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On the Attainability Problem under State Constraints with Piecewise Smooth Boundary

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Abstract—The paper is devoted to the problem of approximating reachable sets for a nonlinear control system with state constraints given as a solution set of a finite system of nonlinear inequalities. Each of these inequalities is given as a level set of a smooth function, but their intersection may have nonsmooth boundary. We study a procedure of eliminating the state constraints based on the introduction of an auxiliary system without constraints such that the right-hand sides of its equations depend on a small parameter. For state constraints with smooth boundary, it was shown earlier that the reachable set of the original system can be approximated in the Hausdorff metric by the reachable sets of the auxiliary control system as the small parameter tends to zero. In the present paper, these results are extended to the considered class of systems with piecewise smooth boundary of the state constraints.
Keywords: reachable set, state constraints, penalty function, approximation, Hausdorff metric.

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1. INTRODUCTION

Reachable sets, solvability sets, tubes of trajectories, and their analogs are used for solving various control problems under uncertainty and differential games (see [1–5]). In the present paper, we consider a method for the description of reachable sets and tubes of trajectories of a control system with state constraints. Questions of the approximate construction of reachable sets, including reachable sets for systems with state constraints, were addressed in many publications [5–12]. A method of eliminating state constraints in the construction of reachable sets for differential inclusions was proposed in [13, 14], where tube trajectories and reachable sets of a differential inclusion with a convex state constraint were approximated by solutions of a family of differential inclusions without state constraints with the right-hand side of the inclusions depending linearly on a matrix parameter. In [15, 16], it was proposed to restrict the set of velocities of the original system near the boundary of the constraints. The right-hand side of the approximating auxiliary system in this procedure depends on a scalar penalty parameter, its trajectories do not intersect the boundary of the constraints, and the reachable set of the system with state constraints is approximated from inside by the reachable set of the approximating system. The proposed method can be considered as an analog of the barrier function method in optimization problems. Its application is limited by the requirement that the inward pointing condition is fulfilled (see,

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e.g., [17–19]): at any boundary point of the state constraints that is reachable from the initial state, there must exist a velocity vector of the control system directed strictly inside the constraints. The convergence of reachable sets can be proved by applying theorems on the approximation of trajectories of the control system by trajectories satisfying the state constraints [17–21]. Such theorems are used in studying the properties of the value function and in applications of the theory of generalized solutions of Hamilton–Jacobi equations [22, 23] in optimal control problems.

In [24], an auxiliary approximating system was obtained by another modification of the set of velocities of the original system. To the right-hand side of equations of the system, a correction term is added, which directs the velocity vector inside the set of constraints when its boundary is intersected. The right-hand side of the auxiliary system depends on a small parameter defining the domain of action of the correction term. The reachable set of this system, which is constructed without regard to the state constraints, contains the reachable set of the original system with state constraints. As the small parameter tends to zero, the reachable sets converge in the Hausdorff metric to the reachable set of the original system. This paper extends [24], where state constraints with smooth boundary were considered. Here, we consider piecewise smooth convex constraints. In contrast to state constraints with smooth boundary, in the piecewise smooth case we have to impose more severe constraints on the right-hand side of the control system in order to obtain estimates for the approximation accuracy. Here, compared to [24], the inward pointing condition is somewhat weakened due to the convexification of the velocity set of the system.

2. DEFINITIONS AND PROBLEM STATEMENT

We consider the control system

$$\dot{x}(t) = f(x(t), u(t)), \quad t_0 \le t \le \theta, \quad x(t_0) = x^0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u(t) \in U$ for a.a. $t \in [t_0, \theta]$ is the control. The set U is compact in $\mathbb{R}^r$, and controls are Lebesgue measurable functions $u : [t_0, \theta] \to U$.

We use the following notation. For a real matrix A, $A^\top$ denotes its transpose, and 0 is either a zero vector of appropriate dimension or the number zero. For $x, y \in \mathbb{R}^n$, $(x, y) = x^\top y$ is the scalar product of vectors and $\|x\| = (x, x)^{1/2}$ is the Euclidean norm. Further, $B_r(\bar{x}) = \{x \in \mathbb{R}^n : \|x - \bar{x}\| \le r\}$ is the ball of radius $r > 0$ centered at the point $\bar{x}$. For $S \subset \mathbb{R}^n$, we denote by ∂S, $\operatorname{int} S$, $\operatorname{cl} S$, and $\operatorname{co} S$ the boundary, interior, closure, and the convex hull of S; $\nabla g(x)$ is the gradient of the function $g(x)$ at the point x; $h(A, B)$ is the Hausdorff distance between the sets $A, B \subset \mathbb{R}^n$; and $\operatorname{conv}(\mathbb{R}^n)$ is the family of convex compact subsets of $\mathbb{R}^n$. We use the notation $\mathcal{U} = \{u(\cdot) \in L_\infty[t_0, \theta] :\ u(t) \in U \text{ for a.a. } t \in [t_0, \theta]\}$ for the set of controls.

