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Convergence of Integrable Operators Affiliated to a Finite von Neumann Algebra

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Abstract—In the Banach space $L_1(\mathcal{M},\tau)$ of operators integrable with respect to a tracial state τ on a von Neumann algebra $\mathcal{M}$, convergence is analyzed. A notion of dispersion of operators in $L_2(\mathcal{M},\tau)$ is introduced, and its main properties are established. A convergence criterion in $L_2(\mathcal{M},\tau)$ in terms of the dispersion is proposed. It is shown that the following conditions for $X \in L_1(\mathcal{M},\tau)$ are equivalent: (i) $\tau(X)=0$, and (ii) $\|I+zX\|_1 \geq 1$ for all $z \in \mathbb{C}$. A.R. Padmanabhan's result (1979) on a property of the norm of the space $L_1(\mathcal{M},\tau)$ is complemented. The convergence in $L_2(\mathcal{M},\tau)$ of the imaginary components of some bounded sequences of operators from $\mathcal{M}$ is established. Corollaries on the convergence of dispersions are obtained.

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1. INTRODUCTION

Let τ be a faithful normal tracial state on a von Neumann algebra $\mathcal{M}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projectors in $\mathcal{M}$, and I be the identity operator in $\mathcal{M}$. We investigate the convergence in the Banach space $L_1(\mathcal{M},\tau)$ of τ-integrable operators [1, 2]. We introduce the dispersion $\mathbb{D}(X)=\|X-\tau(X)I\|_2^2$ of operators $X \in L_2(\mathcal{M},\tau)$ and establish its main properties (Theorem 4.1 and Corollary 4.2). We show that $\inf_{a\in\mathbb{C}}\|X-aI\|_2^2=\mathbb{D}(X)$ for all $X \in L_2(\mathcal{M},\tau)$ (Theorem 4.4). We propose a convergence criterion for sequences of operators in $L_2(\mathcal{M},\tau)$ in terms of the dispersion (Theorem 4.5). Let $\mathcal{K}_0=\{X \in L_2(\mathcal{M},\tau)\colon \tau(X)=0\}$. For $X_n, X \in \mathcal{K}_0$ $(n \in \mathbb{N})$, we prove the equivalence of the following conditions (Corollary 4.6):

(i) $X_n \xrightarrow{\|\cdot\|_2} X$ as $n\to\infty$, and

(ii) $X_n \xrightarrow{\tau} X$ and $\mathbb{D}(X_n)\to\mathbb{D}(X)$ as $n\to\infty$.

In Theorem 4.8, we show that the following conditions for $X \in L_1(\mathcal{M},\tau)$ are equivalent:

(i) $\tau(X)=0$, and

(ii) $\|I+zX\|_1 \geq 1$ for all $z\in\mathbb{C}$.

We complement Padmanabhan's result from [3] on a property of the norm of the space $L_1(\mathcal{M},\tau)$: if an operator $A \in L_1(\mathcal{M},\tau)^+$ is nonsingular, then

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall P\in\mathcal{M}^{\mathrm{pr}} \qquad \big(\tau(P)\geq\varepsilon \ \Rightarrow\ \|PAP\|_1\geq\delta\big)$$

(Theorem 4.9). We establish the convergence in $L_2(\mathcal{M},\tau)$ of the imaginary components of some bounded sequences of operators in $\mathcal{M}$ (Theorem 4.13) and apply the result to the convergence of dispersions (Corollaries 4.7 and 4.14).

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2. NOTATION AND DEFINITIONS

Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projectors in $\mathcal{M}$, I be the identity operator in $\mathcal{M}$, and $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. For $P, Q \in \mathcal{M}^{\mathrm{pr}}$, we write $P \sim Q$ (Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}$. The projector $P \wedge Q$ is defined by the equality $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$, and $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$ projects onto $\overline{\mathrm{Lin}(P\mathcal{H} \cup Q\mathcal{H})}$. Let $\mathcal{M}^{\mathrm{u}}$ and $\mathcal{M}^{+}$ be the subset of unitary operators and the cone of positive elements of $\mathcal{M}$, respectively.

A positive linear functional φ on $\mathcal{M}$ is called

- *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal* if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+)$ $\Rightarrow$ $\varphi(X) = \sup \varphi(X_i)$;
- *tracial* if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$;
- a *state* if $\varphi(I) = 1$.

An operator in $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to a von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A self-adjoint operator is affiliated to $\mathcal{M}$ if and only if all projectors from its spectral decomposition of unity belong to $\mathcal{M}$.

The set $\widetilde{\mathcal{M}}$ of all closed operators that are affiliated to $\mathcal{M}$ and densely defined in $\mathcal{H}$ is a $*$-algebra with respect to taking the adjoint operator, multiplication by scalars, and the operations of strong addition and multiplication obtained by the closure of the ordinary operations [1, 2]. For a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{sa}}$ its positive and Hermitian parts, respectively. The partial order in $\widetilde{\mathcal{M}}^{\mathrm{sa}}$ generated by the proper cone $\widetilde{\mathcal{M}}^+$ is denoted by $\leq$. Let $i \in \mathbb{C}$ with $i^2 = -1$ and $X \in \widetilde{\mathcal{M}}$. For $\mathrm{Re}\, X = (X + X^*)/2$ and $\mathrm{Im}\, X = (X - X^*)/(2i)$, we have $X = \mathrm{Re}\, X + i\, \mathrm{Im}\, X$ and $\mathrm{Re}\, X, \mathrm{Im}\, X \in \widetilde{\mathcal{M}}^{\mathrm{sa}}$.

If X is a closed densely defined linear operator affiliated to $\mathcal{M}$ and $|X| = \sqrt{X^*X}$, then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$. If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^+$.

