem 9, combined with the calculations at the end of the proof of Lemma 13, gives the result.

Thus, we have obtained an analog of Lemma 13.

Lemma 16. *Let $A, B \subset \mathbb{R}$ be two finite sets. Then for any real number $\tau \ge 1$ one has*

$$\big|\{s \in AB^{-1}\colon\ |A \cap sB| \ge \tau\}\big| \ll \frac{d_+(A)|A| \cdot |B|^2}{\tau^3}. \tag{2.13}$$

So, one can say that a set $A \subset \mathbb{R}$ is of multiplicative Szemerédi–Trotter type with a parameter d if inequality (2.13) with $d_+(A)$ replaced by d and with the symbol $\ll$ replaced by $\le$ holds for all $B \subset \mathbb{R}$ and every real number $\tau \ge 1$.

We will consider further generalizations of the quantities $d_*(A)$ and $d_+(A)$ in our forthcoming paper.

3. PROOF OF THE MAIN RESULTS

We need two technical lemmas from [4].

Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set and $\tau > 0$ be a real number. Let also S'_τ be a set such that

$$S'_\tau \subset S_\tau := \{\lambda\colon\ \tau < |A_\lambda| \le 2\tau\} \subseteq A/A$$

and for any nonzero α_1, α_2, and α_3 and different $\lambda_1, \lambda_2, \lambda_3 \in S'_\tau$ one has

$$\sigma(\alpha_1 A_{\lambda_1}, \alpha_2 A_{\lambda_2}, \alpha_3 A_{\lambda_3}) \le \sigma.$$

Lemma 17. *Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set, $\tau > 0$ be a real number,*

$$32\sigma \le \tau^2 \le |A + A|\sqrt{\sigma}, \tag{3.1}$$

and S'_τ and σ be as defined above. Then

$$|A + A|^2 \ge \frac{\tau^3 |S'_\tau|}{128\sqrt{\sigma}}. \tag{3.2}$$

Lemma 18. *Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set and $L \ge 1$ be a real number. Suppose that*

$$|A + A|^2 |A/A| \le L|A|^4. \tag{3.3}$$

Then there are $\tau \ge \mathsf{E}^\times(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^\times(A)$, $|S'_\tau| \ge |S_\tau|/2$, such that for any element λ from S'_τ one has

$$|A_\lambda / A_\lambda| \gtrsim \tau^2 L^{-16}. \tag{3.4}$$

Similarly, if

$$|A + A|^2 |AA| \le L|A|^4, \tag{3.5}$$

then there exist $\tau \ge \mathsf{E}^\times(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^\times(A)$, $|S'_\tau| \ge |S_\tau|/2$, such that for any $\lambda \in S'_\tau$ one has

$$|A_\lambda A_\lambda| \gtrsim \tau^2 L^{-16}. \tag{3.6}$$

Proof of Theorem 3. Consider the situation with A/A, because the case of AA is similar. By Π we denote A/A. Without loss of generality, suppose that $0 \notin A$. Now assume that inequality (3.3) holds with some parameter L. Let also $|A/A|^3 \le L'|A|^4$. Our task is to find a lower bound for the quantities L and L'. Using Lemma 18, we find a number $\tau \ge \mathsf{E}^\times(A)/(2|A|^2)$ and sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^\times(A)$, $|S'_\tau| \gtrsim |S_\tau|$, such that for any element λ from S'_τ one has $|A_\lambda/A_\lambda| \gtrsim \tau^2 L^{-16}$. Using the Katz–Koester inclusion, namely, $A_\lambda/A_\lambda \subseteq \Pi \cap \lambda\Pi^{-1}$, $\lambda \in \Pi$ (see [3]), we get

$$|\Pi \cap \lambda\Pi^{-1}| \ge |A_\lambda/A_\lambda| \gtrsim \tau^2 L^{-16} := t$$

for any $\lambda \in S'_\tau$. In particular, $S'_\tau \subseteq \mathsf{Sym}^\times_t(\Pi,\Pi)$. Since $S'_\tau \subseteq S_\tau$, we obtain

