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Weighted Extrapolation in Iwaniec–Sbordone Spaces. Applications to Integral Operators and Approximation Theory

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Abstract—We prove extrapolation theorems in weighted Iwaniec–Sbordone spaces and apply them to one-weight inequalities for several integral operators of harmonic analysis. In addition, in weighted grand Lebesgue spaces, we establish Bernstein and Nikol'skii type inequalities and prove direct and inverse theorems on the approximation of functions.

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INTRODUCTION

At the end of the 20th century, it became clear that classical function spaces are insufficient for solving a number of problems both in mathematics itself and in applied sciences. The necessity of introducing new function spaces arose, for example, in mathematical models of nonlinear elasticity theory, incompressible fluid dynamics, and physics (Lavrent'ev phenomenon). One of the new function spaces that appeared in the 1990s is the space $L^{p)}$ introduced by Iwaniec and Sbordone [13] in connection with finding minimal conditions for the integrability of the Jacobian. A somewhat more general function space $L^{p),\theta}$ was considered in [10], where the authors analyzed the inhomogeneous n-harmonic equation div $A(x,\Delta u)=\mu$. These spaces are now called the Iwaniec–Sbordone spaces or grand Lebesgue spaces in the literature. The intensive study of these spaces has also been motivated by the fact that they are well suited for solving the problems of existence, uniqueness, and regularity of solutions to a wide class of nonlinear partial differential equations.

The structural properties of grand Lebesgue spaces were studied in [3, 7]. In [8], the authors proved that the maximal Hardy–Littlewood operator is bounded in the space $L^{p)}_w([0,1])$, $1<p<\infty$, if and only if the weight w belongs to the Muckenhoupt class. A similar statement for the Hilbert transforms was obtained in [17] for $L^{p),\theta}_w([0,1])$. One-weight inequalities for various singular integrals and maximal functions in the spaces $L^{p),\theta}_w$ were later established in [14–16, 29]. In [24, 19, 20], criteria for the existence of analogs of the well-known Sobolev theorem were proved. These criteria have the form of necessary and sufficient conditions for the weights and the parameter θ that guarantee the validity of the relevant inequality. In [18], for fractional integrals defined on quasimetric spaces with measure satisfying the doubling condition, we proved a criterion for the validity of a trace inequality in the spaces $L^{p),\theta}_w$.

The present paper is devoted to weighted extrapolation theorems in Iwaniec–Sbordone spaces and to their applications to the mapping properties of a number of integral operators generated by integral transformations defined on general quasimetric spaces with measure satisfying the doubling condition. In addition, applying boundedness theorems for the above-mentioned operators, we also

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prove direct and inverse theorems on the approximation of 2π-periodic functions by trigonometric polynomials in a weighted grand Lebesgue space (more precisely, in its subspace that is the closure of smooth functions with respect to the norm of the original space).

The paper is organized as follows. In Section 1, we give preliminary definitions and present some known results. Section 2 is devoted to extrapolation theorems in weighted Iwaniec–Sbordone spaces. Extrapolation theory is known to go back to Rubio de Francia's paper [27]. Intensive studies on extrapolation problems have been continued to date. We have proved that if some pair of functions satisfies a one-weight inequality in a Lebesgue space with exponent r for all weights of the Muckenhoupt class A_r, then a one-weight inequality in Iwaniec–Sbordone spaces holds for the same pair of functions. Our result covers both cases: when the weight in the definition of the norm of a space occupies the position of a generating measure, and when the weight appears as a factor of an element of the space. Below, these spaces are denoted by $L^{p),\theta}_w$ and $\mathcal{L}^{p),\theta}_w$, respectively. In Section 3, from extrapolation theorems we derive boundedness results for a series of integral operators in weighted grand Lebesgue spaces. In Section 4, we apply the results of Section 3 to the proof of analogs of well-known fundamental inequalities for trigonometric polynomials. For the Weyl fractional derivatives of trigonometric polynomials in the space $\mathcal{L}^{p),\theta}_w$, we establish an inequality analogous to the well-known Bernstein inequality [2]. Next, based on this inequality and Sobolev type theorems presented in Section 3, we prove an inequality of the same type as Nikol'skii's important inequality [25] for the norms of trigonometric polynomials in different metrics. In Sections 5 and 6, using the above-mentioned inequalities, we establish direct and inverse approximation theorems for the fractional derivatives of 2π-periodic functions in the subspace of approximable functions of the space $\mathcal{L}^{p),\theta}_w$.

In this paper, we denote by the same symbols c and C generally different constants that are independent of functions and (in appropriate places) of n.

1. PRELIMINARY INFORMATION AND SOME KNOWN RESULTS

Let (X,d,μ) be a quasimetric space with measure μ and quasimetric d, which means that the following conditions hold:

(i) $d(x,y)=0$ if and only if $x=y$;

(ii) there exists a constant $k>1$ such that the inequality $d(x,y)\le k(d(x,z)+d(z,y))$ is valid for arbitrary $x,y,z\in X$;

(iii) $d(x,y)=d(y,x)$ for arbitrary $x,y\in X$.

For any $x\in X$ and $r>0$, the set $B(x,r):=\{y\in X\colon d(x,y)<r\}$ is called a *ball* in X.

It is assumed that the measure μ is finite and is defined on a σ-algebra of subsets of X that contains all balls. Everywhere below, we also assume that $0<\mu B(x,r)<\infty$ and $\mu\{x\}=0$ for all $x\in X$ and $r>0$. If the measure μ satisfies the doubling condition

$$\mu B(x,2r)\le C\mu B(x,r), \tag{1.1}$$

where the constant C is independent of $x\in X$ and $r>0$, then (X,d,μ) is called a *space of homogeneous type*. It is well known [23] that for an arbitrary quasimetric space (X,d), there exists a continuous quasimetric ρ on X such that it is equivalent to d, each ball with respect to ρ is open in the topology induced by the quasimetric ρ, and there exist constants C and $\theta\in(0,1)$ such that

$$|\rho(x,z)-\rho(y,z)|\le C\rho^\theta(x,y)\big(\rho(x,z)+\rho(y,z)\big)^{1-\theta}.$$

Without loss of generality, we will assume that the quasimetric d is continuous and all balls are open with respect to d. Suppose also that the space of continuous functions with compact support is everywhere dense in the space of μ-measurable functions. For the corresponding definitions, examples, and some properties of spaces of homogeneous type, see [23, 4, 30].

If C_μ is the least constant for which (1.1) holds, then the number

$$D_\mu := \log_2 C_\mu \tag{1.2}$$

is called the *doubling order of the measure* μ.

Let $1 \le r < \infty$. Denote by $L^r(X,\mu)$ the Lebesgue space on X. If w is a weight function (i.e., a positive function on X that is locally integrable almost everywhere in the sense of the measure μ), then we denote by $L^p_w(X,\mu)$ the Lebesgue space with weight w, i.e., $f \in L^r_w(X,\mu)$ if $\|f\|_{L^r_w(X,\mu)} = \|f\|_{L^r(X,w\,d\mu)} < \infty$.

Let $\mu X < \infty$ and $1 < p < \infty$. Let, next, φ be a positive continuous function on the interval $(0,p-1)$ such that it does not decrease on $(0,\sigma)$ for some sufficiently small positive σ and $\lim_{x\to 0+}\varphi(x) = 0$.

The *generalized grand Lebesgue space* $L^{p),\varphi}(X,\mu)$ is defined as the set of those $f\colon X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi}(X)} = \|f\|_{L^{p),\varphi}(X,\mu)} = \sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\int\limits_X |f(x)|^{p-\varepsilon}\,d\mu(x)\right)^{1/(p-\varepsilon)}$$

is finite. The *weighted grand Lebesgue space*, denoted by $L^{p),\varphi}_w(X,\mu)$, coincides with $L^{p),\varphi}(X,w\,d\mu)$, i.e., $\|f\|_{L^{p),\varphi}_w(X,\mu)} = \|f\|_{L^{p),\varphi}(X,w\,d\mu)}$.

Along with the space $L^{p),\varphi}_w(X,\mu)$, we also consider the space $\mathcal{L}^{p),\varphi}_w(X,\mu)$ defined by the norm $\|wf\|_{L^{p),\varphi}(X,d\mu)}$.

For $\varphi(x) = x^\theta$, $\theta > 0$, we will denote the spaces $L^{p),\varphi}(X,\mu)$ and $L^{p),\varphi}_w$ by $L^{p),\theta}(X,\mu)$ and $L^{p),\theta}_w(X,\mu)$, respectively. The same also applies to the spaces $\mathcal{L}^{p),\varphi}(X,\mu)$ and $\mathcal{L}^{p),\varphi}_w(X,\mu)$ for $\varphi(x) = x^\theta$.

It is easy to see that in general $\|w^{1/p}f\|_{L^{p),\varphi}(X)} \ne \|f\|_{L^{p),\varphi}_w(X)}$, so the weighted space $\mathcal{L}^{p),\varphi}_{w^{1/p}}(X,\mu)$ is different from the space $L^{p),\varphi}_w(X,\mu)$.

The space $L^{p),\theta}(X,\mu)$ is known to be a Banach function space (see, for example, [7]). One can easily verify that the following continuous embeddings hold for $0 < \varepsilon \le p-1$ and $\theta_1 < \theta_2$:

$$L^p(X,\mu) \hookrightarrow L^{p),\theta_1}(X,\mu) \hookrightarrow L^{p),\theta_2}(X,\mu) \hookrightarrow L^{p-\varepsilon}(X,\mu).$$

Let $1 < r < \infty$. A weight w is said to belong to the *Muckenhoupt class* $A_r(X)$ if

$$[w]_{A_r} := \sup_B \left(\frac{1}{\mu B}\int\limits_B w(x)\,d\mu\right)\left(\frac{1}{\mu B}\int\limits_B w^{1-r'}(x)\,d\mu\right)^{r-1} < \infty, \qquad r' = \frac{r}{r-1}.$$

Next, $w \in A_1(X)$ if

$$Mw(x) \le Cw(x) \tag{1.3}$$

almost everywhere in the sense of the measure μ, where M denotes the maximal Hardy–Littlewood operator defined on a space of homogeneous type as

$$Mg(x) = \sup_{B\ni x}\frac{1}{\mu B}\int\limits_B |g(y)|\,d\mu(y).$$

Denote the best constant in inequality (1.3) by $[w]_{A_1(X)}$.

By Hölder's inequality, we have

$$[w]_{A_r} \le [w]_{A_s}, \qquad 1 < s < r < \infty. \tag{1.4}$$

The Muckenhoupt classes possess the following important property of openness: if $w \in A_r(X)$, then there exists a positive number σ, $0 < \sigma < r-1$, such that $w \in A_{r-\sigma}(X)$.

The equality

$$\left[w^{1-p'}\right]_{A_{p'}(X)} = [w]^{p'-1}_{A_p(X)}, \qquad 1 < p < \infty, \quad p' = \frac{p}{p-1}, \tag{1.5}$$

is verified directly.

The class $A_\infty(X)$ is defined as $A_\infty(X) = \bigcup_{r>1} A_r(X)$.

Let $1 < p, q < \infty$ and ρ be a function that is positive almost everywhere in the sense of the measure μ and such that ρ^q is locally integrable. It is said that $\rho \in \mathcal{A}_{p,q}(X)$ if

$$[\rho]_{\mathcal{A}_{p,q}(X)} := \sup_B \left(\frac{1}{\mu B} \int\limits_B \rho^q(x)\, d\mu \right) \left(\frac{1}{\mu B} \int\limits_B \rho^{-p'}(x)\, d\mu \right)^{q/p'} < \infty.$$

For $p = q$, we set $\mathcal{A}_{p,q} = \mathcal{A}_p$.