Everywhere below, τ is a faithful normal state on $\mathcal{M}$. Denote by $\mu_t(X)$ the *rearrangement* of the operator $X \in \widetilde{\mathcal{M}}$, i.e., a nonincreasing right continuous function $\mu(X)\colon (0,1] \to [0,\infty)$ defined by the formula

$$\mu_t(X) = \inf\bigl\{\|XP\| \colon\ P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^{\perp}) \leq t\bigr\}, \qquad 0 < t \leq 1.$$

Let m be the linear Lebesgue measure on the interval $[0,1]$. One can define a noncommutative Lebesgue L_p-space $(1 \leq p < \infty)$ affiliated to $(\mathcal{M}, \tau)$ as $L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}}\colon \mu(X) \in L_p([0,1], m)\}$ with the norm $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. We have $L_q(\mathcal{M}, \tau) \subset L_p(\mathcal{M}, \tau)$ and $\|\cdot\|_p \leq \|\cdot\|_q$ on $L_q(\mathcal{M}, \tau)$ for all $1 \leq p \leq q \leq \infty$ (assume that $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ and $\|\cdot\|_\infty = \|\cdot\|$). The state τ can be uniquely extended to a bounded linear functional on $L_1(\mathcal{M}, \tau)$, which will be denoted by the same letter τ.

The $*$-algebra $\widetilde{\mathcal{M}}$ is equipped with the topology t_τ of convergence in measure [2, 4], for which a fundamental system of neighborhoods of zero is formed by the sets

$$U(\varepsilon, \delta) = \bigl\{X \in \widetilde{\mathcal{M}}\colon \exists\, P \in \mathcal{M}^{\mathrm{pr}}\ \ (\|XP\| \leq \varepsilon \ \text{ and } \ \tau(P^{\perp}) \leq \delta)\bigr\}, \qquad \varepsilon > 0, \quad \delta > 0.$$

It is known that $\langle \widetilde{\mathcal{M}}, t_\tau \rangle$ is a complete metrizable topological $*$-algebra, and $\mathcal{M}$ is dense in $\langle \widetilde{\mathcal{M}}, t_\tau \rangle$. To denote the convergence of a net $\{X_j\}_{j \in J} \subset \widetilde{\mathcal{M}}$ to $X \in \widetilde{\mathcal{M}}$ in the topology t_τ, one writes $X_j \xrightarrow{\tau} X$; in this case, $\{X_j\}_{j \in J}$ is said to converge to X in the measure τ. The topology t_τ is independent of the specific choice of the tracial state τ and is a minimal topology among all metrizable topologies consistent with the ring structure on $\widetilde{\mathcal{M}}$ (see [5]).

3. LEMMAS

Lemma 3.1 [6]**.** *If* $X \in \mathcal{M}$ *and* $Y \in \widetilde{\mathcal{M}}$, *then* $\mu_t(XY) \le \|X\|\mu_t(Y)$ *for all* $t > 0$. *If* $X, Y \in \widetilde{\mathcal{M}}^+$ *and* $X \le Y$, *then* $\mu_t(X) \le \mu_t(Y)$ *for all* $t > 0$.

Lemma 3.2 [7, p. 1463]**.** *We have* $|\tau(X)| \le \tau(|X|)$ *for all* $X \in L_1(\mathcal{M}, \tau)$.

Lemma 3.3 [8, Theorem 2.3]**.** *We have* $\tau(X^*) = \overline{\tau(X)}$ *for all* $X \in L_1(\mathcal{M}, \tau)$.

Lemma 3.4. *If* $X_n, X \in L_1(\mathcal{M}, \tau)$ *and* $X_n \xrightarrow{\|\cdot\|_1} X$, *then* $\tau(X_n) \to \tau(X)$ *as* $n \to \infty$.

Proof. We have $\tau(X_n) - \tau(X) = \tau(X_n - X)$ and $|\tau(X_n - X)| \le \tau(|X_n - X|) = \|X_n - X\|_1$ by Lemma 3.2. The lemma is proved. □

Lemma 3.5. *Let a number* $C > 0$ *and a sequence* $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *be such that* $\|Z_n\| \le C$ $(n \in \mathbb{N})$ *and* $Z_n \xrightarrow{\tau} Z \in \widetilde{\mathcal{M}}$ *as* $n \to \infty$. *Then* $Z \in \mathcal{M}$ *and* $Z_n \xrightarrow{\|\cdot\|_p} Z$ *as* $n \to \infty$ *for all* $1 \le p < \infty$.

Proof. Since $Z_n \xrightarrow{\tau} Z$, the sequence $\{\mu_t(Z_n)\}_{n=1}^{\infty}$ converges to $\mu_t(Z)$ at all continuity points t of the rearrangement of the operator Z (see [9, Lemma 1.2(iii)]). Next, $X \in \mathcal{M} \Leftrightarrow X \in \widetilde{\mathcal{M}}$ and

$$\mu_{0+}(X) \equiv \lim_{t \to 0+} \mu_t(X) = \sup_{0 < t \le 1} \mu_t(X) < \infty;$$

in this case, $\|X\| = \mu_{0+}(X)$ (see [10, Lemma 1.1(5)]). Therefore, $\|Z\| \le C$ and $\|Z_n - Z\| \le 2C$ for all $n \in \mathbb{N}$.

Let $X_n, X \in \widetilde{\mathcal{M}}$, $n \in \mathbb{N}$. We have $X_n \xrightarrow{\tau} X \Leftrightarrow \lim_{n\to\infty} \mu_t(X_n - X) = 0$ for every $t > 0$. Let $\varepsilon > 0$ be an arbitrary number. There exists an $N \in \mathbb{N}$ such that $\mu_\varepsilon(Z_n - Z) \le \varepsilon$ for all $n \ge N$. Since the rearrangement of an operator is a nonincreasing function, we have

$$\|Z_n - Z\|_p^p = \int_0^1 \mu_t(Z_n - Z)^p \, dt \le \int_0^\varepsilon \|Z_n - Z\|^p \, dt + \int_\varepsilon^1 \varepsilon \, dt < (2^p C^p + 1)\varepsilon$$

for all $n \ge N$. The lemma is proved. □

Lemma 3.5 can easily be extended to the noncommutative Orlicz spaces $L_f(\mathcal{M}, \tau)$ introduced in [11].

