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Nonlinear Trigonometric Approximations of Multivariate Function Classes

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Abstract—Order-sharp estimates are established for the best N-term approximations of functions from Nikol'skii–Besov type classes $\mathrm{B}^{sm}_{pq}(\mathbb{T}^k)$ with respect to the multiple trigonometric system $\mathfrak{T}^{(k)}$ in the metric of $L_r(\mathbb{T}^k)$ for a number of relations between the parameters s, p, q, r, and m ($s = (s_1, \dots, s_n) \in \mathbb{R}^n_+$, $1 \le p, q, r \le \infty$, $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, $k = m_1 + \ldots + m_n$). Constructive methods of nonlinear trigonometric approximation—variants of the so-called greedy algorithms—are used in the proofs of upper estimates.

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1. INTRODUCTION

Let X be a Banach space with norm $\|\cdot|X\|$ and $\mathcal{G} = (\phi_\jmath\colon \jmath \in \mathcal{J})$ be a system of elements in X ($\mathcal{J}$ is a countable set of indices). The quantity

$$\sigma_N(f, \mathcal{G}, X) = \inf\left\{ \left\| f - \sum_{\imath\in\mathcal{I}} c_\imath \phi_\imath \right| X \right\| \,\Big|\, c_\imath \in \mathbb{C}\ (\imath \in \mathcal{I}),\ \mathcal{I} \subset \mathcal{J}\colon \#\mathcal{I} = N \right\}$$

is called the *best N-term approximation of an element $f \in X$ with respect to the system* $\mathcal{G}$ ($N \in \mathbb{N}_0$). For a set $\mathrm{F} \subset X$, we define

$$\sigma_N(\mathrm{F}, \mathcal{G}, X) = \sup\{\sigma_N(f, \mathcal{G}, X) \mid f \in \mathrm{F}\}. \tag{1.1}$$

Note that the best N-term approximations of elements f of the Hilbert space L_2 with respect to an orthonormal basis appeared for the first time in [21]: it is these quantities in terms of which S.B. Stechkin formulated his well-known criterion of absolute convergence of the series of Fourier coefficients of an element f with respect to such a basis.

There are a lot of publications devoted to the study of the best N-term approximations and various nonlinear approximation methods for some or other sets F, systems $\mathcal{G}$ (bases and dictionaries), and ambient spaces X. The comprehensive surveys [8, 25, 27] and monographs [28, 32] show that interest in nonlinear problems in approximation theory has not waned; in these sources, one can also find a fairly detailed history of the problem and extensive bibliography.

In the present paper, we study the best N-term approximations (1.1) of classes F of smooth functions with respect to the multiple trigonometric system

$$\mathfrak{T}^{(k)} = \left\{ e^{2\pi i\,\xi x} \mid \xi \in \mathbb{Z}^k \right\}$$

in classical function spaces. Namely, we give order-sharp estimates for (1.1) in the case when $X = L_r(\mathbb{T}^k)$ ($1 < r < \infty$) is the Lebesgue function space on the k-dimensional torus, $\Phi = \mathfrak{T}^{(k)}$, and F are Nikol'skii–Besov type function classes $\mathrm{B}^{sm}_{pq}(\mathbb{T}^k)$ for various relations between the parameters of the classes and spaces.

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The first nontrivial result on the best N-term trigonometric approximations of a second-order Bernoulli function in the uniform metric was obtained by Ismagilov in 1974 [11]. A systematic study of the best N-term trigonometric approximations of classes of periodic functions with bounded mixed derivative or difference was started by Temlyakov in 1984 [22]. Further, important results in this direction were obtained by Kashin, Temlyakov, DeVore, Belinskii, and others (see, e.g., [23, 13, 9, 24, 7]). A detailed study of the best N-term trigonometric approximations of Besov classes of mixed smoothness ($\mathrm{B}^{s\mathbf{1}}_{pq}(\mathbb{T}^k)$ with $1 \le q < \infty$ in our notation) was carried out by Romanyuk (see [17, 18] and references therein). Recently, these classical problems have again come to attention in view of the newly developed general theory of greedy algorithms in Banach spaces (see, e.g., [29, 30] and the surveys [25, 27]; for the theory of greedy algorithms, see the monographs [28, 32]).