Further, the right-hand side of system (2.1) is assumed to obey Assumption 1.

Assumption 1. *The mapping $f(x, u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the following conditions*:

(1) *$f(x, u)$ is continuous in (x, u) and locally Lipschitz in x uniformly in $u \in U$*;

(2) *the sublinear growth condition is fulfilled*: *there exists $C > 0$ such that*

$$\|f(x, u)\| \le C(1 + \|x\|), \quad (x, u) \in \mathbb{R}^n \times U.$$

Under the specified conditions, the set of trajectories of system (2.1) corresponding to the given initial condition $x(t_0) = x^0$ is bounded. By B_R, we denote the ball $B_R(\bar{x})$ that contains all trajectories of the system. System (2.1) can be represented in the form of the equivalent differential inclusion

$$\dot{x} \in F(x), \quad x(t_0) = x^0,$$

where $F(x) := f(x, U)$ is the set of velocities of system (2.1) for given $x \in \mathbb{R}^n$. The multivalued mapping $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is compact-valued and locally Lipschitz in the Hausdorff metric. Solutions of the differential inclusion are absolutely continuous functions $x : [t_0, \theta] \to \mathbb{R}^n$ satisfying the condition $\dot{x}(t) \in F(x(t))$ for almost all t.

The state constraints have the form

$$x(t) \in S, \quad t \in [t_0, \theta], \tag{2.2}$$

where S is a closed set in $\mathbb{R}^n$ containing the vector x^0. In what follows, we consider as S the set

$$S = \{x \in \mathbb{R}^n \colon g_i(x) \le 0,\ i = 1, \dots k\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex continuously differentiable functions with locally Lipschitz gradients.

Denote by $x(t, u(\cdot), x^0)$ a solution of system (2.1) with the initial condition $x(t_0) = x^0$. The reachable set (domain) of system (2.1) with state constraint (2.2) at time θ is the set

$$G_0(\theta) = \big\{x \in \mathbb{R}^n \colon\ \exists u(\cdot) \in \mathcal{U},\ x = x(\theta, u(\cdot), x^0),\ x(t, u(\cdot), x^0) \in S, \quad t_0 \le t \le \theta\big\};$$

it is the set of all points to which system (2.1) can be taken at time θ from the initial state x^0 under constraints (2.2). In the present paper, we consider the problem of the approximate construction of $G_0(\theta)$. The original control system is replaced by a family of control systems without state constraints depending on a penalty parameter ε:

$$\dot{x}(t) = f_\varepsilon(x(t), u(t)), \quad x(t_0) = x^0. \tag{2.3}$$

The reachable sets of these systems are constructed without regard to the state constraints, and they approximate $G_0(\theta)$ as $\varepsilon \to 0$.

3. APPROXIMATION OF REACHABLE SETS

In further constructions, we use the following inward pointing condition (see [17–20]).

Assumption 2. *For any* $x \in \partial S \cap B_R$,

$$\operatorname{co} F(x) \cap \operatorname{int} T_S(x) \ne \varnothing. \tag{3.1}$$

Here, $T_S(x)$ is the tangent cone to the set S at the point x, defined as follows:

$$T_S(x) = \big\{d \in \mathbb{R}^n : \lim_{\xi \to +0} \xi^{-1} d(x + \xi d, S) = 0\big\},$$

where $d(x, S)$ is the distance from x to the set S:

$$d(x, S) = \min_{y \in S} \|x - y\|.$$

This condition provides the nonemptiness of the reachable set $G_0(\theta)$.

Define a function $g : \mathbb{R}^n \to \mathbb{R}$:

$$g(x) = \max_{1 \le i \le k} g_i(x); \tag{3.2}$$

the function $g(x)$ is convex and, obviously, $S = \{x \in \mathbb{R}^n : g(x) \le 0\}$.