Lemma 3.6 [12, Theorem 17]**.** *If* $A, B \in \widetilde{\mathcal{M}}$ *and* $AB, BA \in L_1(\mathcal{M}, \tau)$, *then* $\tau(AB) = \tau(BA)$.

Lemma 3.7 [13, Theorem 2.2]**.** *If* $A, B \in \widetilde{\mathcal{M}}$, *then there exist* $U, V \in \mathcal{M}^{\mathrm{u}}$ *such that* $|A + B| \le U|A|U^* + V|B|V^*$.

Lemma 3.8 [14, Ch. V, Proposition 1.6]**.** *We have* $P \vee Q - Q \sim P - P \wedge Q$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

4. MAIN RESULTS

Theorem 4.1. *Let* $\mathbb{D}(X) = \|X - \tau(X)I\|_2^2$ *be the dispersion of operators* $X \in L_2(\mathcal{M}, \tau)$. *Then*

(i) $\mathbb{D}(X) = \mathbb{D}(X^*) = \mathbb{D}(UXU^*)$ *for all* $X \in L_2(\mathcal{M}, \tau)$ *and* $U \in \mathcal{M}^{\mathrm{u}}$;

(ii) $\mathbb{D}(X^*X) = \mathbb{D}(XX^*)$ *for all* $X \in L_4(\mathcal{M}, \tau)$;

(iii) $\mathbb{D}(|X|) \le \mathbb{D}(X)$ *for all* $X \in L_2(\mathcal{M}, \tau)$;

(iv) $\sqrt{\mathbb{D}(X + Y)} \le \sqrt{\mathbb{D}(X)} + \sqrt{\mathbb{D}(Y)}$ *for all* $X, Y \in L_2(\mathcal{M}, \tau)$;

(v) $\mathbb{D}(X + Y) \le 2(\mathbb{D}(X) + \mathbb{D}(Y))$ *for all* $X, Y \in L_2(\mathcal{M}, \tau)$;

(vi) $\mathbb{D}(aI + bX) = |b|^2\mathbb{D}(X)$ *for all* $a, b \in \mathbb{C}$ *and* $X \in L_2(\mathcal{M}, \tau)$;

(vii) $\mathbb{D}(Z) = \mathbb{D}(\operatorname{Re} Z) + \mathbb{D}(\operatorname{Im} Z)$ *for all* $Z \in L_2(\mathcal{M}, \tau)$;

(viii) $\mathbb{D}(aP + bP^{\perp}) = |a - b|^2\tau(P)\tau(P^{\perp}) = \mathbb{D}(bP + aP^{\perp})$ *for all* $a, b \in \mathbb{C}$ *and* $P \in \mathcal{M}^{\mathrm{pr}}$; *in particular,* $\mathbb{D}(P) = \tau(P)\tau(P^{\perp}) = \mathbb{D}(P^{\perp})$ *for all* $P \in \mathcal{M}^{\mathrm{pr}}$.

Proof. By Lemma 3.3 and the definition of dispersion, we have

$$\mathbb{D}(X) = \tau(X^*X) - |\tau(X)|^2 \qquad \text{for all} \quad X \in L_2(\mathcal{M}, \tau). \tag{4.1}$$

Assertion (i) follows from Lemma 3.3 and the unitary invariance of the trace τ on $L_1(\mathcal{M}, \tau)$.

Let us prove (ii). Setting $A = X^*XX^*$ and $B = X$ and applying Lemma 3.6, relation (4.1), and the definition of the trace τ, we have the equalities

$$\mathbb{D}(X^*X) = \tau(X^*XX^* \cdot X) - |\tau(X^*X)| = \tau(X \cdot X^*XX^*) - |\tau(XX^*)| = \mathbb{D}(XX^*)$$

for all $X \in L_4(\mathcal{M}, \tau)$.

Assertion (iii) follows from (4.1) and Lemma 3.2.

Assertion (iv) follows from the additivity of τ and the triangle inequality for $\|\cdot\|_2$.

Assertion (v) follows from (iv). By (v), we have $\overline{\tau(X)}\tau(Y) + \tau(X)\overline{\tau(Y)} \le \|X + Y\|_2^2$ for all $X, Y \in L_2(\mathcal{M}, \tau)$.

Assertion (vi) follows from the definition of $\mathbb{D}$, relation (4.1), and Lemma 3.3.

Let us prove (vii). Let $Z \in L_2(\mathcal{M}, \tau)$; then the operators $X = \operatorname{Re} Z$ and $Y = \operatorname{Im} Z$ belong to $L_2(\mathcal{M}, \tau)^{\text{sa}}$. Since $XY, YX \in L_1(\mathcal{M}, \tau)$, we have $\tau(XY - YX) = 0$ by Lemma 3.6. Since $\tau(X), \tau(Y) \in \mathbb{R}$, by virtue of (4.1) we obtain

$$\begin{aligned}\mathbb{D}(X + iY) &= \tau((X - iY)(X + iY)) - |\tau(X + iY)|^2 \\ &= \tau(X^2) + \tau(Y^2) - \tau(X)^2 - \tau(Y)^2 = \mathbb{D}(X) + \mathbb{D}(Y).\end{aligned}$$

Assertion (viii) follows from (4.1). The theorem is proved. $\square$

Corollary 4.2. *Let* $P, Q \in \mathcal{M}^{\text{pr}}$. *Then*

(i) $\mathbb{D}(P \vee Q) \le \mathbb{D}(P) + \mathbb{D}(Q)$;

(ii) $\mathbb{D}(P \wedge Q) \le \mathbb{D}(P) + \mathbb{D}(Q)$;

(iii) $\mathbb{D}(P) = \mathbb{D}(Q) \Leftrightarrow \tau(P) \in \{\tau(Q), \tau(Q^\perp)\}$.