We will need auxiliary definitions and notation.

Let $k \in \mathbb{N}$, $\mathrm{z}_k = \{1, \ldots, k\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_+ = (0, +\infty)$. For arbitrary elements $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, we set

$$xy = x_1 y_1 + \ldots + x_k y_k, \qquad |x| = |x_1| + \ldots + |x_k|, \qquad |x|_\infty = \max\{|x_\kappa| \colon \kappa \in \mathrm{z}_k\},$$

$$x \le y \quad (x < y) \qquad \Leftrightarrow \qquad x_\kappa \le y_\kappa \quad (x_\kappa < y_\kappa) \quad \text{for all} \quad \kappa \in \mathrm{z}_k.$$

For a number $a \in \mathbb{R}$, let $a_+ = \max\{a, 0\}$ and $\lfloor a \rfloor$ be its integer part; denote by $\#\Gamma$ the number of elements of a finite set Γ; in particular, $\Gamma = \varnothing \Leftrightarrow \#\Gamma = 0$.

Let, as usual, $L_r = L_r(\mathbb{T}^k)$ be the space of rth power integrable functions $f \colon \mathbb{T}^k \to \mathbb{C}$ (for $r = \infty$, the space of essentially bounded functions) on the k-dimensional torus $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$ with the norm

$$\|f|L_r\| = \|f|L_r(\mathbb{T}^k)\| = \left(\int\limits_{\mathbb{T}^k} |f(x)|^r \, dx \right)^{\frac{1}{r}}, \qquad 1 \le r < \infty,$$

$$\|f|L_\infty\| = \|f|L_\infty(\mathbb{T}^k)\| = \operatorname{ess\,sup}\{|f(x)| \colon x \in \mathbb{T}^k\}.$$

For a pair of measurable functions $f \colon \mathbb{T}^k \to \mathbb{C}$ and $g \colon \mathbb{T}^k \to \mathbb{C}$ such that $f\overline{g} \in L_1$, we set

$$\langle f, g \rangle = \int\limits_{\mathbb{T}^k} f(x) \overline{g}(x) \, dx.$$

Let $\mathcal{S}(\mathbb{R}^k)$ be the Schwartz space of test (infinitely differentiable and rapidly decreasing) functions; let $\widehat{\varphi} \equiv \mathcal{F}_k(f)$ be the (direct) Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^k)$:

$$\widehat{\varphi}(\xi) = \int\limits_{\mathbb{R}^k} \varphi(x) e^{-2\pi i\, \xi x} \, dx, \qquad \xi \in \mathbb{R}^k;$$

next, let

$$\widehat{f}(\xi) = \int\limits_{\mathbb{T}^k} f(x) e^{-2\pi i\, \xi x} dx, \qquad \xi \in \mathbb{Z}^k,$$

be the Fourier coefficients of a function $f \in L_1$ with respect to the trigonometric system $\mathfrak{T}^{(k)}$.

Let $A = A(\mathbb{T}^k)$ be the space of functions $f \colon \mathbb{T}^k \to \mathbb{C}$ with absolutely converging Fourier series with the norm

$$\|f|A\| := \sum_{\xi \in \mathbb{Z}^k} |\widehat{f}(\xi)|.$$

Let $1 \le p, q \le \infty$ and $\mathbb{J} \ne \varnothing$ be an at most countable set; then $\ell_q(\mathbb{J})$ is the space of (complex) number sequences $(c_j) = (c_j \mid j \in \mathbb{J})$ with the finite norm

$$\|(c_j)|\ell_q(\mathbb{J})\| = \left(\sum_{j\in\mathbb{J}} |c_j|^q\right)^{\frac{1}{q}}, \quad 1 \le q < \infty, \qquad \|(c_j)|\ell_\infty(\mathbb{J})\| = \sup_{j\in\mathbb{J}} |c_j|;$$

if $\mathbb{J} = \mathbb{N}_0^n$, then $\ell_q \equiv \ell_q(\mathbb{N}_0^n)$ is the space of (multiple complex) number sequences $(c_\alpha) = (c_\alpha \mid \alpha \in \mathbb{N}_0^n)$ with the norm $\|(c_\alpha)|\ell_q\|$.