$$\sum_{a\in A} |A \cap aS'_\tau| = \sum_{\lambda \in S'_\tau} |A \cap \lambda A| \gg \tau |S'_\tau|$$

and hence there is $a \in A$ such that for the set $A' := A \cap aS'_\tau$ one has

$$|A'| \gg \tau |S'_\tau| \cdot |A|^{-1}. \tag{3.7}$$

We know that $S'_\tau \subseteq \mathsf{Sym}^\times_t(\Pi,\Pi)$. Hence $A' \subseteq \mathsf{Sym}^\times_t(a\Pi,\Pi)$. Applying formula (2.7) with $Q = a\Pi$ and $R = \Pi$, we obtain

$$d_*(A') \lesssim \frac{|\Pi|^4}{|A'|t^3} \ll \frac{|\Pi|^4 L^{48}}{|A'|\tau^6} \ll \frac{L^{48}|A| \cdot |\Pi|^4}{|S_\tau|\tau^7}. \tag{3.8}$$

Using Theorem 12 and Lemma 13 as well as inequalities (2.3), (3.7), and (3.8), we get

$$\begin{aligned}
|A+A| &\ge |A'+A'| \gtrsim |A'|^{58/37} d_*(A')^{-21/37} \gtrsim \big(\tau|S_\tau| \cdot |A|^{-1}\big)^{58/37} \big(|S_\tau|\tau^7 L^{-48}|A|^{-1}|\Pi|^{-4}\big)^{21/37} \\
&\gtrsim (\mathsf{E}^\times(A))^{79/37} \tau^{47/37} L^{-1008/37} |A|^{-79/37} |\Pi|^{-84/37} \\
&\gtrsim (\mathsf{E}^\times(A))^{126/37} L^{-1008/37} |A|^{-173/37} |\Pi|^{-84/37} \\
&\ge \big(|A|^4|\Pi|^{-1}\big)^{126/37} L^{-1008/37} |A|^{-173/37} |\Pi|^{-84/37} \\
&\ge |A|^{331/37} L^{-1008/37} |\Pi|^{-210/37} \ge L^{-1008/37} (L')^{-70/37} |A|^{51/37}.
\end{aligned}$$

The last estimate is greater than $|A|^{4/3}$ by some power of $|A|$. Easy calculations show that one can take any number less than 5/9813 for the constant c. This completes the proof. ☐

Proof of Theorem 5. Let $\Pi = A/A$ and $|\Pi| = K|A|$. In the proof we can restrict ourselves to the case $|A|^{5/23} \le K \le \gamma |A|^{1/4}$ where $\gamma > 0$ is a small constant. Without loss of generality, suppose that $0 \notin A$. Using the pigeonhole principle, we find $\tau \ge |A|/(2K)$ and S_τ with $|S_\tau|\tau \gtrsim |A|^2$. Consider two subsets S'_τ and S''_τ of S_τ such that $|S'_\tau| = |S''_\tau| \ge |S_\tau|/2$ and for some parameter $\kappa \in (0,1]$ the following holds: $|A_\lambda/A| \le \kappa|\Pi|$ for all $\lambda \in S'_\tau$ and $|A_\lambda/A| \ge \kappa|\Pi|$ for all $\lambda \in S''_\tau$. For any $\lambda \in S'_\tau$ one has

$$d_*(A_\lambda) \le d(A_\lambda) \le \frac{\kappa^2|\Pi|^2}{|A_\lambda| \cdot |A|} \le \kappa^2 |\Pi|^2 \tau^{-1} |A|^{-1}. \tag{3.9}$$

Thus, applying Corollary 11 and Lemma 17, we see that

$$|A+A|^2 \gg \tau^3 |S_\tau| \big(\kappa^2|\Pi|^2\tau^{-1}|A|^{-1}\big)^{-1/6} \tau^{-5/6} = \tau^{7/3}|S_\tau| \cdot |A|^{1/6} |\Pi|^{-1/3} \kappa^{-1/3},$$

provided that conditions (3.1) hold. Using the inequalities $\tau \ge |A|/(2K)$ and $|S_\tau|\tau \gtrsim |A|^2$, we obtain

$$|A+A|^2 \gtrsim |A|^2 (|A|K^{-1})^{4/3} |A|^{1/6} (|A|K)^{-1/3} \kappa^{-1/3} \gg |A|^{19/6} K^{-5/3} \kappa^{-1/3}.$$