Proof. Let us prove (i). By Lemma 3.8 we obtain $\tau(P \vee Q) \le \tau(P) + \tau(Q)$ for all $P, Q \in \mathcal{M}^{\text{pr}}$. Taking into account the inequality $\max\{\tau(P), \tau(Q)\} \le \tau(P \vee Q)$ and assertion (viii) of Theorem 4.1, we have

$$\begin{aligned}\mathbb{D}(P \vee Q) &= \tau(P \vee Q)(1 - \tau(P \vee Q)) \le (\tau(P) + \tau(Q))(1 - \tau(P \vee Q)) \\ &= \tau(P)(1 - \tau(P \vee Q)) + \tau(Q)(1 - \tau(P \vee Q)) \\ &\le \tau(P)(1 - \tau(P)) + \tau(Q)(1 - \tau(Q)) = \mathbb{D}(P) + \mathbb{D}(Q).\end{aligned}$$

Let us prove (ii). By De Morgan's duality law, we have $(P \wedge Q)^\perp = P^\perp \vee Q^\perp$ for all $P, Q \in \mathcal{M}^{\text{pr}}$. By assertion (i) for the pair of projectors $P^\perp$, $Q^\perp$ and by assertion (viii) of Theorem 4.1, we obtain

$$\mathbb{D}(P \wedge Q) = \mathbb{D}((P \wedge Q)^\perp) = \mathbb{D}(P^\perp \vee Q^\perp) \le \mathbb{D}(P^\perp) + \mathbb{D}(Q^\perp) = \mathbb{D}(P) + \mathbb{D}(Q).$$

Assertion (iii) follows from assertion (viii) of Theorem 4.1. In particular, if $P \sim Q$, then $\mathbb{D}(P) = \mathbb{D}(Q)$. The corollary is proved. $\square$

The set $\mathcal{K}_0 = \{X \in L_2(\mathcal{M}, \tau) \colon \ \tau(X) = 0\}$ is a closed subspace in $L_2(\mathcal{M}, \tau)$: if $X_n \in \mathcal{K}_0$ $(n \in \mathbb{N})$, $X \in L_2(\mathcal{M}, \tau)$, and $X_n \xrightarrow{\|\cdot\|_2} X$ as $n \to \infty$, then $X_n, X \in L_1(\mathcal{M}, \tau)$ $(n \in \mathbb{N})$ and $X_n \xrightarrow{\|\cdot\|_1} X$ as $n \to \infty$; therefore, $X \in \mathcal{K}_0$ in view of Lemma 3.4. The orthogonal complement $\mathcal{K}_0^\perp$ of the subspace $\mathcal{K}_0$ in $L_2(\mathcal{M}, \tau)$ is one-dimensional and is generated by the operator I. Indeed, since by Lemma 3.3 the inner product $(I, X)_{L_2(\mathcal{M}, \tau)}$ is equal to $\tau(I \cdot X^*) = \overline{\tau(X)} = 0$ for $X \in \mathcal{K}_0$, it follows that $I \perp \mathcal{K}_0$. Therefore, for every $X \in L_2(\mathcal{M},\tau)$, the decomposition $X = (X - \tau(X)\cdot I) + \tau(X)\cdot I$ is valid, where the first term on the right-hand side belongs to $\mathcal{K}_0$.

Example 4.3. Let $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$ be the full matrix algebra and $\tau = \mathrm{tr}_n$ be the normalized trace on $\mathcal{M}$. There is a well-known Jacobi formula: $\det e^X = e^{n\tau(X)}$ for all $X \in \mathcal{M}$. In particular, if $\det e^X = 1$, then $\tau(X) = 0$. For $X \in \mathcal{M}$, the following conditions are equivalent:

(i) X is unitarily equivalent to a matrix with zero diagonal;

(ii) $\tau(X) = 0$;

(iii) X is a commutator.

For the proof of the equivalence (i) ⇔ (ii), see [15, Ch. II, Problem 209], and for the equivalence (ii) ⇔ (iii), see [16, Problem 182]. Thus, $\mathcal{K}_0$ coincides with the set of all commutators, and $L_2(\mathcal{M},\tau)$ coincides with $\mathcal{M}$. Therefore, every matrix $A \in \mathbb{M}_n(\mathbb{C})$

(1) is representable as a sum $A = \lambda I + X$ with $X \in \mathcal{K}_0$ and $\lambda = \mathrm{tr}_n A$;

(2) is unitarily equivalent to a matrix with "constant" diagonal;

(3) has a rearrangement

$$\mu_t(A) = \sum_{k=1}^{n} s_k(A)\chi_{[(k-1)/n,\,k/n)}(t), \qquad 0 < t \le 1,$$

where $\{s_k(A)\}_{k=1}^n$ is the set of s-numbers of the matrix A, i.e., the set of eigenvalues of $|A|$ taken in decreasing order and counted with their multiplicities; $\chi_{\mathcal{B}}$ is the indicator of a set $\mathcal{B} \subset \mathbb{R}$.

Theorem 4.4. *If $X \in L_2(\mathcal{M},\tau)$, then $\inf_{a\in\mathbb{C}} \|X - aI\|_2^2 = \mathbb{D}(X)$, i.e.,*

$$\arg\inf_{a\in\mathbb{C}} \|X - aI\|_2^2 = \tau(X).$$

Proof. Let $X \in L_2(\mathcal{M},\tau)$ and $b = \tau(X)$; then $\tau(X - bI) = 0$ and by Lemma 3.3 we obtain

$$\begin{aligned}\|X - aI\|_2^2 &= \|(X - bI) - (a - b)I\|_2^2 = \tau\big((X - bI) - (a - b)I\big)^*\big((X - bI) - (a - b)I\big)\\ &= \tau\big((X - bI)^*(X - bI)\big) - \overline{(a - b)}\tau(X - bI) - (a - b)\overline{\tau(X - bI)} + |a - b|^2\\ &= \mathbb{D}(X) + |a - b|^2 \ge \mathbb{D}(X),\end{aligned}$$

where the equality is attained if and only if $a = b = \tau(X)$. The theorem is proved. □

Theorem 4.5. *Let $1 \le p < 2$ and $X_n, X \in L_2(\mathcal{M},\tau)$, $n \in \mathbb{N}$. The following conditions are equivalent*:

(i) $X_n \xrightarrow{\|\cdot\|_2} X$ *as* $n \to \infty$;

(ii) $X_n \xrightarrow{\|\cdot\|_p} X$ *and* $\mathbb{D}(X_n) \to \mathbb{D}(X)$ *as* $n \to \infty$;

(iii) $X_n \xrightarrow{\tau} X$ *and* $\mathbb{D}(X_n) \to \mathbb{D}(X)$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty}|\tau(X_n)| \le |\tau(X)|$.