Denote by $\ell_q(L_p) \equiv \ell_q(L_p(\mathbb{T}^k))$ and $L_p(\ell_q) \equiv L_p(\mathbb{T}^k; \ell_q)$ the spaces of function sequences $(g_\alpha(x)) = (g_\alpha(x) \mid \alpha \in \mathbb{N}_0^n)$ $(x \in \mathbb{T}^k)$ with the finite norms

$$\|(g_\alpha(x))|\ell_q(L_p)\| = \big\|(\|g_\alpha|L_p\|)\big|\ell_q\big\| \qquad \text{and} \qquad \|(g_\alpha(x))|L_p(\ell_q)\| = \big\|\|(g_\alpha(\cdot))|\ell_q\|\big|L_p\big\|,$$

respectively.

In conclusion, recall the definition of the function spaces $B^{sm}_{pq}(\mathbb{T}^k)$ and $L^{sm}_{pq}(\mathbb{T}^k)$ (and classes) of the Nikol'skii–Besov and Lizorkin–Triebel types, which are the main object of study in the present paper.

Let $n \in \mathbb{N}$ and $n \le k$. We fix a multiindex $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ with $|m| = k$ (if $n = 1$, then $m = k$, and if $n = k$, then $m = \mathbf{1} = (1, \dots, 1) \in \mathbb{N}^k$) and represent $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ as $x = (x^1, \dots, x^n)$, where $x^\nu = (x_\kappa \mid \kappa \in \mathrm{k}_\nu) = (x_{\kappa_{\nu-1}+1}, \dots, x_{\kappa_\nu}) \in \mathbb{R}^{m_\nu}$; here

$$\mathrm{k}_\nu = \{\kappa_{\nu-1} + 1, \dots, \kappa_\nu\}, \qquad \kappa_0 = 0, \qquad \kappa_\nu = m_1 + \ldots + m_\nu, \qquad \nu \in \mathrm{z}_n.$$

Next, we introduce a smooth "m-fold" partition of unity η on $\mathbb{R}^k$.

To this end, we take functions $\eta_0^\nu \in \mathcal{S}(\mathbb{R}^{m_\nu})$, $\nu \in \mathrm{z}_n$, such that

$$0 \le \widehat{\eta_0^\nu}(\xi^\nu) \le 1, \quad \xi^\nu \in \mathbb{R}^{m_\nu}, \qquad \widehat{\eta_0^\nu}(\xi^\nu) = \begin{cases} 1 & \text{if } |\xi^\nu|_\infty \le 1, \\ 0 & \text{if } |\xi^\nu|_\infty \ge \dfrac{3}{2} \end{cases}$$

and set

$$\widehat{\eta^\nu}(\xi^\nu) = \widehat{\eta_0^\nu}(2^{-1}\xi^\nu) - \widehat{\eta_0^\nu}(\xi^\nu), \qquad \widehat{\eta_j^\nu}(\xi^\nu) = \widehat{\eta^\nu}(2^{-j+1}\xi^\nu), \quad j \in \mathbb{N}.$$

Then

$$\eta^{(m_\nu)} := \big\{\widehat{\eta_j^\nu}(\xi^\nu),\ \xi^\nu \in \mathbb{R}^{m_\nu},\ j \in \mathbb{N}_0\big\}$$

is a smooth partition of unity (along "corridors") on $\mathbb{R}^{m_\nu}$, and

$$\eta \equiv \eta^{(m)} := \left\{\widehat{\eta}_\alpha(\xi) = \prod_{\nu=1}^n \widehat{\eta_{\alpha_\nu}^\nu}(\xi^\nu),\ \xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^k,\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n\right\}$$

is a smooth ("m-fold") partition of unity on $\mathbb{R}^k$.