It is easy to see that

$$[\rho]_{\mathcal{A}_{p,q}(X)} = [\rho^q]_{A_{1+q/p'}(X)}, \qquad 1 < p \le q < \infty. \tag{1.6}$$

For $p = q$, equality (1.6) takes the form

$$[\rho]_{\mathcal{A}_p(X)} = [\rho^p]_{A_p(X)}, \qquad 1 < p < \infty.$$

By the Lebesgue theorem on the differentiation of an indefinite integral for (X, d, μ), the following inequalities are valid:

$$[w]_{A_p(X)} \ge 1, \qquad [\rho]_{\mathcal{A}_{p,q}(X)} \ge 1.$$

For the maximal Hardy–Littlewood operator defined on a space of homogeneous type, there exists a constant $\overline{c}$ for which Buckley's estimate

$$\|M\|_{L^p_w(X) \to L^p_w(X)} \le \overline{c} p' [w]^{1/(p-1)}_{A_p(X)}, \qquad 1 < p < \infty, \tag{1.7}$$

holds (see [12]).

Now we formulate known extrapolation theorems for weighted Lebesgue spaces.

The following statements in the case of Euclidean spaces were proved in [6] (see also [11] in the nondiagonal case). For operators in the diagonal case, a similar theorem is contained in [5], and in the nondiagonal case, in [21]. See also [9, p. 548] for similar results.

Theorem A (diagonal case). *Let (X, d, μ) be a space of homogeneous type. Suppose that for some family of pairs of nonnegative measurable functions (f, g), a number $p_0 \in [1, \infty)$, and any $w \in A_{p_0}(X)$, the inequality*

$$\left(\int\limits_X g^{p_0} w\, d\mu \right)^{1/p_0} \le CN\big([w]_{A_{p_0}(X)}\big) \left(\int\limits_X f^{p_0} w\, d\mu \right)^{1/p_0} \tag{1.8}$$

is satisfied, where N is an increasing function and the constant C is independent of w. Then, for an arbitrary p, $1 < p < \infty$, and any $w \in A_p(X)$, we have

$$\left(\int\limits_X g^p w\,d\mu\right)^{1/p} \le CK(w)\left(\int\limits_X f^p w\,d\mu\right)^{1/p}, \tag{1.9}$$

where C is the same constant as in (1.8) *and*

$$K(w) = \begin{cases} N\Big([w]_{A_p(X)}\big(2\|M\|_{L^p_w(X)\to L^p_w(X)}\big)^{p_0-p}\Big), & p < p_0,\\ N\Big([w]_{A_p(X)}^{(p_0-1)/(p-1)}\Big(2\|M\|_{L^{p'}_{w^{1-p'}}(X)\to L^{p'}_{w^{1-p'}}(X)}\Big)^{(p-p_0)/(p-1)}\Big), & p > p_0.\end{cases}$$

Theorem B (nondiagonal case)**.** *Let (X,d,μ) be a space of homogeneous type. Suppose that for some family of pairs (f,g) of nonnegative measurable functions, some $p_0 \in [1,\infty)$ and $q_0 \in (0,\infty)$, and any $w \in \mathcal{A}_{p_0,q_0}(X)$, we have*

$$\left(\int\limits_X g^{q_0} w^{q_0}\,d\mu\right)^{1/q_0} \le CN\big([w]_{\mathcal{A}_{p_0,q_0}(X)}\big)\left(\int\limits_X f^{p_0} w^{p_0}\,d\mu\right)^{1/p_0}, \tag{1.10}$$

where N is an increasing function and the constant C is independent of w. Then, for all p, $1 < p < \infty$, and q, $0 < q < \infty$, such that

$$\frac{1}{q_0} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p}$$

and arbitrary $w \in \mathcal{A}_{p,q}(X)$, we have

$$\left(\int\limits_X g^q w^q\,d\mu\right)^{1/q} \le CK(w)\left(\int\limits_X f^p w^p\,d\mu\right)^{1/p}, \tag{1.11}$$

where C is the same constant as in (1.10) *and*

$$K(w) = \begin{cases} N\Big([w]_{\mathcal{A}_{p,q}(X)}\big(2\|M\|_{L^{\gamma q}_{w^q}(X)\to L^{\gamma q}_{w^q}(X)}\big)^{\gamma(q-q_0)}\Big), & q < q_0,\\ N\Big([w]_{\mathcal{A}_{p,q}(X)}^{(\gamma q_0-1)/(\gamma q-1)}\Big(2\|M\|_{L^{\gamma p'}_{w^{-p'}}(X)\to L^{\gamma p'}_{w^{-p'}}(X)}\Big)^{\gamma(q-q_0)/(\gamma q-1)}\Big), & q > q_0,\end{cases}$$

with $\gamma := 1/q_0 + 1/p_0'$.

Remark. It is easy to see that inequality (1.7) implies the estimate

$$K(w) \le \begin{cases} N\bigg[\Big(2\overline{c}\Big(1+\dfrac{p'}{q}\Big)\Big)^{\gamma(q-q_0)}[w^q]_{A_{1+q/p'}(X)}^{1+\gamma(q-q_0)p'/q}\bigg], & q < q_0,\\ N\bigg[\Big(2\overline{c}\Big(1+\dfrac{q}{p'}\Big)\Big)^{\gamma(q-q_0)/(\gamma q-1)}[w^q]_{A_{1+q/p'}(X)}\bigg], & q > q_0.\end{cases} \tag{1.12}$$

The proofs of Theorems A and B are based on the arguments used in [6]; however, for completeness, below we present a detailed proof, because here we deal with a space of homogeneous type and, moreover, we are interested in exact constants.

The proofs of Theorems A and B are based on the following lemmas (see [6] in the case of Euclidean spaces).

Lemma 1.1. (a) *Let* $1\le p<p_0<\infty$. *If* $w\in A_p(X)$ *and* $u\in A_1(X)$, *then* $wu^{p-p_0}\in A_{p_0}(X)$ *and*

$$\|wu^{p-p_0}\|_{A_{p_0}(X)}\le \|w\|_{A_p(X)}\|u\|^{p_0-p}_{A_1(X)}.$$

(b) *Let* $1<p_0<p<\infty$. *If* $w\in A_p(X)$ *and* $u\in A_1(X)$, *then* $(w^{p_0-1}u^{p-p_0})^{1/(p-1)}\in A_{p_0(X)}$ *and*

$$\big\|(w^{p_0-1}u^{p-p_0})^{1/(p-1)}\big\|_{A_p(X)}\le \|w\|^{(p_0-1)/(p-1)}_{A_p(X)}\|u\|^{(p-p_0)/(p-1)}_{A_1(X)}.$$

The proof of the lemma follows from Hölder's inequality and the obvious inequalities

$$\frac{1}{\mu B}\le\int\limits_B u(y)\,d\mu(y)\le Mu(x)\le\|u\|_{A_1(X)}u(x)$$

for almost every $x\in B$.

The following lemma is known as the Rubio de Francia algorithm.

Lemma 1.2. *Let* $p>1$. *Let* f *be a nonnegative function in* $L^p_w(X)$ *and* $w\in A_p(X)$. *Denote by* M^k *the* k*-th iteration of the operator* M *and set* $M^0f=f$. *Define the operator*

$$Rf(x)=\sum_{k=0}^{\infty}\frac{M^kf(x)}{(2\|M\|_{L^p_w\to L^p_w})^k}.$$

Then $f(x)\le Rf(x)$ *almost everywhere,* $\|Rf\|_{L^p_w(X)}\le 2\|f\|_{L^p_w(X)}$, Rf *is the weight of class* $A_1(X)$, *and*

$$[Rf]_{A_1(X)}\le 2\|M\|_{L^p_w(X)\to L^p_w(X)}. \tag{1.13}$$

This lemma follows from the estimate

$$\|M^k\|_{L^p_w(X)\to L^p_w(X)}\le\|M\|^k_{L^p_w(X)\to L^p_w(X)}$$

and the condition in the definition of the class $A_1(X)$.

When proving Theorem A, we will follow the scheme of the proof of Theorem 1.3 from [6].

Proof of Theorem A. First, let $p<p_0$ and $f\in L^p_w(X)$. By Hölder's inequality, assertion (a) of Lemma 1.1, and estimate (1.13), we obtain

$$\begin{aligned}\int\limits_X g^pw(x)\,d\mu&=\int\limits_X g^pw(Rf)^{p(p-p_0)/p_0}(Rf)^{p(p_0-p)/p_0}\,d\mu\\
&\le\Bigg(\int\limits_X g^{p_0}w(Rf)^{p-p_0}\,d\mu\Bigg)^{p/p_0}\Bigg(\int\limits_X (Rf)^pw(x)\,d\mu\Bigg)^{1-p/p_0}\\
&\le C^pN\big([(Rf)^{p-p_0}]_{A_{p_0}(X)}\big)^p\Bigg(\int\limits_X f^{p_0}w(Rf)^{p-p_0}\,d\mu\Bigg)^{p/p_0}\Bigg(\int\limits_X|f|^pw(x)\,d\mu\Bigg)^{1-p/p_0}\\
&\le C^pN\big([w]_{A_p(X)}[(Rf)^{p-p_0}]_{A_1(X)}\big)^p\int\limits_X|f|^pw(x)\,d\mu\\
&\le C^pN\big([w]_{A_p(X)}\big(2\|M\|_{L^p_w\to L^p_w}\big)^{p_0-p}\big)^p\int\limits_X|f|^pw(x)\,d\mu.\end{aligned}$$

Now, let $p > p_0$. Let, next, $h \in L_w^{p/(p-p_0)}(X)$ with the norm equal to 1. Define a function H so that $H^{p'} w^{1-p'} = h^{p/(p-p_0)} w$. Then the function H belongs to $L^{p'}_{w^{1-p'}}(X)$ and its norm in this space is 1. Using the pointwise estimate $H \le RH$ (which is valid almost everywhere) and the factorization Lemma 1.1 (assertion (b)), we obtain

$$\int\limits_X g^{p_0} h w \, d\mu \le \int\limits_X g^{p_0} w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)} \, d\mu$$

$$\le C^{p_0} N\Bigl(\bigl[w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)}\bigr]_{A_{p_0}(X)}\Bigr)^{p_0} \int\limits_X f^{p_0} w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)} \, d\mu$$

$$\le C^{p_0} N\Bigl([w]_{A_p(X)}^{(p_0-1)/(p-1)} \Bigl(2\|M\|_{L^{p'}_{w^{1-p'}} \to L^{p'}_{w^{1-p'}}}\Bigr)^{(p-p_0)/(p-1)}\Bigr)^{p_0}$$

$$\times \Biggl(\int\limits_X f^p w \, d\mu\Biggr)^{p_0/p} \Biggl(\int\limits_X (RH)^{p'} w^{1-p'} \, d\mu\Biggr)^{(1-p_0)/p}.$$

Taking the least upper bound with respect to h, we arrive at estimate (1.9). $\square$

Proof of Theorem B. First, notice that $[w]_{\mathcal{A}_{p_0,q_0}(X)} = [w^{q_0}]_{A_{q_0\gamma}(X)}$. On the other hand, $[w]_{\mathcal{A}_{p,q}(X)} = [w^q]_{A_{q\gamma}(X)}$, since we also have $\gamma = 1/q + 1/p'$.