For $x \in \mathbb{R}^n$, we set

$$I(x) = \{i \in \{1, \dots k\} \colon g_i(x) = g(x)\};$$

$I(x)$ is the set of indices i at which the maximum is attained in (3.2). In the sequel, we assume that the following condition is satisfied.

Assumption 3. *At the points $x \in \partial S \cap B_R$, the gradients $\nabla g_i(x)$, $i \in I(x)$, are positively linearly independent.*[2]

Under this assumption, condition (3.1) can be written in an equivalent form (see [24]):

$$\max_{\lambda\in\Lambda(x)} \min_{f\in \operatorname{co} F(x)} \Big(\sum_{i=1}^{k} \lambda_i \nabla g_i(x), f\Big) < 0 \quad \forall x \in \{x \in \mathbb{R}^n\colon g(x) = 0\} \cap B_R, \tag{3.3}$$

where

$$\Lambda(x) = \Big\{\lambda \in \mathbb{R}^k\colon \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1;\ \lambda_i = 0 \text{ for } i \notin I(x)\Big\}.$$

Applying the minimax theorem and interchanging the minimum and maximum in (3.3), we get

$$\min_{f\in \operatorname{co} F(x)} \max_{\lambda\in\Lambda(x)} \Big(\sum_{i=1}^{k} \lambda_i \nabla g_i(x), f\Big) < 0.$$

Since

$$\max_{\lambda\in\Lambda(x)} \Big(\sum_{i=1}^{k} \lambda_i \nabla g_i(x), f\Big) = \max_{i\in I(x)} (\nabla g_i(x), f),$$

we find that inequality (3.3) is equivalent to the condition

$$\min_{f\in \operatorname{co} F(x)} \max_{i\in I(x)} \big(\nabla g_i(x), f\big) < 0 \quad \forall x \in \{x \in \mathbb{R}^n\colon g(x) = 0\} \cap B_R.$$

Assertion. *If condition (3.3) holds, then there exist $\sigma > 0$ and $\rho > 0$ such that the inequality*

$$\min_{f\in \operatorname{co} F(x)} \max_{i\in I(x)} (\nabla g_i(x), f) < -\rho \tag{3.4}$$

is valid for all points of the set

$$S_R^\sigma = \{x\colon 0 \leq g(x) \leq \sigma\} \cap B_R.$$

Proof. By contradiction, assume that, for any $\sigma > 0$ and $\rho > 0$, there exists a vector $x^{\sigma,\rho} \in B_R$ such that

$$\min_{f\in \operatorname{co} F(x^{\sigma,\rho})} \max_{i\in I(x^{\sigma,\rho})} (\nabla g_i(x^{\sigma,\rho}), f) \geq -\rho, \quad 0 \leq g(x^{\sigma,\rho}) \leq \sigma. \tag{3.5}$$

Choose sequences of positive numbers σ_m and ρ_m such that $\sigma_m \to 0$ and $\rho_m \to 0$ as $m \to \infty$, and define $x^m = x^{\sigma_m,\rho_m}$. The sequence $x^m \in B_R$ contains a convergent subsequence; without loss of generality, we can assume that $x^m \to \bar{x} \in B_R$. It follows from the continuity of $g(x)$ that $g(\bar{x}) = 0$. Choose $f^m \in \operatorname{co} F(x^m)$ that realizes the minimum in the left-hand side of inequality (3.5) for $\sigma = \sigma_m$, $\rho = \rho_m$, and $x^{\sigma,\rho} = x^m$. The sequence f^m is bounded and satisfies the inequality

$$\max_{i\in I(x^m)} (\nabla g_i(x^m), f^m) \geq -\rho^m;$$

[2] Vectors $a^i \in \mathbb{R}^n$, $i = 1,\dots,m$, are called positively linearly independent if, for any $\alpha_i \geq 0$, $i = 1,\dots,m$, the equality $\sum_{i=1}^m \alpha_i a^i = 0$ implies that $\alpha_i = 0$, $i = 1,\dots,m$.