Denote by $\Delta_\alpha^\eta \equiv \Delta_\alpha^{\eta m}$ the following operators $(\alpha \in \mathbb{N}_0^n)$:

$$\Delta_\alpha^\eta(f; x) \equiv \Delta_\alpha^{\eta m}(f; x) = \sum_{\xi\in\mathbb{Z}^k} \widehat{\eta}_\alpha(\xi)\widehat{f}(\xi)e^{2\pi i\,\xi x}.$$

Definition 1.1. Let $s = (s_1, \dots, s_n) \in \mathbb{R}_+^n$ and $1 \le p, q \le \infty$. Then

(i) the *Nikol'skii–Besov type space* $B^{sm}_{pq} \equiv B^{sm}_{pq}(\mathbb{T}^k)$ consists of all functions $f \in L_p$ for which the norm

$$\|f|B^{sm}_{pq}\| = \|(2^{\alpha s}\Delta_\alpha^\eta(f; x))|\ell_q(L_p)\|$$

is finite;

(ii) the *Lizorkin–Triebel type space* $L^{sm}_{pq} \equiv L^{sm}_{pq}(\mathbb{T}^k)$ (provided that $p < \infty$) consists of all functions $f \in L_p$ for which the norm

$$\|f|L^{sm}_{pq}\| = \|(2^{\alpha s}\Delta^{\eta}_{\alpha}(f;x))|L_p(\ell_q)\|$$

is finite.

The unit balls $\mathrm{B}^{sm}_{pq} \equiv \mathrm{B}^{sm}_{pq}(\mathbb{T}^k)$ and $\mathrm{L}^{sm}_{pq} \equiv \mathrm{L}^{sm}_{pq}(\mathbb{T}^k)$ of these spaces are called the *Nikol'skii–Besov* and *Lizorkin–Triebel* classes, respectively.

Remark 1.1. Comments and bibliography on the spaces B^{sm}_{pq} and L^{sm}_{pq} are given in [2]. Here we just mention the following. For $n = k$ (i.e., $m = \mathbf{1} = (1,\dots,1) \in \mathbb{N}^k$), the spaces $B^{s\mathbf{1}}_{pq}(\mathbb{T}^k)$ and $L^{s\mathbf{1}}_{pq}(\mathbb{T}^k)$ are spaces of mixed smoothness; in particular, $MW^s_p(\mathbb{T}^k) = L^{s\mathbf{1}}_{p2}(\mathbb{T}^k)$ is the space of functions with L_p-bounded dominating mixed derivative (for $1 < p < \infty$) and $MH^s_p(\mathbb{T}^k) \equiv B^{s\mathbf{1}}_{p\infty}(\mathbb{T}^k)$ is the space of functions with L_p-bounded dominating mixed difference (for $1 \le p \le \infty$). For $n = 1$ (i.e., $m = k$), $B^s_{pq}(\mathbb{T}^k) \equiv B^{sk}_{pq}(\mathbb{T}^k)$ and $L^s_{pq}(\mathbb{T}^k) \equiv L^{sk}_{pq}(\mathbb{T}^k)$ are isotropic Nikol'skii–Besov and Lizorkin–Triebel spaces, respectively.

2. BEST N-TERM TRIGONOMETRIC APPROXIMATIONS

Here we formulate and discuss the main result of the paper, namely, estimates for the best N-term approximations of the classes B^{sm}_{pq} with respect to the system $\mathfrak{T}^{(k)}$ in the metric of L_r for certain relations between the parameters s, p, q, r, and m ($s \in \mathbb{R}^n_+$, $1 \le p,q,r \le \infty$, $m = (m_1,\dots,m_n) \in \mathbb{N}^n$, $k = m_1 + \ldots + m_n$).

Let $s = (s_1,\dots,s_n) \in \mathbb{R}^n_+$ and $m = (m_1,\dots,m_n) \in \mathbb{N}^n$. Set $\varsigma_\nu = s_\nu/m_\nu$, $\nu \in \mathrm{z}_n$; without loss of generality, we will assume that

$$\varsigma \equiv \min\{\varsigma_\nu \mid \nu \in \mathrm{z}_n\} = \varsigma_1 = \ldots = \varsigma_\omega < \varsigma_\nu, \qquad \nu \in \mathrm{z}_n \setminus \mathrm{z}_\omega,$$

for some $\omega \in \mathrm{z}_n$.