Case of $q < q_0$. In this case, $p < p_0$. Set $f \in L^p_{w^p}(X)$. Define a function H by the equality $H^{q\gamma} w^q = f^p w^p$. Then $H \in L^{q\gamma}_{w^q}(X)$. Let us construct RH following the Rubio de Francia algorithm and apply Hölder's inequality. Then we have

$$\int\limits_X (gw)^q \, d\mu = \int\limits_X (gw)^q (RH)^{q\gamma(q-q_0)/q_0} (RH)^{q\gamma(q_0-q)/q_0} \, d\mu$$

$$\le \Biggl(\int\limits_X g^{q_0} w^q (RH)^{\gamma(q-q_0)} \, d\mu\Biggr)^{q/q_0} \Biggl(\int\limits_X \bigl((RH)^\gamma w\bigr)^q \, d\mu\Biggr)^{1-q/q_0}. \qquad (1.14)$$

Using Lemma 1.1, we find that $w^q (RH)^{\gamma(q-q_0)} \in A_{q_0\gamma}(X)$ and

$$\bigl[w^q (RH)^{\gamma(q-q_0)}\bigr]_{A_{q_0\gamma}(X)} \le [w^q]_{A_{q\gamma}(X)} [(RH)]_{A_1(X)}^{\gamma(q_0-q)}.$$

Now, according to (1.8), we derive the estimate

$$\Biggl(\int\limits_X g^{q_0} w^q (RH)^{\gamma(q-q_0)} \, d\mu\Biggr)^{1/q_0} \le CN\Bigl(\bigl[w^r (RH)^{\gamma(q-q_0)}\bigr]_{A_{q_0\gamma}(X)}\Bigr) \Biggl(\int\limits_X f^{p_0} w^q (RH)^{\gamma(q-q_0)} \, d\mu\Biggr)^{1/p_0}.$$

Taking into account this inequality in (1.14) and using the estimates

$$(RH)^{-1} \le H^{-1},$$

$$\|RH\|_{L^{\gamma q}_{w^q}(X)} \le 2\|H\|_{L^{\gamma q}_{w^q}(X)} = 2\|f\|^{p/(q\gamma)}_{L^p_{w^p}(X)},$$

$$[RH]_{A_1(X)} \le 2\|M\|_{L^{\gamma q}_{w^q}(X) \to L^{\gamma q}_{w^q}(X)},$$

which follow from the Rubio de Francia algorithm, we can see that

$$\left(\int_X (gw)^q \, d\mu\right)^{1/q} \le CN\Big([w]_{A_{q\gamma}(X)} \big(2\|M\|_{L^{\gamma q}_{w^q}(X)}\big)^{\gamma(q-q_0)}\Big) \left(\int_X (fw)^p \, d\mu\right)^{1/p}.$$

Case of $q > q_0$. In this case, $p > p_0$. Now we apply the dual argument. Let $h \in L^{q/(q-q_0)}_{w^q}(X)$ be a nonnegative function with the norm equal to 1. We can easily see that $[w]_{\mathcal{A}_{p,q}(X)} = [w^{-p'}]^{p'\gamma-1}_{A_{p'\gamma}(X)}$. Define a function $H \in L^{p'\gamma}_{w^{-p'}}$ by the equality $H^{p'\gamma}_{w^{-p'}} = h^{q/(q-q_0)} w^q$. If we define RH by the Rubio de Francia algorithm, then $RH \in A_1(X)$ and $H \le RH$. Hence,

$$\int_X g^{q_0} h w^q \, d\mu = \int_X g^{q_0} \big(H^{p'\gamma} w^{-(p'+q)}\big)^{(q-q_0)/q} w^q \, d\mu \le \int_X g^{q_0} w^{q_0 p'/p_0'} (RH)^{\gamma(q-q_0)/(q\gamma-1)} \, d\mu.$$

Notice that $(w^q)^{(q_0\gamma-1)/(q\gamma-1)}(RH)^{\gamma(q-q_0)/(q\gamma-1)}$ is a weight of class $A_{q_0\gamma}(X)$. Moreover,

$$\big[(w^q)^{(q_0\gamma-1)/(q\gamma-1)}(RH)^{\gamma(q-q_0)/(q\gamma-1)}\big]_{A_{q_0\gamma}(X)} \le [w^r]^{(q_0\gamma-1)/(q\gamma-1)}_{A_{r\gamma}(X)} [RH]^{(q-q_0)\gamma/(q\gamma-1)}_{A_1(X)}. \tag{1.15}$$

If we take into account (1.8), we obtain

$$\int_X g^{q_0} h w^q \, d\mu \le CN\Big(\big[(w^r)^{(q_0\gamma-1)/(q\gamma-1)}(RH)^{(q-q_0)\gamma/(q\gamma-1)}\big]_{A_{q_0\gamma}}\Big)^{q_0}$$
$$\times \left(\int_X f^{q_0} w^{p'(p_0-1)} (RH)^{p_0\gamma(q-q_0)/(q_0(q\gamma-1))}\right)^{q_0/p_0}.$$

Now, it suffices to apply Hölder's inequality for the exponent p/p_0 and take account of (1.15) and the inequality $[RH]_{A_1(X)} \le \|M\|_{L^{p'\gamma}_{w^{-p'}}(X)}$ to complete the proof of Theorem B. Inequality (1.7) yields the desired estimate for the constant in this case as well. □

2. EXTRAPOLATION IN WEIGHTED IWANIEC–SBORDONE SPACES

In this section, just as in the previous one, (X, d, μ) denotes a space of homogeneous type. Assume that $\mu X < \infty$.

Theorem 2.1 (diagonal case). *Let* $p_0 \in [1, \infty)$ *and* $\mathcal{F}(X)$ *be a class of pairs of nonnegative measurable functions defined on* X. *Suppose that for arbitrary* $(f, g) \in \mathcal{F}(X)$ *and all* $w \in A_{p_0}(X)$, *the inequality*

$$\left(\int_X g^{p_0} w \, d\mu\right)^{1/p_0} \le CN\big([w]_{A_{p_0}(X)}\big) \left(\int_X f^{p_0} w \, d\mu\right)^{1/p_0} \tag{2.1}$$

is valid, where N *is an increasing function and* C *is a positive constant independent of* (f, g) *and* w. *Then the following inequality holds for all* p, $1 < p < \infty$, $\theta > 0$, $w \in A_p(X)$, *and* $(f, g) \in \mathcal{F}$:

$$\|g\|_{L^{p),\theta}_w(X)} \le \overline{C} \|f\|_{L^{p),\theta}_w(X)}, \tag{2.2}$$

where $\overline{C}$ *is independent of* (f, g) *but depends on* w.

Theorem 2.2 (nondiagonal case)**.** *Let $1 < p_0 < \infty$ and $1 < q_0 < \infty$. Let $\mathcal{F}(X)$ denote a family of pairs of nonnegative measurable functions defined on X. Suppose that for all $(f,g) \in \mathcal{F}(X)$ and $w \in \mathcal{A}_{p_0,q_0}(X)$,*

$$\left(\int\limits_X (gw)^{q_0}\,d\mu\right)^{1/q_0} \le CN\big([w]_{\mathcal{A}_{p_0,q_0}(X)}\big)\left(\int\limits_X (fw)^{p_0}\,d\mu\right)^{1/p_0}, \tag{2.3}$$

where N is an increasing function and the constant C is independent of (f,g) and w. Then, for numbers p and q, $1 < p < \infty$, $1 < q < \infty$, such that

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p},$$

and for any $\theta > 0$, an arbitrary $w \in \mathcal{A}_{p,q}(X)$, and all $(f,g) \in \mathcal{F}$, we have

$$\|g\|_{\mathcal{L}_w^{q),\theta q/p}(X)} \le \overline{C}\,\|f\|_{\mathcal{L}_w^{p),\theta}(X)}, \tag{2.4}$$

where the constant $\overline{C}$ is independent of (f,g) but depends on w.

Proof of Theorem 2.1. Let (2.1) hold for $w \in A_{p_0}(X)$. By Theorem A and inequality (1.7), we have

$$\left(\int\limits_X g^r w\,d\mu\right)^{1/r} \le C_1 C(w,r)\left(\int\limits_X f^r w\,d\mu\right)^{1/r} \tag{2.5}$$

for all $w \in A_r(X)$, where

$$C(w,r) = \begin{cases} N\Big((2\overline{c}\,r')^{p_0-r}[w]_{A_r(X)}^{(p_0-1)/(r-1)}\Big), & r < p_0,\\ N\big((2\overline{c}\,r)^{(r-p_0)/(r-1)}[w]_{A_r(X)}\big), & r > p_0,\end{cases}$$

and the positive constant C_1 is independent of r. Set $r = p - \varepsilon$ in (2.5). By Hölder's inequality, we have

$$\begin{aligned}\|g\|_{L_w^{p),\theta}(X)} &\le C \sup_{0<\varepsilon<\varepsilon_0}\left(\varepsilon^\theta \int\limits_X |g(x)|^{p-\varepsilon} w(x)\,d\mu(x)\right)^{1/(p-\varepsilon)}\\ &\le C \sup_{0<\varepsilon<\varepsilon_0} C(w,p-\varepsilon)\left(\varepsilon^\theta \int\limits_X |f(x)|^{p-\varepsilon} w(x)\,d\mu(x)\right)^{1/(p-\varepsilon)}\\ &\le C \sup_{0<\varepsilon<\varepsilon_0} C(w,p-\varepsilon)\|f\|_{L_w^{p),\theta}(X)},\end{aligned}$$

where the constant C depends on ε_0, p_0, and p and ε_0 is a sufficiently small positive number such that $w \in A_{p-\varepsilon_0}$. Such a number ε_0 exists in view of the openness of the class $A_p(X)$. In this case, $[w]_{A_r(X)} \ge 1$ and $[w]_{A_{p-\varepsilon}(X)} \le [w]_{A_{p-\varepsilon_0}(X)}$ (see (1.4)). Finally,

$$\sup_{0<\varepsilon<\varepsilon_0} C(w,p-\varepsilon) < \infty. \qquad \square$$

To prove Theorem 2.2, we introduce some notation. Let $1 < p < q < \infty$, $\varepsilon_0 \in (0, q-1)$, and $\varepsilon \in (0,\varepsilon_0)$. Set

$$\Phi(x) = \left[\frac{x-q}{1-A(x-q)} + p\right]^{1-(x-q)A}, \qquad \text{where} \quad A = \frac{1}{p} - \frac{1}{q}. \tag{2.6}$$

Next, let

$$\Psi(x) := \Phi(x^{\theta}), \qquad \theta > 0. \tag{2.7}$$

Proof of Theorem 2.2. First, let us prove the theorem for $p < q$. Suppose that (2.3) holds for arbitrary $w \in \mathcal{A}_{p_0,q_0}(X)$. Now, let $w \in \mathcal{A}_{p,q}(X)$. Then, in view of (1.6), the equality $[w]_{\mathcal{A}_{p,q}(X)} = [w^q]_{A_{1+q/p'}(X)}$ is valid. According to the openness of the Muckenhoupt class, we have $w^q \in A_{1+(q-\varepsilon_0)/(p-\eta_0)'}(X)$ for some positive ε_0 and η_0 satisfying the condition

$$\frac{1}{p-\eta_0} - \frac{1}{q-\varepsilon_0} = \frac{1}{p} - \frac{1}{q} = A.$$

Then $w^q \in A_{1+(q-\varepsilon)/(p-\eta)'}(X)$ for all ε and η such that $0 < \varepsilon < \varepsilon_0$, $0 < \eta < \eta_0$, and

$$\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = A. \tag{2.8}$$

Moreover,

$$[w^q]_{A_{1+(q-\varepsilon)/(p-\eta)'}(X)} \le [w^q]_{A_{1+(q-\varepsilon_0)/(p-\eta_0)'}(X)}.$$

Next, by Hölder's inequality, we have

$$[w]_{\mathcal{A}_{p-\varepsilon,q-\eta}(X)} \le [w^q]^{1-\varepsilon/q}_{A_{1+(q-\varepsilon)/(p-\eta)'}(X)} = \big[w^{q/(q-\varepsilon)}\big]^{1-\varepsilon/q}_{\mathcal{A}_{p-\eta,q-\varepsilon}(X)}.$$

Let functions Φ and Ψ be defined by equalities (2.6) and (2.7), respectively.