without loss of generality, we assume that $f^m \to \bar{f} \in \operatorname{co} F(\bar{x})$. Let $i \notin I(\bar{x})$; then $g_i(\bar{x}) < g(\bar{x})$. By the continuity of the functions $g_i(x)$ and $g(x)$, for sufficiently large m, we have $g_i(x^m) < g(x^m)$, which is equivalent to the condition $i \notin I(x^m)$. Consequently, $I(x^m) \subset I(\bar{x})$ and

$$\max_{i\in I(\bar{x})}(\nabla g_i(x^m), f^m) \geq \max_{i\in I(x^m)}(\nabla g_i(x^m), f^m) \geq -\rho^m.$$

The function $\Psi(x, f) = \max_{i\in I(\bar{x})}(\nabla g_i(x), f)$ is continuous. Therefore, passing in the last inequality to the limit as $m \to \infty$, we find that

$$\max_{i\in I(\bar{x})}(\nabla g_i(\bar{x}), \bar{f}) \geq 0, \quad \bar{x} \in \operatorname{co} F(\bar{x}), \quad g(\bar{x}) = 0,$$

which contradicts condition (3.5). □

In what follows, we will use the following strengthening of condition (3.4).

Assumption 4. *There exist $\sigma > 0$, $\rho > 0$, and a Lipschitz function $\bar{f}(x)$ defined on the set S_R^σ such that*

$$\max_{i\in I(x)}(\nabla g_i(x), \bar{f}(x)) < -\rho, \quad \bar{f}(x) \in \operatorname{co} F(x) \quad \forall x \in S_R^\sigma.$$

Under this assumption, we define the right-hand side $f_\varepsilon(x, u)$ of control system (2.3) on the set $\{x \in \mathbb{R}^n : g(x) \leq \sigma\} \cap B_R$ as follows. Choose $0 < \varepsilon < \sigma$. Let $h_\varepsilon(\tau) : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $0 \leq h_\varepsilon(\tau) \leq 1$, $h_\varepsilon(\tau) = 1$ for $\tau < 0$, and $h_\varepsilon(\tau) = 0$ for $\tau > \varepsilon$. Define

$$f_\varepsilon(x, u) = \begin{cases} h_\varepsilon(g(x))f(x, u) + (1 - h_\varepsilon(g(x)))\bar{f}(x) & \text{for} \quad g(x) > 0, \\ f(x, u) & \text{for} \quad g(x) \leq 0. \end{cases}$$

As $h_\varepsilon(\tau)$, we can take the linear–quadratic function

$$h_\varepsilon(\tau) = \begin{cases} 1 & \text{for} \quad \tau < 0, \\ 1 - \dfrac{2\tau^2}{\varepsilon^2} & \text{for} \quad 0 \leq \tau \leq \varepsilon/2, \\ \dfrac{2(\tau - \varepsilon)^2}{\varepsilon^2} & \text{for} \quad \varepsilon/2 \leq \tau \leq \varepsilon, \\ 0 & \text{for} \quad \tau > \varepsilon. \end{cases}$$

Theorem 1. *Let $f(x, u)$ and the constraints of the problem satisfy Assumptions* 1, 3, *and* 4. *Then*:

(1) *for $0 < \varepsilon < \sigma$, the mapping $f_\varepsilon(x, u)$ is continuous on $\{x \in \mathbb{R}^n : g(x) \leq \sigma\} \cap B_R \times U$ and Lipschitz in x uniformly in $u \in U$;*

(2) *for any $u(\cdot) \in \mathcal{U}$, the solution $x_\varepsilon(t)$ of system* (2.3) *with the initial condition $x_\varepsilon(t_0) = x^0$ can be continued to $[t_0, \theta]$ and satisfies the inequality*

$$g(x_\varepsilon(t)) \leq \varepsilon, \quad t \in [t_0, \theta].$$

Proof. The first part of the proof repeats almost word-for-word the proof of Theorem 1 in [24]. Fix $\varepsilon > 0$. On the set $S_1 \times U$, where $S_1 = \{x : g(x) \leq 0\} \cap B_R$, the function $f_\varepsilon(x, u)$ coincides with $f(x, u)$; hence, it is continuous. For $(x, u) \in S_2 \times U$, where $S_2 = \{x : 0 \leq g(x) \leq \sigma\} \cap B_R$, $f_\varepsilon(x, u)$ is continuous as the superposition of continuous functions. The points x where $g(x) = 0$ belong to each of the sets S_1 and S_2; therefore, the continuity of $f_\varepsilon(x, u)$ at these points follows from its continuity on these sets. To prove the Lipschitz condition for $f_\varepsilon(x,u)$, we note that there exist constants $L_1, L_2 > 0$, independent of u, such that $\forall i = 1,2$