Proof. (i) ⇒ (ii). Since $X_n \xrightarrow{\|\cdot\|_2} X$ as $n \to \infty$, it follows that $\|X_n\|_2 \to \|X\|_2$ and $X_n \xrightarrow{\|\cdot\|_p} X$ as $n \to \infty$. Therefore, $X_n \xrightarrow{\|\cdot\|_1} X$ and $\tau(X_n) \to \tau(X)$ as $n \to \infty$ by Lemma 3.4. In view of (4.1), we have

$$\mathbb{D}(X_n) = \|X_n\|_2^2 - |\tau(X_n)|^2 \to \|X\|_2^2 - |\tau(X)|^2 = \mathbb{D}(X) \qquad \text{as} \quad n \to \infty.$$

(ii) ⇒ (iii). If $X_n \xrightarrow{\|\cdot\|_p} X$ as $n \to \infty$, then $X_n \xrightarrow{\tau} X$ as $n \to \infty$ in view of [5, Theorem 1]. Since $X_n \xrightarrow{\|\cdot\|_1} X$ as $n \to \infty$, it follows that $\tau(X_n) \to \tau(X)$ as $n \to \infty$ by Lemma 3.4.

(iii) ⇒ (i). We have $X_n^* \xrightarrow{\tau} X^*$ as $n \to \infty$ in view of the t_τ-continuity of the involution from $\widetilde{\mathcal{M}}$ to $\widetilde{\mathcal{M}}$. Therefore, $X_n^* X_n \xrightarrow{\tau} X^* X$ as $n \to \infty$ due to the joint t_τ-continuity of the product from $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$ to $\widetilde{\mathcal{M}}$. Since $X_n^* X_n, X^* X \in L_1(\mathcal{M}, \tau)^+$, we have

$$\tau(X^* X) \le \liminf_{n\to\infty} \tau(X_n^* X_n)$$

by Fatou's lemma [6, Theorem 3.5(i)]. Now, in view of the properties of the lower limit of number sequences, we have

$$|\tau(X)|^2 - \liminf_{n\to\infty} \tau(X_n^* X_n) \le |\tau(X)|^2 - \tau(X^* X) = \liminf_{n\to\infty} \big(|\tau(X_n)|^2 - \tau(X_n^* X_n)\big)$$
$$\le \liminf_{n\to\infty} |\tau(X_n)|^2 - \liminf_{n\to\infty} \tau(X_n^* X_n).$$

Therefore, $|\tau(X)|^2 \le \liminf_{n\to\infty} |\tau(X_n)|^2$, and since the real function $t \mapsto \sqrt{t}$ $(t \ge 0)$ is monotone and continuous, we obtain $|\tau(X)| \le \liminf_{n\to\infty} |\tau(X_n)|$. Hence, $|\tau(X)| = \lim_{n\to\infty} |\tau(X_n)|$ and

$$\|X_n\|_2^2 = \mathbb{D}(X_n) + |\tau(X_n)|^2 \to \mathbb{D}(X) + |\tau(X)|^2 = \|X\|_2^2 \qquad \text{as} \quad n \to \infty.$$

Since the real function $t \mapsto \sqrt{t}$ $(t \ge 0)$ is continuous, we find that $\|X_n\|_2 \to \|X\|_2$ as $n \to \infty$. Consequently, $X_n \xrightarrow{\|\cdot\|_2} X$ as $n \to \infty$ in view of [6, Theorem 3.7]. The theorem is proved. □

Corollary 4.6. *Let $X_n, X \in \mathcal{K}_0$, $n \in \mathbb{N}$. The following conditions are equivalent*:

(i) $X_n \xrightarrow{\|\cdot\|_2} X$ *as* $n \to \infty$;

(ii) $X_n \xrightarrow{\tau} X$ *and* $\mathbb{D}(X_n) \to \mathbb{D}(X)$ *as* $n \to \infty$.

Theorem 4.5 and Lemmas 3.4 and 3.5 imply

Corollary 4.7. *Under the hypotheses of Lemma* 3.5, *it holds that* $\mathbb{D}(Z_n) \to \mathbb{D}(Z)$ *and* $\tau(Z_n) \to \tau(Z)$ *as* $n \to \infty$.

Theorem 4.8. *For* $X \in L_1(\mathcal{M}, \tau)$, *the following conditions are equivalent*:

(i) $\tau(X) = 0$;

(ii) $\|I + zX\|_1 \ge 1$ *for all* $z \in \mathbb{C}$.