Now, we apply (1.12), where p and q are replaced with $p-\eta$ and $q-\varepsilon$, respectively. As a result, we obtain the estimates

$$\begin{aligned}\|g\|_{\mathcal{L}^{q),\Phi(\varepsilon)}_w(X)} &\le C \sup_{0<\varepsilon<\sigma_0} \Psi(\varepsilon)^{1/(q-\varepsilon)} \|wg\|_{L^{q-\varepsilon}(X)} \\ &\le C \sup_{0<\eta<\sigma_1} C_1(w,p-\eta,q-\varepsilon)\eta^{\theta/(p-\eta)} \|wf\|_{L^{p-\eta}(X)} \le \overline{C}\|f\|_{\mathcal{L}^{p),\theta}_w(X)},\end{aligned}$$

where σ_0 and σ_1 are sufficiently small positive numbers chosen so that $\sigma_0 < \varepsilon_0$ and $\sigma_1 < \eta_0$, and $C_1(w,p-\eta,q-\varepsilon)$ is equal to the expression of $K(w)$ in which p and q are replaced with $p-\eta$ and $q-\varepsilon$, respectively; the constant C depends on σ_0 and q. In the last inequality, we used estimate (1.12). Finally, obvious arguments and estimate (1.12) complete the proof of the theorem for $p < q$.

The proof of the theorem for $p = q$ is quite analogous. In this case, we follow the line of reasoning from the proof of Theorem 2.1 and apply the easily verifiable inequalities

$$[w^{p-\varepsilon}]_{A_{p-\varepsilon}(X)} \le [w^p]^{1-\varepsilon/p}_{A_{p-\varepsilon}(X)} \le [w^p]^{1-\varepsilon/p}_{A_{p-\varepsilon_0}(X)} \le [w^p]_{A_{p-\varepsilon_0}(X)},$$

where $w \in \mathcal{A}_p(X)$, ε_0 is chosen so that $w^p \in A_{p-\varepsilon_0}(X)$, and $0 < \varepsilon < \varepsilon_0$. $\square$

3. APPLICATIONS TO WEIGHTED INEQUALITIES FOR INTEGRAL OPERATORS

In this section, we apply the results of the previous section to one-weight inequalities for integral transformations defined on spaces of homogeneous type. Below, $\mathcal{D}(X)$ denotes the set of essentially bounded functions defined on X. As above, we will assume that $\mu X < \infty$.

Recall the well-known definition of the Calderon–Zygmund kernels in quasimetric spaces with measure.

Let $k\colon X \times X \setminus \{(x,x)\colon x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions

$$|k(x,y)| \le \frac{c}{\mu B(x, d(x,y))}, \qquad x, y \in X, \quad x \ne y,$$

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le c\omega\biggl(\frac{d(x_2,x_1)}{d(x_2,y)}\biggr)\frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2, and y such that $d(x_2,y) > d(x_1,x_2)$, where ω is a positive nondecreasing function on $(0,\infty)$ satisfying the Δ_2-condition ($\omega(2t) \le c\omega(t)$, $t > 0$) and the Dini condition $\int_0^1 \omega(t)t^{-1}\,dt < \infty$.

In addition, we assume that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}(X,\mu)$, the limit

$$(Tf)(x) = \lim_{\varepsilon\to 0} \int\limits_{X\setminus B(x,\varepsilon)} k(x,y)f(y)\,d\mu(y)$$

exists almost everywhere on X and the operator T is bounded in $L^{p_0}(X,\mu)$.

It is known (see [26]) that there exists a constant $\widetilde{c}_0 := \widetilde{c}_0([w]_{A_\infty})$, independent of f and depending on $[w]_{A_\infty}$, such that

$$\|Tf\|_{L^{p_0}_w(X,\mu)} \le \widetilde{c}_0 \|Mf\|_{L^{p_0}_w(X,\mu)}, \qquad f \in \mathcal{D}(X), \quad w \in A_\infty(X,\mu),$$

where the mapping $x \to \widetilde{c}_0(x)$ does not decrease on $(1,\infty)$.

Using the extrapolation Theorem 2.1, we establish the validity of the following statement.

Theorem 3.1. *Let $1 < p < \infty$ and $\theta > 0$. Then there exists a positive constant C such that*

$$\|Tf\|_{L^{p),\theta}_w(X)} \le C\|Mf\|_{L^{p),\theta}_w(X)}$$

for all $f \in \mathcal{D}(X)$ and any $w \in A_p(X)$.

Let $b \in \mathrm{BMO}(X)$, $m \in \mathbb{N} \cup \{0\}$, and let

$$T_b^m f(x) = \int\limits_X [b(x) - b(y)]^m k(x,y) f(y)\,d\mu(y),$$

where k is a Calderon–Zygmund kernel.

It is known (see [26]) that if $1 < r < \infty$, $b \in \mathrm{BMO}(X)$, and $w \in A_\infty(X)$, then the following one-weight inequality is valid:

$$\|T_b^m f\|_{L^r_w(X)} \le C\|b\|^m_{\mathrm{BMO}(X)} \|M^{m+1} f\|_{L^r_w(X)}, \qquad f \in \mathcal{D}(X),$$

where M^{m+1} is the $(m+1)$th iteration of the maximal function M. This result and the extrapolation theorem imply the following statement.

Theorem 3.2. *Let $1 < p < \infty$ and $\theta > 0$. Then there exists a positive constant C such that for any $w \in A_p(X)$*

$$\|T_b^m f\|_{L^{p),\theta}_w(X)} \le C\|b\|^m_{\mathrm{BMO}(X)} \|M^{m+1} f\|_{L^{p),\theta}_w(X)}, \qquad f \in \mathcal{D}(X).$$

Let

$$I_\alpha f(x) = \int\limits_X K_\alpha(x,y) f(y)\,d\mu(y), \qquad x \in X,$$

where

$$K_\alpha(x,y)=\begin{cases}\mu(B_{xy})^{\alpha-1}, & x\neq y,\\ \mu\{x\}, & x=y,\ \mu\{x\}>0,\end{cases}$$

$0<\alpha<1$, and $B_{xy}:=B(x,d(x,y))$.

Consider also

$$M_\alpha f(x)=\sup_{B\ni x}\frac{1}{(\mu B)^{1-\alpha}}\int_B |f(y)|\,d\mu(y),\qquad 0<\alpha<1.$$

It is known (see [1]) that if $0<r<\infty$ and $w\in A_\infty(X)$, then the inequality

$$\|I_\alpha f\|_{L^r_w(X)}\le C\|M_\alpha f\|_{L^r_w(X)}$$

holds with some constant C independent of f. For $b\in \mathrm{BMO}(X)$, set

$$I^m_{\alpha,b}f(x)=\int_X [b(x)-b(y)]^m K_\alpha(x,y)\,d\mu(y),\qquad 0<\alpha<1,$$

$$\mathcal{I}^m_{\alpha,b}f(x)=\int_X |b(x)-b(y)|^m K_\alpha(x,y)\,d\mu(y),\qquad 0<\alpha<1.$$

It is clear that $|I^m_{\alpha,b}f(x)|\le \mathcal{I}^m_{\alpha,b}f(x)$ for $f\ge 0$. In [1], it is proved that if $0<p<\infty$, $0<\alpha<1$, $m\in\mathbb{N}\cup\{0\}$, $w\in A_\infty(X)$, and $b\in\mathrm{BMO}(X)$, then there exists a constant C such that

$$\int_X |\mathcal{I}^m_{\alpha,b}f(x)|^p w(x)\,d\mu(x)\le C\|b\|^{mp}_{\mathrm{BMO}(X)}\int_X \big[M_\alpha(M^m f)(x)\big]^p w(x)\,d\mu(x).$$

The extrapolation theorem in the diagonal case allows us to derive the following statements from the indicated result.

Theorem 3.3. *Let $1<p<\infty$, $\theta>0$, and $w\in A_p(X)$. Then there exists a positive constant C such that*

$$\|I_\alpha f\|_{L^{p),\theta}_w(X)}\le C\|M_\alpha f\|_{L^{p),\theta}_w(X)}$$

for all $f\in\mathcal{D}(X)$.

Theorem 3.4. *Let $1<p<\infty$, $\theta>0$, and $w\in A_p(X)$. Then there exists a positive constant C such that*

$$\|\mathcal{I}^m_{\alpha,b}f\|_{L^{p),\theta}_w(X)}\le C\|b\|^m_{\mathrm{BMO}(X)}\|M_\alpha(M^m f)\|_{L^{p),\theta}_w(X)}$$

for all $f\in\mathcal{D}(X)$.

Next, the following statements are valid for semilinear operators.

Theorem 3.5 (diagonal case)**.** *Let $p_0\in(1,\infty)$. Suppose that the following inequality holds for a semilinear operator S, any f from the domain of the operator S, and any $w\in A_{p_0}(X)$:*

$$\|Sf\|_{L^{p_0}_w(X)}\le CN\big([w]_{A_{p_0}(X)}\big)\|f\|_{L^{p_0}_w(X)},$$

where N is an increasing function and the constant C is independent of w. Then, for arbitrary p, $1<p<\infty$, $\theta>0$, $w\in A_p(X)$, and $f\in L^{p),\theta}_w(X)$, we have

$$\|Sf\|_{L^{p),\theta}_w(X)}\le C\|f\|_{L^{p),\theta}_w(X)},$$

where C is independent of f.

Theorem 3.6 (nondiagonal case)**.** *Let $1 < p_0 < \infty$ and $1 < q_0 < \infty$. Suppose that for a semilinear operator S, any f from the domain of the operator S, and any $w \in \mathcal{A}_{p_0,q_0}(X)$, we have*

$$\|Sf\|_{L^{q_0}_{w^{q_0}}} \le CN\big([w]_{\mathcal{A}_{p_0,q_0}(X)}\big)\|f\|_{L^{p_0}_{w^{p_0}}(X)},$$

where N is an increasing function and the constant C is independent of w. Then for $1 < p < \infty$ and $1 < q < \infty$ subject to the condition

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p}$$

and for any $\theta > 0$, $w \in \mathcal{A}_{p,q}(X)$, and $f \in \mathcal{L}^{p),\theta}_w(X)$, the following inequality is valid:

$$\|Sf\|_{\mathcal{L}^{q),\theta q/p}_w(X)} \le C\|f\|_{\mathcal{L}^{p),\theta}_w(X)},$$

where the positive constant C is independent of f.

Proof of Theorem 3.5. First, notice that according to Theorem A

$$\|Sf\|_{L^r_w(X)} \le C_1 C(w,r)\|f\|_{L^r_w(X)} \tag{3.1}$$

for $r \in (1,\infty)$, $w \in A_r(X)$, and any $f \in L^r_w(X)$, where $C(w,r)$ is defined as in Theorem 2.1 and C_1 is independent of r.