Hence

$$|A+A| \gtrsim |A|^{19/12} K^{-5/6} \kappa^{-1/6}. \tag{3.10}$$

For the set S''_τ we use the arguments as in the proof of Theorem 3. Using the Katz–Koester inclusion, namely, $A_\lambda/A \subseteq \Pi \cap \lambda\Pi^{-1}$, we get

$$|\Pi \cap \lambda\Pi^{-1}| \geq |A_\lambda/A| \geq \kappa|\Pi| := t$$

for any $\lambda \in S''_\tau$. In particular, $S''_\tau \subseteq \mathsf{Sym}^\times_t(\Pi,\Pi)$. Since $S''_\tau \subseteq S_\tau$, we obtain

$$\sum_{a\in A} |A \cap aS''_\tau| = \sum_{\lambda \in S''_\tau} |A \cap \lambda A| \gg \tau|S''_\tau| \gg \eta\tau|S_\tau|,$$

and hence there is $a \in A$ such that for the set $A' := A \cap aS''_\tau$ one has

$$|A'| \gg \tau|S_\tau| \cdot |A|^{-1}. \tag{3.11}$$

We know that $S''_\tau \subseteq \mathsf{Sym}^\times_t(\Pi,\Pi)$. Hence $A' \subseteq \mathsf{Sym}^\times_t(a\Pi,\Pi)$. Applying formula (2.7) with $Q = a\Pi$ and $R = \Pi$, we obtain

$$d_*(A') \leq \frac{|\Pi|^4}{|A'|t^3} = \frac{|\Pi|}{|A'|\kappa^3} \ll \frac{|A|\cdot|\Pi|}{\kappa^3|S_\tau|\tau}. \tag{3.12}$$

Using Theorem 12 and Lemma 13 as well as inequalities (2.3), (3.11), and (3.12), we get

$$\begin{aligned} |A+A| &\geq |A'+A'| \gtrsim |A'|^{58/37} d_*(A')^{-21/37} \gtrsim \big(\tau|S_\tau|\cdot|A|^{-1}\big)^{58/37} \big(\kappa^3\tau|S_\tau|\cdot|A|^{-1}|\Pi|^{-1}\big)^{21/37} \\ &= (|S_\tau|\tau)^{79/37}|A|^{-79/37}|\Pi|^{-21/37}\kappa^{63/37} \gtrsim |A|^{79/37}|\Pi|^{-21/37}\kappa^{63/37} \\ &\geq |A|^{58/37}K^{-21/37}\kappa^{63/37}. \end{aligned} \tag{3.13}$$

Combining bound (3.13) with (3.10), we find that the optimal choice of κ is

$$\kappa = |A|^{7/830} K^{-59/415} \leq 1, \tag{3.14}$$

because $|A|^{5/23} \leq K$. Substituting the last inequality into (3.10), we obtain

$$|A+A| \gtrsim |A|^{19/12}K^{-5/6}\big(|A|^{7/830}K^{-59/415}\big)^{-1/6} = |A|^{1313/830}K^{-336/415}.$$

We only need to check conditions (3.1). The inequality $\tau^2 \geq 32\sigma$ follows easily from (3.9) and the inequality $K \lesssim |A|^{1/4}$. Indeed, by Corollary 11, inequality (3.9), and the bound $\tau \geq |A|/(2K)$, we have

$$\sigma \leq \big(K^2|A|\tau^{-1}\big)^{1/3}\tau^{5/3} \ll \gamma^{2/3}\tau^2$$

and $\sigma \leq \tau^2/32$ if γ is small enough. It remains to check that $\tau^2 \leq |A+A|\sqrt{\sigma}$. We have taken $\sigma = \tau^{4/3}K^{2/3}|A|^{1/3}\kappa^{2/3}$. Thus we need to verify the inequality