Below, we will use the signs $\ll$ and $\asymp$ of order inequality and equality: for functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $H\colon \mathbb{R}_+ \to \mathbb{R}_+$, we write $F(u) \ll H(u)$ as $u \to \infty$ if there exists a constant $C = C(F,H) > 0$ such that the inequality $F(u) \le CH(u)$ holds for $u \ge u_0 > 0$, and $F(u) \asymp H(u)$ if $F(u) \ll H(u)$ and $H(u) \ll F(u)$ simultaneously.

Everywhere below, $p_* = \min\{2,p\}$, $p^* = \max\{2,p\}$, and log is the binary logarithm.

Theorem 2.1. I. *Let $1 \le p \le r \le 2$, $r > 1$, $1 \le q \le \infty$, and $s \in \mathbb{R}^n_+$ be such that $\varsigma > \frac{1}{p} - \frac{1}{r}$. Then*

$$\sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \asymp N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{\omega-1} N)^{(\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q})_+}. \tag{2.1}$$

II. *Let $1 \le p \le 2 \le r < \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}^n_+$ be such that $\varsigma > \frac{1}{p}$. Then*

$$\sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \asymp N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{\omega-1} N)^{\varsigma-\frac{1}{p}+1-\frac{1}{q}}.$$

III. *Let $2 \le p \le r < \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}^n_+$ be such that $\varsigma > \frac{1}{2}$. Then*

$$\sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \asymp N^{-\varsigma}(\log^{\omega-1} N)^{\varsigma+\frac{1}{2}-\frac{1}{q}}.$$

IV. *Let $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}^n_+$ be such that $\varsigma > \frac{1}{p_*}$. Then*

$$\sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_\infty\big) \ll N^{-\varsigma+\frac{1}{p_*}-\frac{1}{2}}(\log^{\omega-1} N)^{\varsigma-\frac{1}{p_*}+1-\frac{1}{q}}(\log N)^{\frac{1}{2}}.$$

We make the following remark concerning this theorem.

Remark 2.1. As mentioned in the Introduction, extensive literature has been devoted to the best N-term approximations. Here we only point out publications directly related to Theorem 2.1, namely, those related to function classes of the Nikol'skii–Besov scale, to the trigonometric system $\mathfrak{T}^{(k)}$, and to the cases considered in Theorem 2.1.

(a) First of all, we mention Temlyakov's papers [22, 23], in which the best N-term trigonometric approximations of classes of functions of mixed smoothness were studied for the first time; in particular, Temlyakov established order-sharp estimates for the classes $\mathrm{MH}_p^s(\mathbb{T}^k)$ in case I. The exact order of these quantities for the same classes $\mathrm{MH}_p^s(\mathbb{T}^k)$ in case II was announced by Belinskii [5].

(b) Next, in 1995, DeVore and Temlyakov [9] found the exact order of $\sigma_N(\mathrm{F}, \mathfrak{T}^{(k)}, L_r)$ in the case of isotropic Nikol'skii–Besov classes $\mathrm{F} = \mathrm{B}_{pq}^{sk}$ for all natural values of the parameters of the class and space (in particular, in all cases I–IV). To prove the lower estimate in Theorem 2.1 (case I with $\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q} < 0$), we will need a particular case of their result:

$$\sigma_N\big(\mathrm{B}_{pq}^{sk}, \mathfrak{T}^{(k)}, L_r\big) \asymp N^{-\frac{\varsigma}{k}+\frac{1}{p}-\frac{1}{r}}, \qquad 1 \le p \le r \le 2, \quad \varsigma > k\left(\frac{1}{p} - \frac{1}{r}\right), \quad 1 \le q \le \infty. \tag{2.2}$$

Here we also mention the earlier paper by Belinskii [6], in which these estimates were obtained in the one-dimensional case.

(c) In [17, 18], Romanyuk carried out a detailed analysis of the best N-term trigonometric approximations of the Nikol'skii–Besov classes $\mathrm{B}_{pq}^{s\mathbf{1}}(\mathbb{T}^k)$, including cases I–IV considered in Theorem 2.1.

(d) Finally, note that Theorem 2.1 is a part of the result announced in [1] for the classes $\mathrm{B}_{pq}^{sm}(\mathbb{T}^k)$.