Let $f \in L^{p),\theta}_w(X)$. Then $f \in L^{p-\varepsilon}_w(X)$ for all $\varepsilon \in (0, p-1)$. In inequality (3.1), we set $r = p - \varepsilon$. Applying the same arguments as in the proof of Theorem 2.1, we find

$$\begin{aligned}
\|Sf\|_{L^{p),\theta}_w(X)} &\le C \sup_{0<\varepsilon<\varepsilon_0} \Bigg(\varepsilon^\theta \int\limits_X |Sf(x)|^{p-\varepsilon} w(x)\, d\mu(x)\Bigg)^{1/(p-\varepsilon)} \\
&\le C \sup_{0<\varepsilon<\varepsilon_0} C(w, p-\varepsilon) \Bigg(\varepsilon^\theta \int\limits_X |f(x)|^{p-\varepsilon} w(x)\, d\mu(x)\Bigg)^{1/(p-\varepsilon)} \\
&\le C \sup_{0<\varepsilon<\varepsilon_0} C(w, p-\varepsilon)\|f\|_{L^{p),\theta}_w(X)},
\end{aligned}$$

where ε_0 is a sufficiently small positive number and C depends on ε_0, p_0, and p. □

Proof of Theorem 3.6. The line of reasoning is similar to that from the proof of Theorem 2.2. In this case, one should use the fact that if $f \in L^{p),\theta}_{w^p}(X)$, then $f \in L^{p-\eta}_{w^p}(X)$ for all $\eta \in (0, p-1)$. Applying the same arguments as in the proof of Theorem 3.5, we find that

$$\|Sf\|_{\mathcal{L}^{q),\theta q/p}_w(X)} \le C \sup_{0<\eta<\sigma_1} C_1(w, p-\eta, q-\varepsilon)\|f\|_{\mathcal{L}^{p),\theta}_w(X)},$$

where σ_1 is a sufficiently small positive number and the numbers ε and η satisfy equality (2.8). □

The proof of the following theorem is also based on the nondiagonal case of the extrapolation theorem.

Theorem 3.7. *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. Suppose that $\theta > 0$. Let T_α denote one of the operators I_α or M_α. If $w \in \mathcal{A}_{p,q}(X)$, $q = p/(1-\alpha p)$, then there exists a positive constant C such that the inequality*

$$\|T_\alpha f\|_{\mathcal{L}^{q),q\theta/p}_w(X)} \le C\|f\|_{\mathcal{L}^{p),\theta}_w(X)}$$

holds for all $f \in \mathcal{L}^{p),\theta}_w(X)$.

4. BERNSTEIN AND NIKOL'SKII TYPE INEQUALITIES IN THE SPACES $\mathcal{L}_w^{p),\theta}$

Bernstein and Nikol'skii inequalities for trigonometric polynomials play an important role in the problems of approximation of functions and in the theory of function spaces. In this section, using the boundedness of integral operators in weighted Iwaniec–Sbordone spaces, we first prove Bernstein type weighted inequalities for the (Weyl) fractional derivatives of trigonometric polynomials and then establish a Nikol'skii type inequality.

For the definition of the Weyl fractional derivative for periodic functions, see, for example, [31, Ch. XII, § 8]. Henceforth, we set $\mathbb{T} = [-\pi, \pi]$.

Theorem 4.1. *Let $1 < p < \infty$, $\theta > 0$, and $w \in \mathcal{A}_p$. The following inequality is valid for an arbitrary trigonometric polynomial T_n and a number $\alpha > 0$:*

$$\|T_n^{(\alpha)}\|_{\mathcal{L}_w^{p),\theta}} \le cn^\alpha \|T_n\|_{\mathcal{L}_w^{p),\theta}}, \tag{4.1}$$

where the constant c is independent of n and T_n.

Proof. First, we prove the inequality for positive integer α. Let $\alpha = 1$. We employ the representation

$$T_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} T_n(t) D_n(t-x)\,dt, \qquad \text{where} \quad D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt. \tag{4.2}$$

Differentiating equality (4.2) yields

$$T_n'(x) = \frac{2n}{\pi}\int_{-\pi}^{\pi} T_n(x+t)\sin nt K_{n-1}(t)\,dt,$$

where K_{n-1} is the Fejer kernel of order $n-1$. Hence,

$$|T_n'(x)| \le 2n\sigma_{n-1}(x, |T_n|).$$

Using the well-known estimate for the Cesàro means,

$$\sigma_{n-1}(x, |T_n|) \le c_1 M(|T_n|)(x),$$

where M is the maximal Hardy–Littlewood function and the constant c_1 is independent of T_n and x, and applying the theorems on the boundedness of the operator M in the spaces $\mathcal{L}_w^{p),\theta}$, we conclude that

$$\|T_n'\|_{\mathcal{L}_w^{p),\theta}} \le cn\|T_n\|_{\mathcal{L}_w^{p),\theta}}.$$

Now, we proceed to the case of fractional derivatives of order $\alpha > 0$. Let, first, $0 < \alpha < 1$. It is known that the fractional derivative of smooth functions (in particular, trigonometric polynomials) coincides with the Marchaud fractional derivative (see, for example, [28, § 19, Remark 19.1]). We avail ourselves of the following representation of the Marchaud fractional derivative:

$$T_n^{(\alpha)}(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{T_n(x) - T_n(x-t)}{t^{1+\alpha}}\,dt, \tag{4.3}$$

where Γ is the Euler gamma function. Let us pass to the absolute value and represent the integral (4.3) as a sum of two terms:

$$\bigl|T_n^{(\alpha)}(x)\bigr| \le \frac{\alpha}{\Gamma(1-\alpha)}\left(\int\limits_0^{2/n}\frac{|T_n(x)-T_n(x-t)|}{t^{1+\alpha}}\,dt+\int\limits_{2/n}^{\infty}\frac{|T_n(x)-T_n(x-t)|}{t^{1+\alpha}}\,dt\right)$$
$$=\frac{\alpha}{\Gamma(1-\alpha)}(I_1+I_2).$$

Applying Minkowski's inequality, we obtain

$$\|I_1\|_{\mathcal{L}_w^{p),\theta}} \le \int\limits_0^{2/n}\frac{\|T_n(x)-T_n(x-t)\|_{\mathcal{L}_w^{p),\theta}}}{t^{1+\alpha}}\,dt \le \int\limits_0^{2/n}\left\|\frac{1}{t}\int\limits_{x-t}^{x}T_n'(u)\,du\right\|_{\mathcal{L}_w^{p),\theta}}\frac{dt}{t^{\alpha}}.$$

In view of the boundedness of the maximal function in $\mathcal{L}_w^{p),\theta}$, from this estimate we deduce

$$\|I_1\|_{\mathcal{L}_w^{p),\theta}} \le c_2\int\limits_0^{2/n}\|T_n'\|_{\mathcal{L}_w^{p),\theta}}\frac{dt}{t^{\alpha}}.$$

Applying the already proved inequality for the first-order derivatives, we have

$$\|I_1\|_{\mathcal{L}_w^{p),\theta}} \le c_3 n^{\alpha}\|T_n\|_{\mathcal{L}^{p),\theta}}. \tag{4.4}$$

Estimating $\|I_2\|_{\mathcal{L}_w^{p),\theta}}$, we obtain

$$\|I_2\|_{\mathcal{L}_w^{p),\theta}} \le 2\int\limits_{2/n}^{\infty}\|T_n\|_{\mathcal{L}_w^{p),\theta}}\frac{dt}{t^{1+\alpha}} \le c_4 n^{\alpha}\|T_n\|_{\mathcal{L}_w^{p),\theta}}. \tag{4.5}$$

Inequalities (4.4) and (4.5) imply the validity of inequality (4.1) in the case of $0<\alpha<1$.

Next, if T_n is a trigonometric polynomial, then $T_n^{(\alpha)}$ is also a trigonometric polynomial of the same degree. Therefore, it follows from what has been proved that an estimate of the form (4.1) holds for all $\alpha>0$. □

Note that inequality (4.1) in the classical Lebesgue spaces with exact constant 1 for $\alpha \ge 1$ was established by Lizorkin [22] in a more general context of "trigonometric integrals"

$$T_n(x)=\int\limits_{-n}^{n}e^{ixt}\,d\sigma(t).$$

Now, we present an analog of the well-known inequality of Nikol'skii for the norms of a trigonometric polynomial in different metrics.

Theorem 4.2. *Let $1<p<q<\infty$, $\theta>0$, and $\theta_1 \ge \theta q/p$. Suppose that $w \in \mathcal{A}_{p,q}$. Then the following inequality holds*:

$$\|T_n\|_{\mathcal{L}_w^{q),\theta_1}} \le c_5 n^{1/p-1/q}\|T_n\|_{\mathcal{L}_w^{p),\theta}}. \tag{4.6}$$

Proof. We set $1/p-1/q=s$ and apply the representation

$$T_n(x)=a_0(T_n)+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}T_n^{(s)}(x-t)\psi_s(t)\,dt,$$

where

$$\psi_s(t) = \sum_{k=-\infty}^{\infty}{}' \frac{e^{ikt}}{(ik)^s} = 2\sum_{k=1}^{\infty} \frac{\cos(kt - s\pi/2)}{k^s}$$

(the prime means that the term with number $k = 0$ is omitted). At the same time,

$$|\psi_s(t)| \le \frac{c}{|t|^{1-s}}$$

(see [28, Theorem 19.3, Lemma 19.1, corollary to Lemma 19.1]). Hence,

$$|T_n(x)| \le |a_0(T_n)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|T_n^{(s)}(x-t)|}{|t|^{1-s}}\,dt \le |a_0(T_n)| + \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \frac{|T_n^{(s)}(t)|}{|x-t|^{1-s}}\,dt. \tag{4.7}$$

Thus,

$$\|T_n\|_{\mathcal{L}_w^{q),\theta}} \le c_6\left(|a_0(T)| + \left\|\int_{-2\pi}^{2\pi} \frac{|T^{(s)}(t)|}{|x-t|^{1-s}}\,dt\right\|_{\mathcal{L}^{q),\theta_1}}\right).$$

Next, since $w^p \in A_p$, it follows from the openness of A_p by Hölder's inequality that $w^{p-\varepsilon_0} \in A_{p-\varepsilon_0}$ for sufficiently small ε_0, $0 < \varepsilon_0 < p-1$. Fix a number ε_0. Again by Hölder's inequality, we have

$$\begin{aligned}|a_0(T)| &\le c_7\int_{-\pi}^{\pi} |T_n(x)|\,dx \\ &\le c_7\left(\int_{-\pi}^{\pi} |T_n(x)w(x)|^{p_0-\varepsilon}\,dx\right)^{1/(p_0-\varepsilon)} \left(\int_{-\pi}^{\pi} w^{-(p-\varepsilon_0)/(p-\varepsilon_0-1)}(x)\,dx\right)^{1-1/(p-\varepsilon_0)}.\end{aligned}$$