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \le L_i\|x-y\| \quad \forall x,y \in S_i \ \ \forall u \in U.$$

For $x, y \in S_1$, the inequality follows from Assumption 1. On $S_2 \times U$, $f_\varepsilon(x,u)$ is the superposition of functions Lipschitz in x (a function convex on $\mathbb{R}^n$ is Lipschitz on any bounded set). We take $x \in S_1$ and $y \in S_2$ and connect these points by a line segment. At the ends of the segment, the function g takes values of different signs; hence, on this segment, there exists a point z at which $g(x) = 0$. In view of the fact that $z \in S_i$, $i = 1,2$, we find that

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \le |f_\varepsilon(x,u) - f_\varepsilon(z,u)| + |f_\varepsilon(z,u) - f_\varepsilon(y,u)|$$

$$\le L_1\|x-z\| + L_2\|y-z\| \le \max\{L_1,L_2\}(\|x-z\| + \|y-z\|) = \max\{L_1,L_2\}\|x-y\| \ \ \forall u \in U.$$

Consider the solution $x_\varepsilon(t)$ of system (2.3) corresponding to the control $u(\cdot) \in \mathcal{U}$. Since $f_\varepsilon(x,u)$ is a convex combination of the vectors $f(x,u)$ and $f(x,\bar{u}(x))$, which belong to the convex set $\operatorname{co} F(x)$, we have the inclusion $\dot{x}_\varepsilon(t) \in \operatorname{co} F(x_\varepsilon(t))$ for almost all t. Let us prove that this trajectory does not leave the set $\{x \in \mathbb{R}^n : g(x) \le \sigma\} \cap B_R$ where the right-hand side $f_\varepsilon(x,u)$ of system (2.3) is defined. Since the solution $x_\varepsilon(t)$ of the differential inclusion $\dot{x}_\varepsilon(t) \in \operatorname{co} F(x_\varepsilon(t))$ can be approximated arbitrarily closely in the uniform metric by solutions of the inclusion $\dot{x}(t) \in F(x(t))$ [25], we have $x_\varepsilon(t) \in B_R$ for all values of t for which the solution is defined. Let γ^* be the maximum among numbers γ not exceeding θ such that the solution $x_\varepsilon(t)$ is defined on $[t_0, \gamma]$. Let us prove that the inequality $g(x_\varepsilon(t)) \le \varepsilon$ is fulfilled at all points $[t_0, \gamma^*]$. Assume by contradiction that $g(x_\varepsilon(t^*)) > \varepsilon$ for some $t^* \in [t_0, \gamma^*]$. Let

$$t_* = \max\{t : t \in [t_0, t^*], g(x_\varepsilon(t)) = \varepsilon\}.$$

The function $g(x_\varepsilon(t))$ is Lipschitz and, hence, is differentiable almost everywhere. We estimate the quantity $\dfrac{d}{dt} g(x_\varepsilon(t))$ at the points $t \in [t_*, t^*]$ where this derivative and the derivative $\dot{x}_\varepsilon(t)$ exist. Define $\Delta x_\varepsilon(t) = x_\varepsilon(t + \delta t) - x_\varepsilon(t)$; then

$$g_i(x_\varepsilon(t+\delta t)) - g_i(x_\varepsilon(t)) = (\nabla g_i(x_\varepsilon(t)), \Delta x_\varepsilon(t)) + o_i(\Delta x_\varepsilon(t)), \quad i = 1,\dots,k, \tag{3.6}$$

where $o_i(\eta)/\eta \to 0$ as $\eta \to 0$. Substituting $\Delta x_\varepsilon(t) = \dot{x}_\varepsilon(t)\Delta t + o(\Delta t)$ $(o(\Delta t)/\Delta t \to 0,\ \Delta t \to 0)$ into equality (3.6), we get

$$g_i(x_\varepsilon(t+\Delta t)) - g_i(x_\varepsilon(t)) = (\nabla g_i(x_\varepsilon(t)), \dot{x}_\varepsilon(t))\Delta t + \alpha_i(\Delta t), \quad i = 1,\dots,k, \tag{3.7}$$

where

$$\alpha_i(\Delta t) = \nabla g_i(x_\varepsilon(t), o(\Delta t)) + o_i(\dot{x}_\varepsilon(t)\Delta t + o(\Delta t)).$$