$$\tau^8 \leq |A+A|^6 K^2 |A| \kappa^2. \tag{3.15}$$

By bound (1.1) one has $|A+A|^2 \gg |A|^3K^{-1}\log^{-1}|A|$. In addition, in view of (3.14) and the estimate $K \ll |A|^{1/4}$, we have $\kappa \gg |A|^{-9/332} \geq |A|^{-3/100}$. Thus,

$$|A+A|^6K^2|A|\kappa^2 \gg |A|^{10-3/50}K^{-1}\log^{-3}|A| \gg |A|^9,$$

and (3.15) is true for large $|A|$ since $\tau \leq |A|$. This completes the proof. $\square$

4. RESULTS ON ENERGIES

In this section we obtain results on the additive and multiplicative energies.

Let us start with a lemma that can be of interest in its own right.

Lemma 19. *Let $A, P \subset \mathbb{R}$ be two finite sets. Put*

$$\sigma_* := \sum_{x\in P} |A \cap xA|.$$

Then there is $A' \subseteq A$ such that A' is of SzT type with $d_*(A') \lesssim |P|^2|A|^2|A'|^2/\sigma_*^3$ and $|A'| \gtrsim \sigma_*|P|^{-1}$.

Similarly, put

$$\sigma_+ := \sum_{x\in P} |A \cap (x + A)|.$$

Then there exists $A'' \subseteq A$ such that A'' is of SzT type with $d_+(A'') \lesssim |P|^2|A|^2|A''|^2/\sigma_+^3$ and $|A''| \gtrsim \sigma_+|P|^{-1}$.

Proof. We have

$$\sigma_* = \sum_{x\in A} |P \cap xA^{-1}|,$$

and thus by the pigeonhole principle there is a set $A' \subseteq A$ and a number $q \le |A|$ such that $|A'|q \sim \sigma_*$ and $q < |P \cap xA^{-1}| \le 2q$ for any $x \in A'$. Since $q \le |P|$, we have $|A'| \gtrsim \sigma_*|P|^{-1}$. Using Lemma 13 with $Q = P$ and $R = A$, we see that the set A' is of SzT type with $d_*(A')$ estimated as

$$d_*(A') \ll \frac{|P|^2|A|^2}{q^3|A'|} \lesssim \frac{|P|^2|A|^2|A'|^2}{\sigma_*^3},$$

as required.

Applying similar arguments and Lemma 16 instead of Lemma 13, we obtain the existence of a set A''. This completes the proof. $\Box$

Now we are ready to formulate the main result of the section, which shows that any set either has small multiplicative energy or contains a large subset with small additive energy, and vice versa. Similar results were obtained in [1] but, as we said in the Introduction, we do not use the Balog–Szemerédi–Gowers theorem in the proof.

Theorem 20. *Let $A \subset \mathbb{R}$ be a set. Then there is $A_1 \subseteq A$ such that $|A_1| \gtrsim \mathsf{E}^\times(A)|A|^{-2}$ and*

$$\mathsf{E}^+(A_1)\mathsf{E}^\times(A) \lesssim |A_1|^{7/2}|A|^2. \tag{4.1}$$

Similarly, there is $A_2 \subseteq A$ such that $|A_2| \gtrsim \mathsf{E}^+(A)|A|^{-2}$ and

$$\mathsf{E}^\times(A_2)\mathsf{E}^+(A) \lesssim |A_2|^{7/2}|A|^2. \tag{4.2}$$

Proof. Put

$$\mathsf{E}_3^\times(A) := \sum_x |A \cap xA|^3.$$

By the pigeonhole principle there is $P \subseteq A/A$ and a number Δ such that $\Delta^3|P| \sim \mathsf{E}_3^*(A)$ and $\Delta < |A \cap xA| \le 2\Delta$ for any $x \in P$. Applying Lemma 19 with $\sigma_* \sim \Delta|P|$, we find a set $A_1 \subseteq A$, $|A_1| \gtrsim \Delta$, such that $d_*(A_1) \lesssim |A|^2|A_1|^2/(|P|\Delta^3)$. We have $\Delta \gtrsim \mathsf{E}_3^\times(A)\Delta^{-2}|P|^{-1}$ and hence, by the Cauchy–Schwarz inequality, we get $|A_1| \gtrsim \mathsf{E}^\times(A)^2|A|^{-2}\Delta^{-2}|P|^{-1}$. Next,