In view of the inclusion $w^{p-\varepsilon_0} \in A_{p-\varepsilon_0}$, the second factor on the right-hand side is finite. On the other hand,

$$\sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{\theta/(p-\varepsilon)} = \varepsilon_0^{\theta/(p-\varepsilon_0)}$$

because the function $x^{1/(p-x)}$ increases for $0 < x < p-1$. Therefore,

$$|a_0(T)| \le c_8\varepsilon_0^{\theta/(p-\varepsilon_0)}\left(\int_{-\pi}^{\pi} |T_n(x)w(x)|^{p-\varepsilon_0}\,dx\right)^{1/(p-\varepsilon_0)} \le c_8\|T_n\|_{\mathcal{L}_w^{p),\theta}} \le c_8 n^{1/p-1/q}\|T_n\|_{\mathcal{L}_w^{p),\theta}}. \tag{4.8}$$

Applying Theorem 3.7 on the boundedness of the fractional integral operator from $\mathcal{L}_w^{p),\theta}$ to $\mathcal{L}_w^{q),\theta_1}$ under the condition $w \in \mathcal{A}_{p,q}$, we have

$$\left\|\int_{-2\pi}^{2\pi} \frac{|T_n^{(s)}(t)|}{|x-t|^{1-s}}\,dt\right\|_{\mathcal{L}_w^{q),\theta_1}} \le c_9\|T_n^{(s)}\|_{\mathcal{L}_w^{p),\theta}}. \tag{4.9}$$

By Hölder's inequality, we deduce from the condition $w \in \mathcal{A}_{p,q}$ that $w \in \mathcal{A}_p$. Hence, according to Theorem 4.1, we can apply the Bernstein type inequality to the right-hand side of (4.9) to obtain

$$\left\|\int_{-2\pi}^{2\pi} \frac{|T_n^{(s)}(t)|}{|x-t|^{1-s}}\,dt\right\|_{\mathcal{L}_w^{q),\theta_1}} \le c_{10}\|T_n\|_{\mathcal{L}_w^{p),\theta}}. \tag{4.10}$$

Now, (4.7), (4.8), and (4.10) imply (4.6). □

5. DIRECT APPROXIMATION THEOREM

It is known [3] that the spaces $\mathcal{L}_w^{p),\theta}$ ($1 < p < \infty$, $\theta > 0$) are nonreflexive, nonseparable, and, for $w \neq 1$, noninvariant with respect to rearrangements. The closure of infinitely differentiable functions in the norm of the space $\mathcal{L}_w^{p),\theta}$ does not coincide with this space. Below, we will denote this closure by $\widetilde{\mathcal{L}}_w^{p),\theta}(\mathbb{T})$, where $\mathbb{T} = [-\pi, \pi]$. In this section, we consider problems of approximation of 2π-periodic functions by trigonometric polynomials. Note that the subspace $\widetilde{\mathcal{L}}_w^{p),\theta}(\mathbb{T})$ is described by the condition

$$\lim_{\varepsilon\to 0} \varepsilon^\theta \int\limits_{\mathbb{T}} |f(x)w(x)|^{p-\varepsilon}\, dx = 0.$$

In the space $\widetilde{\mathcal{L}}_w^{p),\theta}(\mathbb{T})$, we introduce structural and constructive characteristics of functions,

$$\Omega(f,\delta)_{\mathcal{L}_w^{p),\theta}} = \sup_{0<h\le\delta} \left\| \frac{1}{2h} \int\limits_{x-h}^{x+h} f(t)\, dt - f(x) \right\|_{\mathcal{L}_w^{p),\theta}},$$

and best approximations by trigonometric polynomials,

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} = \sup_{T_k,\ k\le n} \|f - T_k\|_{\mathcal{L}_w^{p),\theta}},$$

where the least upper bound is taken over all trigonometric polynomials T_k of degree $k \le n$. For $f \in \widetilde{\mathcal{L}}_w^{p),\theta}(\mathbb{T})$, we have

$$\lim_{n\to\infty} E_n(f)_{\mathcal{L}_w^{p),\theta}} = 0.$$

In view of well-known general arguments, for any $f \in \widetilde{\mathcal{L}}_w^{p),\theta}$ and a given n there exists a polynomial of best approximation T_n, i.e.,

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} = \|f - T_n\|_{\mathcal{L}_w^{p),\theta}}.$$

Due to the boundedness of the conjugation operator in $\mathcal{L}_w^{p),\theta}$ for $1 < p < \infty$, $\theta > 0$, and $w \in \mathcal{A}_p(\mathbb{T})$, just as for the spaces $\mathcal{L}^p$ ($1 < p < \infty$), one can prove that the deviations of partial Fourier sums from $f \in \mathcal{L}_w^{p),\theta}$ have the same order as the best approximation.

Given $\alpha \ge 0$, $1 < p < \infty$, $\theta > 0$, and a weight $w \in \mathcal{A}_p(\mathbb{T})$, we denote by $W^\alpha_{p),\theta,w}$ the subset of those functions in $\mathcal{L}_w^{p),\theta}$ for which

$$\|f\|_{W^\alpha_{p),\theta,w}} = \|f\|_{\mathcal{L}_w^{p),\theta}} + \big\|f^{(\alpha)}\big\|_{\mathcal{L}_w^{p),\theta}} < +\infty,$$

where $f^{(\alpha)}(x)$ denotes the Weyl fractional derivative of f of order α.

Denote by $\widetilde{W}^\alpha_{p),\theta,w}$ the subset of those functions in $\widetilde{\mathcal{L}}_w^{p),\theta}$ for which the expression on the right-hand side of the above equality is finite.

Our immediate goal is to prove analogs of the direct Jackson inequalities for $\mathcal{L}_w^{p),\theta}(\mathbb{T})$. First, we prove the following statement.

Theorem 5.1. *Let $1 < p < \infty$, $\theta > 0$, $\alpha > 0$, and $w \in \mathcal{A}_p(\mathbb{T})$. Then the following inequality holds for some positive constant c and all $f \in \widetilde{W}^\alpha_{p),\theta,w}$:*

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} \le \frac{c}{(n+1)^\alpha} E_n\big(f^{(\alpha)}\big)_{\mathcal{L}_w^{p),\theta}}. \tag{5.1}$$

Proof. For $f \in \mathcal{L}_w^{p),\theta}$, we denote by $S_n(x,f)$ the partial sum of the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} A_k(x,f), \qquad A_k(x,f) = a_k \cos kx + b_k \sin kx.$$

Let us show that

$$f(x) - S_n(x,f) = \cos\frac{\pi\alpha}{2} \sum_{k=n+1}^{\infty} k^{-\alpha} A_k\big(x, f^{(\alpha)}\big) + \sin\frac{\pi\alpha}{2} \sum_{k=n+1}^{\infty} k^{-\alpha} A_k\big(x, f^{(\alpha)}\big). \tag{5.2}$$

By definition,

$$A_k\big(x, f^{(\alpha)}\big) = k^{\alpha} A_k\Big(x + \frac{\pi\alpha}{2k}, f\Big).$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} A_k(x,f) &= A_0(x,f) + \cos\frac{\pi\alpha}{2} \sum_{k=1}^{\infty} A_k\Big(x + \frac{\pi\alpha}{2}, f\Big) + \sin\frac{\pi\alpha}{2} \sum_{k=n+1}^{\infty} A_k\Big(x + \frac{\pi\alpha}{2k}, f\Big) \\ &= A_0(x,f) + \cos\frac{\pi\alpha}{2} \sum_{k=1}^{\infty} k^{-\alpha} A_k\big(x, f^{(\alpha)}\big) + \sin\frac{\pi\alpha}{2} \sum_{k=1}^{\infty} k^{-\alpha} A_k\big(x, f^{(\alpha)}\big).\end{aligned}$$

This implies (5.2).

On the other hand, applying Abel's transformation, we can write

$$\begin{aligned}\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\big(x, f^{(\alpha)}\big) &= \sum_{k=n+1}^{\infty} k^{-\alpha} \Big[S_k\big(x, f^{(\alpha)}\big) - f^{(\alpha)}(x) - \big(S_{k-1}\big(x, f^{(\alpha)}\big) - f^{(\alpha)}(x)\big)\Big] \\ &= \sum_{k=n+2}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \big(S_k\big(x, f^{(\alpha)}\big) - f^{(\alpha)}(x)\big).\end{aligned}$$

A similar equality is valid for $\widetilde{f}^{(\alpha)}$. Passing to the norms and using the fact that

$$\|f(x) - S_n(x,f)\|_{\mathcal{L}_w^{p),\theta}} \le c E_n(f)_{\mathcal{L}_w^{p),\theta}},$$

we obtain

$$\begin{aligned}\|f(x) - S_n(x,f)\|_{\mathcal{L}_w^{p),\theta}} &\le c\Bigg(E_n\big(f^{(\alpha)}\big)\Bigg[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\Bigg] \\ &\qquad + E_n\big(\widetilde{f}^{(\alpha)}\big)\big[(k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\big]\Bigg) \\ &\le \frac{c}{(n+1)^{\alpha}} E_n(f)_{\mathcal{L}_w^{p),\theta}}.\end{aligned}$$

Notice that in view of the boundedness of the conjugation operator in $\mathcal{L}_w^{p),\theta}$ for $w \in \mathcal{A}_p(\mathbb{T})$, we have

$$E_n\big(\widetilde{f}^{(\alpha)}\big) \le c E_n\big(f^{(\alpha)}\big).$$

The theorem is proved. $\square$

For any $f \in \mathcal{L}_w^{p),\theta}$ ($1 < p < \infty$, $\theta > 0$) and $w \in \mathcal{A}_p(\mathbb{T})$, we introduce a so-called $\mathcal{K}_2$-functional defined as

$$\mathcal{K}_2\bigl(f,\delta;\mathcal{L}_w^{p),\theta},W^2_{p),\theta,w}\bigr) = \inf_{g\in W^2_{p),\theta,w}} \Bigl\{\|f-g\|_{\mathcal{L}_w^{p),\theta}} + \delta^2\|g''\|_{\mathcal{L}_w^{p),\theta}}\Bigr\}, \qquad \delta > 0.$$

Theorem 5.2. *Let $1 < p < \theta$, $\theta > 0$, and $w \in \mathcal{A}_p(\mathbb{T})$. Then there exist positive constants c_1 and c_2 such that*

$$c_1\Omega(f,\delta) \le \mathcal{K}_2\bigl(f,\delta;\mathcal{L}_w^{p),\theta},W^2_{p),\theta,w}\bigr) \le c_2\Omega(f,\delta) \tag{5.3}$$

for an arbitrary $f \in \mathcal{L}_w^{p),\theta}$ and a positive δ.

Proof. Given a $\delta > 0$, we take a positive integer n such that $1/n < \delta < 2/n$. Consider the sequence of operators

$$(K_nf)(x) = n^2\int\limits_0^{1/n}\int\limits_0^t\int\limits_{-u}^u f(x+\tau)\,d\tau\,du\,dt, \qquad x\in\mathbb{T}, \quad f\in\mathcal{L}_w^{p),\theta}.$$

Then

$$(K_nf)''(x) = cn^2(I-m_{1/n})f(x), \tag{5.4}$$

where I is the identity operator,

$$(m_{1/n}f)(x) = \frac{n}{2}\int\limits_{x-1/n}^{x+1/n} f(t)\,dt,$$

and c is a constant independent of f and n.

In view of the boundedness of the maximal function in $\mathcal{L}_w^{p),\theta}$ and the Minkowski type inequality, we can conclude that the sequence K_nf is uniformly bounded in the norm. Indeed,

$$\|K_nf\|_{\mathcal{L}_w^{p),\theta}} \le cn^2\int\limits_0^{1/n}\int\limits_0^t 2u\|m_hf\|_{\mathcal{L}_w^{p),\theta}}\,du\,dt \le cn^2\|f\|_{\mathcal{L}_w^{p),\theta}}\int\limits_0^{1/n}\int\limits_0^t 2u\,du\,dt \le c\|f\|_{\mathcal{L}_w^{p),\theta}}. \tag{5.5}$$

At the same time, in view of (5.4), we have $(K_nf)'' \in \mathcal{L}_w^{p),\theta}$ for any fixed n.