Proof. (i) ⇒ (ii). By Lemma 3.2, we have

$$\|I + zX\|_1 = \tau(|I + zX|) \ge |\tau(I + zX)| = |1 + z\tau(X)| = 1.$$

(ii) ⇒ (i). Let us rewrite inequality (ii) as $\tau(\rho^{-1}(|I + \rho e^{i\theta} X| - I)) \ge 0$, where $\rho > 0$ and $\theta \in \mathbb{R}$. Since

$$\frac{1}{\rho}(|I + \rho Z| - I) = (2\operatorname{Re} Z + \rho |Z|^2)(|I + \rho Z| + I)^{-1}$$

for all $\rho > 0$ and $Z \in \widetilde{\mathcal{M}}$, and since the involution from $\widetilde{\mathcal{M}}$ to $\widetilde{\mathcal{M}}$ is t_τ-continuous, the product from $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$ to $\widetilde{\mathcal{M}}$ is jointly t_τ-continuous, and the operator function $Z \mapsto \sqrt{Z}$ from $\widetilde{\mathcal{M}}^+$ to $\widetilde{\mathcal{M}}^+$ is t_τ-continuous, by [3, Theorem 2.1] we have

$$\frac{1}{\rho}(|I + \rho e^{i\theta} X| - I) \xrightarrow{\tau} \operatorname{Re}(e^{i\theta} X) \qquad \text{as} \quad \rho \to 0+ \tag{4.2}$$

for all $\theta \in \mathbb{R}$. Let $\rho > 0$ and $\theta \in \mathbb{R}$. By Lemma 3.7, there exists an operator $U_{\rho,\theta} \in \mathcal{M}^{\mathrm{u}}$ such that $|\rho^{-1}I + e^{i\theta}X| \le \rho^{-1}I + U_{\rho,\theta}|X|U^*_{\rho,\theta}$; therefore,

$$\left|\frac{1}{\rho}I + e^{i\theta}X\right| - \frac{1}{\rho}I \le U_{\rho,\theta}|X|U^*_{\rho,\theta}. \tag{4.3}$$

Applying once again Lemma 3.7, we find $V_{\rho,\theta}, W_{\rho,\theta} \in \mathcal{M}^{\mathrm{u}}$ such that

$$\frac{1}{\rho}I = \left|\frac{1}{\rho}I + e^{i\theta}X - e^{i\theta}X\right| \le V_{\rho,\theta}\left|\frac{1}{\rho}I + e^{i\theta}X\right|V^*_{\rho,\theta} + W_{\rho,\theta}|X|W^*_{\rho,\theta};$$

hence,

$$\left|\frac{1}{\rho}I + e^{i\theta}X\right| - \frac{1}{\rho}I \ge -V^*_{\rho,\theta}W_{\rho,\theta}|X|W^*_{\rho,\theta}V_{\rho,\theta}. \tag{4.4}$$

From (4.3) and (4.4), for the operator $Y_{\rho,\theta} \equiv U_{\rho,\theta}|X|U^*_{\rho,\theta} + V^*_{\rho,\theta}W_{\rho,\theta}|X|W^*_{\rho,\theta}V_{\rho,\theta} \in L_1(\mathcal{M},\tau)^+$ we obtain

$$-Y_{\rho,\theta} \le \left|\frac{1}{\rho}I + e^{i\theta}X\right| - \frac{1}{\rho}I \le Y_{\rho,\theta};$$

therefore, by [17, Theorem 1], for some $S_{\rho,\theta} \in \mathcal{M}^{\mathrm{u}} \cap \mathcal{M}^{\mathrm{sa}}$ we have

$$\left|\left|\frac{1}{\rho}I + e^{i\theta}X\right| - \frac{1}{\rho}I\right| \le \frac{1}{2}(Y_{\rho,\theta} + S_{\rho,\theta}Y_{\rho,\theta}S_{\rho,\theta}). \tag{4.5}$$

Recall [6] that $\mu_{t+s}(A+B) \le \mu_t(A) + \mu_s(B)$ and $\mu_t(A) = \mu_t(|A|)$ for all $A, B \in \widetilde{\mathcal{M}}$ and $t, s > 0$. Consequently, for all $t > 0$, by Lemma 3.1 and the unitary invariance of rearrangements, we have

$$\begin{aligned}\mu_t\left(\left|\frac{1}{\rho}I + e^{i\theta}X\right| - \frac{1}{\rho}I\right) &\le \frac{1}{2}\mu_t(Y_{\rho,\theta} + S_{\rho,\theta}Y_{\rho,\theta}S_{\rho,\theta}) \le \frac{1}{2}\big(\mu_{t/2}(Y_{\rho,\theta}) + \mu_{t/2}(S_{\rho,\theta}Y_{\rho,\theta}S_{\rho,\theta})\big)\\ &= \mu_{t/2}(Y_{\rho,\theta}) \le \mu_{t/4}\big(U_{\rho,\theta}|X|U^*_{\rho,\theta}\big) + \mu_{t/4}\big(V^*_{\rho,\theta}W_{\rho,\theta}|X|W^*_{\rho,\theta}V_{\rho,\theta}\big)\\ &= 2\mu_{t/4}(X).\end{aligned}$$

Let us apply the dominated convergence theorem (see [18, Proposition 3.3(ii)]): if $\{A\}_{n=1}^{\infty} \in L_1(\mathcal{M},\tau)$, $A_n \xrightarrow{\tau} A \in \widetilde{\mathcal{M}}$ as $n \to \infty$, and $\mu(A_n) \le f \in L_1(\mathbb{R}^+, m)$ for all $n \in \mathbb{N}$, then $A \in L_1(\mathcal{M},\tau)$ and $A_n \xrightarrow{\|\cdot\|_1} A$ as $n \to \infty$. Therefore, from (4.2) and (4.5) with $f(t) \equiv 2\mu_{t/4}(X)$, taking account of Lemma 3.3, we obtain

$$\mathrm{Re}(e^{i\theta}\tau(X)) = \tau(\mathrm{Re}(e^{i\theta}X)) = \lim_{\rho\to 0+} \tau\left(\frac{1}{\rho}(|I + \rho e^{i\theta}X| - I)\right) \ge 0,$$

and this relation holds for all $\theta \in \mathbb{R}$. Choosing θ such that $e^{i\theta}\tau(X) = -|\tau(X)|$, we get $\tau(X) = 0$. The theorem is proved. ☐

For $A \in L_1(\mathcal{M},\tau)$, it was essentially established in [3, Proposition 2] that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall P \in \mathcal{M}^{\mathrm{pr}} \qquad \big(\tau(P) \le \delta \;\Rightarrow\; \|PAP\|_1 \le \varepsilon\big).$$