$$\mathsf{E}^\times(A) \ge \sum_{x\in P} |A \cap xA|^2 \ge \Delta^2|P|.$$

Therefore, $|A_1| \gtrsim \mathsf{E}^\times(A)|A|^{-2}$. Using Corollary 11, we get

$$(\mathsf{E}^+(A_1))^2\mathsf{E}^\times_3(A) \lesssim (\mathsf{E}^+(A_1))^2|P|\Delta^3 \ll |A_1|^7|A|^2.$$

Finally, applying the Cauchy–Schwarz inequality again, we obtain (4.1), as required.

By similar arguments we obtain the existence of a set A_2. This completes the proof. □

Now we can prove Theorem 7 from the Introduction.

Proof of Theorem 7. Let $M \ge 1$ be a parameter, which we will choose later. Our arguments are a sort of an algorithm. We construct a decreasing sequence of sets $C_1 = A \supseteq C_2 \supseteq \ldots \supseteq C_k$ and an increasing sequence of sets $B_0 = \varnothing \subseteq B_1 \subseteq \ldots \subseteq B_{k-1} \subseteq A$ such that for any $j = 1, 2, \ldots, k$ the sets C_j and B_{j-1} are disjoint and, moreover, $A = C_j \sqcup B_{j-1}$. If at some step j we have $\mathsf{E}^\times(C_j) \le |A|^3/M$, then we stop our algorithm and put $C = C_j$, $B = B_{j-1}$, and $k = j-1$. In the opposite situation, where $\mathsf{E}^\times(C_j) > |A|^3/M$, we apply Theorem 20 to the set C_j and find a subset D_j of C_j such that $|D_j| \gtrsim |A|/M$ and

$$\mathsf{E}^+(D_j) \lesssim |D_j|^{7/2}M|A|^{-1}. \tag{4.3}$$

After that we put $C_{j+1} = C_j \setminus D_j$ and $B_j = B_{j-1} \sqcup D_j$ and repeat the procedure. Clearly, $B_k = \bigsqcup_{j=1}^k D_j$ and, since $|D_j| \gtrsim |A|/M$, we have $k \lesssim M$. Finally, by the Hölder inequality, Lemma 8, and (4.3), we get

$$\begin{aligned}(\mathsf{E}^+(B_k))^{1/4} &\le \sum_{j=1}^k (\mathsf{E}^+(D_j))^{1/4} \lesssim (M|A|^{-1})^{1/4}\sum_{j=1}^k |D_j|^{7/8} \le (M|A|^{-1})^{1/4}\biggl(\sum_{j=1}^k |D_j|\biggr)^{7/8} k^{1/8}\\ &\lesssim (M|A|^{-1})^{1/4}|A|^{7/8}M^{1/8} = M^{3/8}|A|^{5/8}.\end{aligned}$$

Hence

$$\mathsf{E}^+(B_k) \lesssim M^{3/2}|A|^{5/2}.$$

Optimizing over M, that is, choosing $M = |A|^{1/5}$, we obtain the result. This completes the proof. □

From Theorem 20 we immediately get

$$\mathsf{E}^+(A_1)\mathsf{E}^\times(A) \lesssim |A|^{11/2} \qquad \text{and} \qquad \mathsf{E}^\times(A_2)\mathsf{E}^+(A) \lesssim |A|^{11/2}.$$

If $\mathsf{E}^\times(A)$ (respectively, $\mathsf{E}^+(A)$) is not too large, then it is not difficult to construct larger sets A_1 and A_2 satisfying these inequalities.