Next, it follows from (5.5) that $f - K_nf \in \mathcal{L}_w^{p),\theta}$ and

$$\begin{aligned}\mathcal{K}_2\bigl(f,\delta;\mathcal{L}_w^{p),\theta},W^2_{p),\theta,w}\bigr) &\le \mathcal{K}_2\biggl(f,\frac{1}{n};\mathcal{L}_w^{p),\theta},W^2_{p),\theta,w}\biggr)\\ &\le c\Bigl(\|f-K_nf\|_{\mathcal{L}_w^{p),\theta}} + n^{-2}\|(K_nf)''\|_{\mathcal{L}_w^{p),\theta}}\Bigr) = c(I_1+I_2).\end{aligned} \tag{5.6}$$

Let us estimate each term in the parentheses:

$$\begin{aligned}I_1 = \|f-K_nf\|_{\mathcal{L}_w^{p),\theta}} &\le cn^2\int\limits_0^{1/n}\int\limits_0^t \|(I-m_u)f\|_{\mathcal{L}_w^{p),\theta}}\,du\,dt\\ &\le c\sup_{0<u\le 1/n}\|I-m_uf\|_{\mathcal{L}_w^{p),\theta}} = c\Omega\biggl(f,\frac{1}{n}\biggr)_{\mathcal{L}_w^{p),\theta}}.\end{aligned} \tag{5.7}$$

Next, it follows from (5.4) that

$$I_2 = n^{-2}\|(K_n f)''\|_{\mathcal{L}_w^{p),\theta}} \le c\|(I - m_{1/n})f\|_{\mathcal{L}_w^{p),\theta}} \le c\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}}. \tag{5.8}$$

Formulas (5.7) and (5.8) imply the validity of the estimate on the right-hand side of (5.3).

Let us proceed to estimating the $\mathcal{K}_2$-functional from below. For $g \in W^2_{p),\theta,w}$, we have

$$(I - m_h g)(x) = \frac{1}{2h}\int_{-h}^{h}\big(g(x) - g(x+t)\big)\,dt = \frac{c}{h}\int_0^h\int_0^t\int_{-u}^{u} g''(x+\tau)\,d\tau\,du\,dt.$$

Hence

$$\begin{aligned}\|(I - m_h g)\|_{\mathcal{L}_w^{p),\theta}} &\le \frac{c}{h}\int_0^h\int_0^t u\left\|\frac{1}{2u}\int_{-u}^{u} g''(x+\tau)\,d\tau\right\|_{\mathcal{L}_w^{p),\theta}} du\,dt \le \frac{c}{h}\int_0^h\int_0^t u\|g''\|_{\mathcal{L}_w^{p),\theta}}\,du\,dt\\ &= ch^2\|g''\|_{\mathcal{L}_w^{p),\theta}}.\end{aligned} \tag{5.9}$$

As a result, for $g \in W^2_{p),\theta,w}$ we obtain the estimate

$$\Omega(g,\delta)_{\mathcal{L}_w^{p),\theta}} \le c\delta^2\|g''\|_{\mathcal{L}_w^{p),\theta}}$$

with a constant c independent of g and δ.

Next, for $f \in \mathcal{L}_w^{p),\theta}$ and an arbitrary $g \in W^2_{p),\theta,w}$, we have

$$\Omega(f,\delta)_{\mathcal{L}_w^{p),\theta}} \le c\Big(\|f - g\|_{\mathcal{L}_w^{p),\theta}} + \delta^2\|g''\|_{\mathcal{L}_w^{p),\theta}}\Big).$$

Hence,

$$\Omega(f,\delta)_{\mathcal{L}_w^{p),\theta}} \le c\mathcal{K}_2\big(f,\delta;\mathcal{L}_w^{p),\theta}, W^2_{p),\theta,w}\big).$$

The theorem is proved. □

Now, let us prove an analog of the Jackson inequality.

Theorem 5.3. *Let $1 < p < \infty$, $\theta > 0$, and $w \in \mathcal{A}_p(\mathbb{T})$. For an arbitrary $\alpha \ge 0$ and all $f \in \widetilde{W}^2_{p),\theta,w}$, the inequality*

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} \le \frac{c}{(n+1)^\alpha}\Omega\Big(f^{(\alpha)}, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}}, \qquad n \in \mathbb{N}, \tag{5.10}$$

holds with a constant c independent of f and n.

Proof. First, we prove (5.10) for $\alpha = 0$. We will apply the estimates established in the proof of Theorem 5.2. Obviously,

$$E_n(f) \le E_n(f - K_n f) + E_n(K_n f). \tag{5.11}$$

In view of (5.7), we have the estimate

$$E_n(f - K_n f) \le \|f - K_n f\| \le c\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}}.$$

According to Theorem 5.1 and in view of estimate (5.8),

$$E_n(K_nf) \le \frac{c}{n^2}E_n((K_nf)'') \le \frac{c}{n^2}\|(K_nf)''\| \le c\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}}.$$

Now, (5.11) implies (5.10) for $\alpha = 0$.

By virtue of Theorem 5.1 and what has been proved above for $f \in \widetilde{W}^{\alpha}_{p),\theta,w}$, we have

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} \le \frac{c}{n^\alpha}E_n\big(f^{(\alpha)}\big)_{\mathcal{L}_w^{p),\theta}} \le \frac{c}{n^\alpha}\Omega\Big(f^{(\alpha)}, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}}.$$

The theorem is proved. □

6. INVERSE APPROXIMATION THEOREMS IN $\widetilde{\mathcal{L}}_w^{p),\theta}$

In this section, we prove the following statements.

Theorem 6.1. *Let $1 < p < \infty$ and $\theta > 0$. Suppose that $w \in \mathcal{A}_p(\mathbb{T})$. Then the following inequality holds for $f \in \widetilde{\mathcal{L}}_w^{p),\theta}(\mathbb{T})$:*

$$\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}} \le \frac{c}{n^2}\sum_{\nu=0}^{n}(\nu+1)E_\nu(f)_{\mathcal{L}_w^{p),\theta}},$$

where the constant c is independent of f and n.

Theorem 6.2. *Let $1 < p < \infty$, $\theta > 0$, and $w \in \mathcal{A}_p(\mathbb{T})$. If the condition*

$$\sum_{\nu=1}^{n}\nu^{\alpha-1}E_\nu(f)_{\mathcal{L}_w^{p),\theta}} < +\infty \tag{6.1}$$

holds for $f \in \widetilde{\mathcal{L}}_w^{p),\theta}$ and some $\alpha > 0$, then $f^{(\alpha)} \in \widetilde{\mathcal{L}}_w^{p),\theta}$ and

$$E_n\big(f^{(\alpha)}\big)_{\mathcal{L}_w^{p),\theta}} \le c_1\Bigg(n^\alpha E_n(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{\mathcal{L}_w^{p),\theta}}\Bigg), \tag{6.2}$$

$$\Omega\Big(f^{(\alpha)}, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}} \le c_2\Bigg(\frac{1}{n^2}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+1}E_\nu(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{\mathcal{L}_w^{p),\theta}}\Bigg), \tag{6.3}$$

where the constants c_1 and c_2 are independent of n and f.

Proof of Theorem 6.1. Let T_n be a trigonometric polynomial of degree n corresponding to the function $f \in \widetilde{\mathcal{L}}_w^{p),\theta}$. We choose m from the condition $2^m \le n < 2^{m+1}$. By Theorem 5.2, we have the estimate

$$\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}} \le c\Big(E_n(f)_{\mathcal{L}_w^{p),\theta}} + n^{-2}\|T_n''\|_{\mathcal{L}_w^{p),\theta}}\Big).$$

Next,

$$\|T_n''\|_{\mathcal{L}_w^{p),\theta}} \le \|T_n'' - T_{2^{m+1}}''\|_{\mathcal{L}^{p),\theta}} + \Bigg(\|T_1'' - T_0''\|_{\mathcal{L}_w^{p),\theta}} + \sum_{i=1}^{m}\|T_{2^{i+1}}'' - T_{2^i}''\|_{\mathcal{L}^{p),\theta}}\Bigg).$$

Applying inequality (2.1), we obtain

$$\|T_n''\|_{\mathcal{L}_w^{p),\theta}} \le cE_{2^n}(f)_{\mathcal{L}_w^{p),\theta}} \cdot 2^{(m+1)2} + \sum_{i=0}^{m}2^{2(i+1)}E_{2^i}(f)_{\mathcal{L}_w^{p),\theta}}.$$

Since the sequence of best approximations is monotone, it holds that

$$2^{2(i+1)} E_{2^i}(f)_{\mathcal{L}_w^{p),\theta}} \le c \sum_{\nu=2^{i-1}-1}^{2^i} \nu E_\nu(f)_{\mathcal{L}_w^{p),\theta}},$$

$$\Omega\Big(f, \frac{1}{n}\Big)_{\mathcal{L}_w^{p),\theta}} \le cn^{-2}\Bigg(E_0(f) + E_1(f) + E_n(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{\nu=1}^{2^m} \nu E_\nu(f)_{\mathcal{L}_w^{p),\theta}}\Bigg) \le \frac{c}{n^2}\sum_{\nu=0}^{n}(\nu+1)E_\nu(f)_{\mathcal{L}_w^{p),\theta}}.$$

The theorem is proved. □

Proof of Theorem 6.2. First, we prove that under condition (6.1) the function f belongs to $W^\alpha_{p),\theta,w}$. To this end, it is necessary to show that the series

$$\sum_{k=1}^{\infty} k^\alpha A_k\Big(x + \frac{\pi\alpha}{2k}, f\Big) \tag{6.4}$$

converges in the norm of the space $\mathcal{L}_w^{p),\theta}$ and, hence, is a Fourier series of the fractional derivative $f^{(\alpha)} \in \mathcal{L}_w^{p),\theta}$. Here $A_k(x,f) := a_k(f)\cos kx + b_k(f)\sin kx$. Let us prove that

$$\lim_{\substack{m,n\to\infty\\ n>m}} \Bigg\|\sum_{k=m}^{n} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\| = 0.$$

Let $[2^l, 2^{l+1})$ and $[2^t, 2^{t+1})$ be the least dyadic intervals containing m and n, respectively. Then

$$\Bigg\|\sum_{k=m}^{n} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}} \le \Bigg\|\sum_{k=2^l}^{2^{t+1}-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}}$$
$$+ \Bigg\|\sum_{k=2^l}^{m-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}} + \Bigg\|\sum_{k=n+1}^{2^{t+1}-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}} = I_1 + I_2 + I_3.$$

Applying the Bernstein type inequality (4.1), we obtain

$$\Bigg\|\sum_{k=2^l}^{2^{t+1}-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}} \le c\sum_{k=l}^{t}\Bigg\|\sum_{k=2^N}^{2^{N+1}-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}}$$
$$= c\sum_{k=l}^{t}\Bigg\|\Bigg(\sum_{j=2^\mu}^{2^{\mu+1}-1} A_k(x,f)\Bigg)^{(\alpha)}\Bigg\|_{\mathcal{L}_w^{p),\theta}} \le c\sum_{k=l}^{t} 2^{\mu\alpha}E_{2^\mu}(f)_{\mathcal{L}_w^{p),\theta}}.$$

In view of condition (6.1), the right-hand side of this estimate tends to zero as $m\to\infty$ and $n\to\infty$.