Theorem 4.9. *Let $A \in L_1(\mathcal{M},\tau)^+$ be a nonsingular operator. Then*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall P \in \mathcal{M}^{\mathrm{pr}} \qquad \big(\tau(P) \ge \varepsilon \;\Rightarrow\; \|PAP\|_1 \ge \delta\big).$$

Proof. Let $\varepsilon > 0$ be an arbitrary number and $\lambda > 0$ be such that the spectral projector $Q = P^A((\lambda, +\infty))$ satisfies the inequality $\tau(Q) > 1 - \varepsilon/2$. Let $P \in \mathcal{M}^{\mathrm{pr}}$ with $\tau(P) \ge \varepsilon$ be arbitrary. By Lemma 3.8, we have $\tau(P \wedge Q) + \tau(P \vee Q) = \tau(P) + \tau(Q)$ and, taking into account the inequality $\tau(P \vee Q) \le 1$, obtain $\tau(P \wedge Q) \ge \varepsilon/2$.

If $X, Y \in \widetilde{\mathcal{M}}^+$ and $Z \in \widetilde{\mathcal{M}}$, then the inequality $X \le Y$ implies that $ZXZ^* \le ZYZ^*$ (see [18, p. 720]). We have $\mu_t(ZZ^*) = \mu_t(Z^*Z)$ for all $t > 0$ and $Z \in \widetilde{\mathcal{M}}$ (see [10, formula (1)]). Then, by Lemma 3.1, we obtain

$$\|PAP\|_1 = \tau(PAP) = \int_0^1 \mu_t(PAP)\,dt = \int_0^1 \mu_t\big(\sqrt{A}\cdot P\cdot\sqrt{A}\big)\,dt \ge \int_0^1 \mu_t\big(\sqrt{A}\cdot P\wedge Q\cdot\sqrt{A}\big)\,dt$$

$$= \int_0^1 \mu_t(P\wedge Q\cdot A\cdot P\wedge Q)\,dt \ge \int_0^1 \mu_t(\lambda\cdot P\wedge Q)\,dt = \lambda\tau(P\wedge Q) \ge \lambda\frac{\varepsilon}{2} \equiv \delta.$$

The theorem is proved. $\square$

The Radon–Nikodym theorem (see [19]) and Theorem 4.9 imply

Corollary 4.10. *Let φ be a faithful normal positive linear functional on $\mathcal{M}$. Then*

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \forall P \in \mathcal{M}^{\mathrm{pr}} \qquad \big(\tau(P) \ge \varepsilon \Rightarrow \varphi(P) \ge \delta\big).$$

Theorem 4.11. *Let φ be a faithful normal state on $\mathcal{M}$ such that the Radon–Nikodym derivative $d\varphi/d\tau$ belongs to $\mathcal{M}$. Then there exists a number $\lambda \in (0,1]$ and a faithful normal state ψ on $\mathcal{M}$ such that $\tau = \lambda\varphi + (1-\lambda)\psi$.*

Proof. Obviously, $A \equiv d\varphi/d\tau \in \mathcal{M}^+$, $\|A\| \ge 1$, and $\tau(A) = 1$. We have $\varphi(X) = \tau(AX)$ for all $X \in \mathcal{M}$. Take a constant $C > \|A\|$ and set $\lambda = C^{-1}$. Let

$$\psi(X) \equiv \frac{1}{1-\lambda}\tau((I-\lambda A)X) \qquad \text{for all} \quad X \in \mathcal{M}.$$

Since $I - \lambda A \in \mathcal{M}^+$ and this operator is invertible, ψ is a faithful normal state on $\mathcal{M}$. The rest is obvious. The theorem is proved. $\square$

Remark 4.12. If the state φ in Theorem 4.11 is tracial, then $A \in \mathcal{M} \cap \mathcal{M}'$ and ψ is also a tracial state. Let $\mu^{(1)}(X)$ and $\mu^{(2)}(X)$ be the rearrangements of the operator $X \in \widetilde{\mathcal{M}}$ with respect to φ and ψ, respectively. We have

$$\mu_t^{(1)}(X) = \inf\big\{\|XP\|\colon\ P \in \mathcal{M}^{\mathrm{pr}},\ \lambda^{-1}\tau(P^\perp) - (\lambda^{-1}-1)\psi(P^\perp) \le t\big\}$$
$$\le \inf\big\{\|XP\|\colon\ P \in \mathcal{M}^{\mathrm{pr}},\ \lambda^{-1}\tau(P^\perp) \le t\big\} = \mu_{\lambda t}(X)$$

for all $0 < t \le 1$. In a similar way we obtain $\mu_t^{(2)}(X) \le \mu_{(1-\lambda)t}(X)$ for all $0 < t \le 1$. Therefore, $L_p(\mathcal{M},\tau) \subset L_p(\mathcal{M},\varphi) \cap L_p(\mathcal{M},\psi)$ for all $1 \le p < \infty$.

Theorem 4.13. *Let numbers $1 \le p < \infty$ and $C > 0$ and a sequence $\{Z_n\}_{n=1}^\infty$ in $\mathcal{M}$ be such that $\|Z_n\| \le C$ ($n \in \mathbb{N}$), $\|\mathrm{Re}\, Z_n\|_1 \to 1$, and $|Z_n| \xrightarrow{\tau} I$ as $n \to \infty$. Then $\|\mathrm{Im}\, Z_n\|_p \to 0$ as $n \to \infty$.*

Proof. By Lemma 3.5, we have $|Z_n| \xrightarrow{\|\cdot\|_1} I$ as $n \to \infty$. Let $Z_n = X_n + iY_n$ with $X_n, Y_n \in \mathcal{M}^{\mathrm{sa}}$, $n \in \mathbb{N}$. By Lemma 3.1, we have

$$\mu_t(I - Z_n^*Z_n) = \mu_t(I - |Z_n|^2) = \mu_t\big((I+|Z_n|)(I-|Z_n|)\big) \le \big\|I + |Z_n|\big\|\mu_t(I - |Z_n|)$$
$$\le (C+1)\mu_t(I - |Z_n|)$$

for all $t > 0$ and $n \in \mathbb{N}$; therefore, $Z_n^*Z_n \xrightarrow{\|\cdot\|_1} I$ as $n \to \infty$. Let $Z_n^* = V_n|Z_n^*|$ be a polar decomposition of the operator Z_n^*; then $|Z_n| = V_n|Z_n^*|V_n^*$ and $Z_n^*Z_n = V_nZ_nZ_n^*V_n^*$ for all $n \in \mathbb{N}$.