Obviously, $\alpha_i(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

For $i \in I(x_\varepsilon(t))$, we have $g_i(x_\varepsilon(t)) = g(x_\varepsilon(t))$. If $i \notin I(x_\varepsilon(t))$, then $g_i(x_\varepsilon(t)) < g(x_\varepsilon(t))$; consequently, by the continuity of $g(x)$, $g_i(x)$, and $x_\varepsilon(t)$ for sufficiently small Δt, we find that $g_i(x_\varepsilon(t+\Delta t)) < g(x_\varepsilon(t+\Delta t))$. Thus, $g(x_\varepsilon(t+\Delta t)) = \max_{i \in I(x_\varepsilon(t))} g_i(x_\varepsilon(t+\Delta t))$. In view of the above, passing in both sides of equality (3.7) to the maximum over $i \in I(x_\varepsilon(t))$, we obtain

$$g(x_\varepsilon(t+\Delta t)) - g(x_\varepsilon(t)) \le \Delta t \max_{i \in I(x_\varepsilon(t))} (\nabla g_i(x_\varepsilon(t)), \dot{x}_\varepsilon(t)) + \max_{i \in I(x_\varepsilon(t))} \alpha_i(\Delta t).$$

In the limit as $\Delta t \to 0$, we obtain from the above inequality that

$$\frac{d}{dt} g(x_\varepsilon(t)) \le \max_{i \in I(x_\varepsilon(t))} (\nabla g_i(x_\varepsilon(t)), \dot{x}_\varepsilon(t)).$$

Since $g(x_\varepsilon(t)) \ge \varepsilon$ on the interval $[t_*, t^*]$, we have $h_\varepsilon g(x_\varepsilon(t)) = 0$ and, consequently,

$$\dot{x}_\varepsilon(t) = f_\varepsilon(x_\varepsilon(t), u(t)) = \bar{f}(x_\varepsilon(t)).$$

From the definition of $\bar{f}(x)$, we find that $\dfrac{d}{dt} g(x_\varepsilon(t)) \le -\rho < 0$ for almost all $t \in [t_*, t^*]$, which implies that $g(x_\varepsilon(t_*)) > g(x_\varepsilon(t^*))$, contrary to the assumption. The theorem is proved.

Lemma 1. *Let S be a set in $\mathbb{R}^n$ given by the system of inequalities $S = \{x \in \mathbb{R}^n \colon g_i(x) \le 0, i = 1, \dots, m\}$, and let the functions $g_i(x)$, $i = 1, \dots, m$, be convex and satisfy Slater's condition*:

$$\exists \bar{x} \in \mathbb{R}^n, \quad g_i(\bar{x}) < 0, \quad i = 1, \dots, m.$$

Let D be a bounded subset of $\mathbb{R}^n$. Then there exists a constant $M > 0$ such that

$$d(x^*, S) \le M \max\big\{ \max_{i=1,\dots,m} g_i(x^*), 0\big\} \ \forall x^* \in D. \tag{3.8}$$

Proof. Assume that $g(x) = \max_{i=1,\dots,m} g_i(x)$, the function $g(x)$ is convex, and $S = \{x \in \mathbb{R}^n : g(x) \le 0,\ i = 1, \dots, m\}$. Define $h = -g(\bar{x}) > 0$. We take an arbitrary point $x^* \notin S$ and connect $\bar{x}$ with x^* by a line segment. Points of this segment have the form $x(\lambda) = x^* + \lambda(\bar{x} - x^*)$, where $0 \le \lambda \le 1$. Since $g(x(0)) > 0$ and $g(x(1)) < 0$, there exists λ in the interval $[0, 1]$ such that $g(x(\lambda)) = 0$. It follows from the convexity of $g(x)$ that

$$0 = g(x(\lambda)) \le \lambda g(\bar{x}) + (1 - \lambda) g(x^*) = -\lambda h + (1 - \lambda) g(x^*),$$

whence

$$\lambda \le \frac{g(x^*)}{h + g(x^*)} \le \frac{g(x^*)}{h}.$$

The equality $x(\lambda) - x^* = \lambda(\bar{x} - x^*)$ implies that $\lambda = \|x(\lambda) - x^*\| / \|\bar{x} - x^*\|$.