The terms I_2 and I_3 are estimated identically. For example,

$$I_2 = \Bigg\|\sum_{k=2^l}^{m-1} k^\alpha A_k\Big(x+\frac{\pi\alpha}{2k}, f\Big)\Bigg\|_{\mathcal{L}_w^{p),\theta}} \le \Bigg\|\Bigg(\sum_{k=2^l}^{m-1} A_k(x,f)\Bigg)^{(\alpha)}\Bigg\|_{\mathcal{L}_w^{p),\theta}} \le cm^\alpha\Bigg\|\sum_{k=2^l}^{m-1} A_k(x,f)\Bigg\|_{\mathcal{L}_w^{p),\theta}}$$
$$\le c2^{l\alpha}E_{2^l}(f)_{\mathcal{L}_w^{p),\theta}}.$$

Since the series (6.1) converges, we have

$$\lim_{m\to\infty} 2^{l\alpha}E_{2^l}(f)_{\mathcal{L}_w^{p),\theta}} = 0.$$

Thus, the series (6.4) is a Fourier series of some function from $\mathcal{L}_w^{p),\theta}$, and this function is precisely $f^{(\alpha)}$.

Now, given a positive integer n, suppose that the number m is chosen so that $2^m \le n < 2^{m+1}$.

Next, if T_n is a trigonometric polynomial of best approximation of a function $f \in \mathcal{L}_w^{p),\theta}$, then, applying inequality (4.1), we have

$$\begin{aligned}\big\|f^{(\alpha)} - T_n^{(\alpha)}\big\|_{\mathcal{L}_w^{p),\theta}} &\le \big\|T_{2^{m+1}}^{(\alpha)} - T_n^{(\alpha)}\big\| + \sum_{j=m}^{\infty}\big\|T_{2^{j+2}}^{(\alpha)} - T_{2^{j+1}}^{(\alpha)}\big\|_{\mathcal{L}_w^{p),\theta}} \\ &\le c\left(2^{m\alpha}\left\|T_{2^{m+1}} - T_n\right\|_{\mathcal{L}_w^{p),\theta}} + \sum_{j=m}^{\infty} 2^{j\alpha} E_{2^{j+1}}(f)_{\mathcal{L}_w^{p),\theta}}\right) \\ &\le c\left(n^\alpha E_n(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{j=m}^{\infty} 2^{j\alpha} E_{2^{j+1}}(f)_{\mathcal{L}_w^{p),\theta}}\right) \\ &\le c\left(n^\alpha E_n(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{j=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{\mathcal{L}_w^{p),\theta}}\right),\end{aligned}$$

which yields the desired estimate (6.2).

Condition (6.1), the monotonicity of $E_n(f)$, and inequality (6.2) imply that

$$\lim_{n\to\infty} E_n\big(f^{(\alpha)}\big) = 0;$$

hence, $f^{(\alpha)} \in \widetilde{\mathcal{L}}_w^{p),\theta}$.

By Theorem 6.1, for $f \in \widetilde{W}^{\alpha}_{p),\theta,w}$ we have

$$\Omega\left(f^{(\alpha)}, \frac{1}{n}\right) \le \frac{c}{n^2}\sum_{\nu=0}^{m}(\nu+1)E_\nu\big(f^{(\alpha)}\big)_{\mathcal{L}_w^{p),\theta}}.$$

Now, it suffices to apply the just-proved inequality (6.2) to the right-hand side and change the order of summation, and we arrive at inequality (6.3). □

Corollary 6.1. *Let $f \in \widetilde{\mathcal{L}}_w^{p),\theta}$, $w \in \mathcal{A}_p$, $1 < p < \infty$, and $\theta > 0$. If*

$$E_n(f)_{\mathcal{L}_w^{p),\theta}} = O\left(\frac{1}{n^\beta}\right), \qquad \beta > 0,$$

then

$$\Omega(f,\delta)_{\mathcal{L}_w^{p),\theta}} = \begin{cases} O(\delta^\beta), & \beta < 2, \\ O\left(\delta^2 \log\dfrac{1}{\delta}\right), & \beta = 2, \\ \delta^2, & \beta > 2. \end{cases}$$

Denote by $H^{\beta}_{p),\theta,w}$ the subclass of functions in $\widetilde{\mathcal{L}}_w^{p),\theta}$ such that

$$\Omega(f,\delta) = O(\delta^\beta).$$

Then, from the direct theorem and Corollary 6.1 we obtain a constructive description of the class $H^{\beta}_{p),\theta,w}$; namely, $f \in H^{\beta}_{p),\theta,w}$, $\beta < 2$, if and only if

$$E_n(f) = O\left(\frac{1}{n^\beta}\right).$$

Applying the Nikol'skii type inequality and the same arguments as above, we prove the following theorem.

Theorem 6.3. *Let $1 < p < q < \infty$, $\theta > 0$, and $\theta_1 \geq \theta q/p$. Suppose that $w \in \mathcal{A}_{p,q}$. If the condition*

$$\sum_{\nu=1}^{\infty} \nu^{1/p-1/q+\alpha-1} E_\nu(f)_{\mathcal{L}_w^{p),\theta}} < \infty$$

is satisfied for some $\alpha \geq 0$, then $f \in \widetilde{W}^{\alpha}_{q),\theta_1,w}$ and the inequalities

$$E_n\big(f^{(\alpha)}\big)_{\mathcal{L}_w^{q),\theta_1}} \leq c\left\{ n^{\alpha+1/p-1/q} E_n(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{\nu=n+1}^{\infty} \nu^{1/p-1/q+\alpha-1} E_\nu(f)_{\mathcal{L}_w^{p),\theta}} \right\},$$

$$\Omega\left(f^{(\alpha)}, \frac{1}{n}\right)_{\mathcal{L}_w^{q),\theta_1}} \leq C\left\{ \frac{1}{n^2} \sum_{\nu=0}^{n} (\nu+1)^{1/p-1/q+\alpha+1} E_\nu(f)_{\mathcal{L}_w^{p),\theta}} + \sum_{\nu=n+1}^{\infty} \nu^{1/p-1/q+\alpha-1} E_\nu(f)_{\mathcal{L}^{p),\theta}} \right\}$$

are valid with constants c and C independent of f and n.

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REFERENCES

1. A. Bernardis, S. Hartzstein, and G. Pradolini, "Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type," J. Math. Anal. Appl. **322**, 825–846 (2006).
2. S. N. Bernstein, *Collected Works* (Akad. Nauk SSSR, Moscow, 1952), Vol. 1 [in Russian].
3. C. Capone and A. Fiorenza, "On small Lebesgue spaces," J. Funct. Spaces Appl. **3**, 73–89 (2005).
4. R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes: Etude de certaines intégrales singulières* (Springer, Berlin, 1971), Lect. Notes Math. **242**.
5. O. Dragičević, L. Grafakos, M. C. Pereyra, and S. Petermichl, "Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces," Publ. Mat. **49** (1), 73–91 (2005).
6. J. Duoandikoetxea, "Extrapolation of weights revisited: New proofs and sharp bounds," J. Funct. Anal. **260** (6), 1886–1901 (2011).
7. A. Fiorenza, "Duality and reflexivity in grand Lebesgue spaces," Collect. Math. **51** (2), 131–148 (2000).
8. A. Fiorenza, B. Gupta, and P. Jain, "The maximal theorem for weighted grand Lebesgue spaces," Stud. Math. **188** (2), 123–133 (2008).
9. L. Grafakos, *Classical Fourier Analysis*, 3rd ed. (Springer, New York, 2014), Grad. Texts Math. **249**.
10. L. Greco, T. Iwaniec, and C. Sbordone, "Inverting the p-harmonic operator," Manuscr. Math. **92**, 249–258 (1997).
11. E. Harboure, R. A. Macías, and C. Segovia, "Extrapolation results for classes of weights," Am. J. Math. **110**, 383–397 (1988).
12. T. P. Hytönen, C. Pérez, and E. Rela, "Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type," J. Funct. Anal. **263** (12), 3883–3899 (2012).
13. T. Iwaniec and C. Sbordone, "On the integrability of the Jacobian under minimal hypotheses," Arch. Ration. Mech. Anal. **119**, 129–143 (1992).
14. V. Kokilashvili, "Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem," Proc. A. Razmadze Math. Inst. **151**, 129–133 (2009).
15. V. Kokilashvili, "Boundedness criteria for singular integrals in weighted grand Lebesgue spaces," J. Math. Sci. **170** (1), 20–33 (2010).
16. V. Kokilashvili, "Singular integrals and strong maximal functions in weighted grand Lebesgue spaces," in *Nonlinear Analysis, Function Spaces and Applications* (Czech Acad. Sci., Inst. Math., Prague, 2011), Vol. 9, pp. 261–269.

17. V. Kokilashvili and A. Meskhi, “A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces,” Georgian Math. J. **16** (3), 547–551 (2009).
18. V. Kokilashvili and A. Meskhi, “Trace inequalities for fractional integrals in grand Lebesgue spaces,” Stud. Math. **210** (2), 159–176 (2012).
19. V. Kokilashvili and A. Meskhi, “Potentials with product kernels in grand Lebesgue spaces: One-weight criteria,” Lith. Math. J. **53** (1), 27–39 (2013).
20. V. Kokilashvili, A. Meskhi, and H. Rafeiro, “Boundedness of commutators of singular and potential operators in generalized grand Morrey spaces and some applications,” Stud. Math. **217** (2), 159–178 (2013).
21. M. T. Lacey, K. Moen, C. Pérez, and R. H. Torres, “Sharp weighted bounds for fractional integral operators,” J. Funct. Anal. **259** (5), 1073–1097 (2010).
22. P. I. Lizorkin, “Estimates for trigonometric integrals and the Bernstein inequality for fractional derivatives,” Izv. Akad. Nauk SSSR, Ser. Mat. **29** (1), 109–126 (1965) [Am. Math. Soc. Transl., Ser. 2, **77**, 45–62 (1968)].
23. R. A. Macías and C. Segovia, “Lipschitz functions on spaces of homogeneous type,” Adv. Math. **33**, 257–270 (1979).
24. A. Meskhi, “Criteria for the boundedness of potential operators in grand Lebesgue spaces,” arXiv: 1007.1185v1 [math.FA].
25. S. M. Nikol’skii, “Inequalities for entire functions of finite degree and their application to the theory of differentiable functions of several variables,” Tr. Mat. Inst. im. V.A. Steklova, Akad. Nauk SSSR **38**, 244–278 (1951) [Am. Math. Soc. Transl., Ser. 2, **80**, 1–38 (1969)].
26. G. Pradolini and O. Salinas, “Commutators of singular integrals on spaces of homogeneous type,” Czech. Math. J. **57** (1), 75–93 (2007).
27. J. L. Rubio de Francia, “Factorization and extrapolation of weights,” Bull. Am. Math. Soc. **7**, 393–395 (1982).
28. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Nauka i Tekhnika, Minsk, 1987; Gordon and Breach, New York, 1993).
29. S. G. Samko and S. M. Umarkhadzhiev, “On Iwaniec–Sbordone spaces on sets which may have infinite measure,” Azerb. J. Math. **1** (1), 67–84 (2011).
30. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces* (Springer, Berlin, 1989), Lect. Notes Math. **1381**.
31. A. Zygmund, *Trigonometric Series* (Cambridge Univ. Press, Cambridge, 1959), Vol. 2.

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