Since the von Neumann algebra $\mathcal{M}$ is finite, the partial isometry V_n can be extended to an operator $U_n \in \mathcal{M}^{\mathrm{u}}$ with the property $Z_n^* = U_n|Z_n^*|$ (see [5, proof of Theorem 2]). Thus, $Z_n^* Z_n = U_n Z_n Z_n^* U_n^*$ for all $n \in \mathbb{N}$. Now,

$$I - Z_n Z_n^* = U_n^*(I - Z_n^* Z_n)U_n \xrightarrow{\|\cdot\|_1} 0 \qquad \text{as} \quad n \to \infty$$

due to the unitary invariance of the norm $\|\cdot\|_1$. Hence,

$$I - (X_n^2 + Y_n^2) = \frac{1}{2}\big((I - Z_n^* Z_n) + (I - Z_n Z_n^*)\big) \xrightarrow{\|\cdot\|_1} 0 \qquad \text{as} \quad n \to \infty.$$

Since $I - \sqrt{X_n^2 + Y_n^2} = (I + \sqrt{X_n^2 + Y_n^2}\,)^{-1}(I - (X_n^2 + Y_n^2))$ and $(I + \sqrt{X_n^2 + Y_n^2})^{-1} \le I$ for all $n \in \mathbb{N}$, by Lemma 3.1 we have

$$\mu_t\big(I - \sqrt{X_n^2 + Y_n^2}\,\big) \le \big\|\big(I + \sqrt{X_n^2 + Y_n^2}\,\big)^{-1}\big\|\, \mu_t\big(I - (X_n^2 + Y_n^2)\big) \le \mu_t\big(I - (X_n^2 + Y_n^2)\big)$$

for all $t > 0$ and $n \in \mathbb{N}$. Therefore, $\sqrt{X_n^2 + Y_n^2} \xrightarrow{\|\cdot\|_1} I$ as $n \to \infty$. Since $X_n^2 \le X_n^2 + Y_n^2$, it follows that $\sqrt{X_n^2} = |X_n| \le \sqrt{X_n^2 + Y_n^2}$ for all $n \in \mathbb{N}$, because the function $\lambda \mapsto \sqrt{\lambda}$ ($\lambda \ge 0$) is operator monotone.

Let $A_n \equiv \sqrt{X_n^2 + Y_n^2} - |X_n|$ and $B_n \equiv \sqrt{X_n^2 + Y_n^2} + |X_n|$ for all $n \in \mathbb{N}$. Then $A_n, B_n \in \mathcal{M}^+$ and $\tau(A_n) = \|A_n\|_1 \to 0$ as $n \to \infty$. Since

$$\big\|\sqrt{X_n^2 + Y_n^2}\,\big\| = \sqrt{\|X_n^2 + Y_n^2\|} \le \sqrt{\|X_n^2\| + \|Y_n^2\|} \le \sqrt{\|X_n\|^2 + \|Y_n\|^2} \le \sqrt{2}C,$$

it follows that $\|B_n\| \le (\sqrt{2} + 1)C$ for all $n \in \mathbb{N}$. It is easy to see that

$$2Y_n^2 = A_n B_n + B_n A_n \qquad (n \in \mathbb{N}).$$

Now, by the triangle inequality for the norm $\|\cdot\|_1$, the equality $\|T\|_1 = \|T^*\|_1$ for all $T \in L_1(\mathcal{M}, \tau)$, and Lemma 3.1, we have

$$\begin{aligned}\|Y_n\|_2^2 = \tau(Y_n^2) = \|Y_n^2\|_1 = \frac{1}{2}\|A_n B_n + B_n A_n\|_1 &\le \frac{1}{2}\big(\|A_n B_n\|_1 + \|B_n A_n\|_1\big) = \|B_n A_n\|_1 \\ &\le \|B_n\| \cdot \|A_n\|_1 \le (\sqrt{2} + 1)C\|A_n\|_1\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $Y_n \xrightarrow{\|\cdot\|_2} 0$ as $n \to \infty$. Therefore, $Y_n \xrightarrow{\tau} 0$ as $n \to \infty$ in view of [5, Theorem 1], and by Lemma 3.5 we have $Y_n \xrightarrow{\|\cdot\|_p} 0$ as $n \to \infty$. The theorem is proved. □

Theorems 4.5 and 4.13 imply

Corollary 4.14. *Under the hypotheses of Theorem* 4.13, *we have* $\mathbb{D}(\operatorname{Im} Z_n) \to 0$ *as* $n \to \infty$.

Remark 4.15. If $\mathcal{M} = L^\infty(\Omega, \Sigma, \nu)$ and $\tau(f) = \int_\Omega f\, d\nu$, where (Ω, Σ, ν) is a probability space, then the $*$-algebra $\widetilde{\mathcal{M}}$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, ν). In this case, the topology t_τ is the ordinary topology of convergence in probability; the rearrangement

$$\mu_t(f) = \inf\big\{s \ge 0\colon\ \nu(\{\omega \in \Omega\colon |f(\omega)| > s\}) \le t\big\}$$

coincides with the nonincreasing rearrangement of the function $|f|$. In this commutative case, Theorem 4.5 is new, while Theorems 4.8 and 4.13 are given in [20, Exercise 2.12.104] and [20, Exercise 2.12.105], respectively.

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