Corollary 21. *Let $A \subset \mathbb{R}$ be a set. Then there is $A_1 \subseteq A$ such that $|A_1| \gg (\mathsf{E}^\times(A))^{1/3}$ and*

$$\mathsf{E}^+(A_1)\mathsf{E}^\times(A) \lesssim |A|^{11/2}. \tag{4.4}$$

Similarly, there is $A_2 \subseteq A$ such that $|A_2| \gg (\mathsf{E}^+(A))^{1/3}$ and

$$\mathsf{E}^\times(A_2)\mathsf{E}^+(A) \lesssim |A|^{11/2}. \tag{4.5}$$

Proof. We proceed as in the proof of Theorem 7.

We construct a decreasing sequence of sets $C_1 = A \supseteq C_2 \supseteq \ldots \supseteq C_k$ and an increasing sequence of sets $B_0 = \varnothing \subseteq B_1 \subseteq \ldots \subseteq B_{k-1} \subseteq A$ such that for any $j = 1, 2, \ldots, k$ the sets C_j and B_{j-1} are disjoint and, moreover, $A = C_j \sqcup B_{j-1}$. If at some step j we have $|B_{j-1}| > (\mathsf{E}^\times(A))^{1/3}/2$, then we stop our algorithm and put $A_1 = B_{j-1}$ and $k = j-1$. In the opposite situation, where

$|B_{j-1}| \le (\mathsf{E}^\times(A))^{1/3}/2$, we apply Theorem 20 to the set C_j and find a subset D_j of C_j such that $|D_j| \gtrsim \mathsf{E}^\times(C_j)/|C_j|^2$ and

$$\mathsf{E}^+(D_j) \lesssim |D_j|^{7/2}|C_j|^2\mathsf{E}^\times(C_j)^{-1}.$$

We observe, however, that the inequality $\mathsf{E}^\times(B_{j-1}) \le \mathsf{E}^\times(A)/8$ due to $|B_{j-1}| \le (\mathsf{E}^\times(A))^{1/3}/2$ implies $\mathsf{E}^\times(C_j) \gg \mathsf{E}^\times(A)$. Therefore,

$$|D_j| \gtrsim \mathsf{E}^\times(A)/|A|^2 \qquad \text{and} \qquad \mathsf{E}^+(D_j) \lesssim |D_j|^{7/2}|A|^2\mathsf{E}^\times(A)^{-1}.$$

Next we put $C_{j+1} = C_j \setminus D_j$ and $B_j = B_{j-1} \sqcup D_j$ and repeat the procedure.

By Lemma 8, we have

$$\begin{aligned}(\mathsf{E}^+(B_{k-1}))^{1/4} &\le \sum_{j=1}^{k-1}(\mathsf{E}^+(D_j))^{1/4} \lesssim |A|^{1/2}\mathsf{E}^\times(A)^{-1/4}\sum_{j=1}^{k-1}|D_j|^{7/8} \\ &\lesssim |A|^{1/2}\mathsf{E}^\times(A)^{-1/4}\sum_{j=1}^{k-1}|D_j|\big(\mathsf{E}^\times(A)|A|^{-2}\big)^{-1/8} \\ &\le |A|^{1/2}\mathsf{E}^\times(A)^{-1/4}\mathsf{E}^\times(A)^{1/3}\big(\mathsf{E}^\times(A)|A|^{-2}\big)^{-1/8} \\ &= |A|^{3/4}\mathsf{E}^\times(A)^{-1/24} \le |A|^{11/8}\mathsf{E}^\times(A)^{-1/4}.\end{aligned}$$

So,

$$\mathsf{E}^+(B_{k-1}) \lesssim |A|^{11/2}\mathsf{E}^\times(A)^{-1}. \tag{4.6}$$

Next,

$$\mathsf{E}^+(D_k) \lesssim |D_k|^{7/2}|A|^2\mathsf{E}^\times(A)^{-1} \le |A|^{11/2}\mathsf{E}^\times(A)^{-1}. \tag{4.7}$$

Since $A_1 = B_k = B_{k-1} \cup D_k$, we combine (4.6) and (4.7) to complete the proof of the first claim of the corollary. The proof of the second claim is similar. $\square$

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