As a result, in view of the inclusion $x(\lambda) \in S$, we come to the inequality

$$d(x^*, S) \le \|x(\lambda) - x^*\| \le \frac{g(x^*)}{h} \|\bar{x} - x^*\| \le M g(x^*)$$

for $M = \max_{x^* \in D} \|\bar{x} - x^*\| / h$, which completes the proof for $x^* \notin S$. For $x^* \in S$, the inequality is obvious. □

Lemma 2. *Assume that $S = \{x \in \mathbb{R}^n \colon g(x) \le 0,\ i = 1, \dots, m\}$, where $g_i(x)$, $i = 1, \dots, m$, are convex continuously differentiable functions satisfying Assumption* 3. *Then, there exists $M > 0$ such that inequality* (3.8) *holds for all $x \in B_R$.*

Proof. Indeed, let us choose any point $x \in \partial S$. By the condition, the gradients $\nabla g_i(x)$, $i \in I(x)$, are positively linearly independent. Then, there exists $h \in \mathbb{R}^n$ such that $(\nabla g_i(x), h) < 0$, $i \in I(x)$ (see [26]). We take $\bar{x} = x + \xi h$; for small positive ξ, we have $g_i(\bar{x}) < 0$, $i = 1, \dots, k$; i.e., Slater's condition holds. To complete the proof, we apply Lemma 1. □

Theorem 2. *Let $f(x, u)$ and the constraints of the problem satisfy Assumptions* 1, 3, *and* 4. *Then, for any $0 < \varepsilon < \sigma$, we have the inclusion $G_0(\theta) \subset G_\varepsilon(\theta)$. There exists a constant $L > 0$ such that*

$$h(G_0(\theta), G_\varepsilon(\theta)) \leq L\varepsilon. \tag{3.9}$$

Proof. The trajectories of auxiliary system (2.3) are trajectories of the differential inclusion

$$\dot{x}(t) \in \operatorname{co} F(x(t)), \quad x(t_0) = x^0,$$

where $F(x) = f(x, U)$ is a compact-valued locally Lipschitz multivalued mapping. Fix $\varepsilon_0 = \sigma/2$ and consider $\varepsilon < \varepsilon_0$. For any $\delta > 0$ and any trajectory $x_\varepsilon(t)$ of system (2.3), there exists a trajectory $\bar{x}_\varepsilon(t)$ of the differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{3.10}$$

(of control system (2.1)) such that (see [25])

$$\max_{t\in[t_0,t_1]} \|\bar{x}_\varepsilon(t) - x_\varepsilon(t)\| \leq \delta.$$

We can choose δ so small that

$$\max_{t\in[t_0,t_1]} g(\bar{x}_\varepsilon(t)) < 3\frac{\varepsilon}{2}.$$

By [21, Theorem 1], there exists a constant $K > 0$ such that, for any trajectory $\bar{x}_\varepsilon(t)$ of (3.10), there exists a trajectory $\hat{x}(t)$ of (3.10) satisfying the state constraints $\hat{x}(t) \in S$ and the inequality

$$\max_{t\in[t_0,t_1]} \|\bar{x}_\varepsilon(t) - \hat{x}(t)\| \leq K \max_{t\in[t_0,t_1]} (d(\bar{x}_\varepsilon(t)), S). \tag{3.11}$$

By Lemmas 1 and 2, there exists $M > 0$ such that

$$d(\bar{x}_\varepsilon(t), S) \leq M \max\big\{ \max_{i=1,\ldots,m} g_i(\bar{x}_\varepsilon(t)), 0\big\} \leq \frac{3M\varepsilon}{2};$$

consequently, inequality (3.11) can be written in the form

$$\max_{t\in[t_0,t_1]} \|\bar{x}_\varepsilon(t)) - \hat{x}(t))\| \leq \frac{3KM\varepsilon}{2}.$$

Since, for $g(x) \leq 0$, we have $f_\varepsilon(x, u) = f(x, u)\ \forall u \in U$, it follows that $G_0(\theta) \subset G_\varepsilon(\theta)$. For $\hat{x}(\theta) \in G_0(\theta)$ and $x_\varepsilon(\theta) \in G_\varepsilon(\theta)$, we have

$$\|\hat{x}(\theta) - x_\varepsilon(\theta)\| \leq \delta + \frac{3KM\varepsilon}{2}.$$

Since δ can be chosen arbitrarily small, we obtain (3.9) for $L = \dfrac{3KM}{2}$. Theorem 2 is proved.

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