When proving Theorem 2.1, we combine the methods and technique of [23] (developed and adapted to the classes $\mathrm{B}_{pq}^{s\mathbf{1}}$ in [17, 18]) and [30, 31] with the methods and technique for studying the classes $\mathrm{B}_{pq}^{sm}(\mathbb{T}^k)$ and $\mathrm{L}_{pq}^{sm}(\mathbb{T}^k)$ from [2–4]. The proof of upper estimates in Theorem 2.1 (in cases II–IV) is based on the theory of greedy algorithms, follows the scheme and style of the recent papers [30, 31], and is of constructive character. Note that until the studies [10, 26] all upper estimates for the best sparse trigonometric approximations in the space L_r for $r > 2$ were in the character of existence theorems and were based on geometric or probabilistic arguments: a very detailed discussion of this topic is given in [30]. In particular, this is true of the proof for the classes $\mathrm{B}_{pq}^{s\mathbf{1}}$ in [17]. The application of the theory of greedy algorithms provides constructive methods of sparse trigonometric approximations and significantly simplifies and clarifies obtaining upper estimates. In [30], this is demonstrated, in particular, for the classes MH_p^s and $\mathrm{B}_{pq}^{s\mathbf{1}}$ (in our notation). In the present paper, we give further examples, including the function classes B_{pq}^{sm}.

(e) In conclusion, note that in [4] the author proved the following estimates for (1.1) with respect to the system of wavelets $\mathcal{W}^m$ (see Section 3 for the definition of $\mathcal{W}^m$):

$$\sigma_N(\mathrm{F}_{pq}^{sm}, \mathcal{W}^m, L_r) \asymp N^{-\varsigma}(\log N)^{(\omega-1)\left(\varsigma+\frac{1}{2}-\frac{1}{q}\right)} \qquad \text{if} \quad 1 < r < \infty, \tag{2.3}$$

$$\sigma_N(\mathrm{F}_{pq}^{sm}, \mathcal{W}^m, L_1) \ll N^{-\varsigma}(\log N)^{(\omega-1)\left(\varsigma+1-\frac{1}{q}\right)}, \tag{2.4}$$

$$\sigma_N(\mathrm{B}_{\infty q}^{sm}, \mathcal{W}^m, L_\infty) \ll N^{-\varsigma}(\log N)^{(\omega-1)\left(\varsigma+1-\frac{1}{q}\right)},$$

and, for any $\delta \in (0, 1)$,

$$\sigma_N(\mathrm{F}_{pq}^{sm}, \mathcal{W}^m, L_\infty) \ll N^{-\varsigma+\frac{\delta}{p}}(\log N)^{(\omega-1)\left(\varsigma-\frac{\delta}{p}+1-\frac{1}{q}\right)} \qquad \text{if} \quad 1 \le p < \infty \tag{2.5}$$

($\mathrm{F} \in \{\mathrm{B}, \mathrm{L}\}$); for the classes $\mathrm{B}_{pq}^{sm}(\mathbb{T}^k)$, a detailed comparison with the estimates from Theorem 2.1 was presented. We put a special emphasis on the following point: estimates (2.4) and (2.5) are established for greedy algorithms in the spaces L_1 and L_∞, respectively, with respect to the system $\mathcal{W}^m$. It is well known that greedy algorithms are not as efficient in these spaces as in smooth Banach spaces, including L_r $(1 < r < \infty)$. As for the best approximations, (2.3) obviously implies the upper estimate (stronger than (2.4))

$$\sigma_N(\mathrm{F}^{sm}_{pq}, \mathcal{W}^m, L_1) \ll N^{-\varsigma}(\log N)^{(\omega-1)\left(\varsigma+\frac{1}{2}-\frac{1}{q}\right)}.$$

3. PRELIMINARIES

In this section, we collect known facts that are important for further consideration.

Let Λ be a finite set in $\mathbb{Z}^k$ and

$$\mathrm{T}(\Lambda) = \left\{ t(x) = \sum_{\xi\in\Lambda} \widehat{t}(\xi) e^{2\pi i\,\xi x} \;\middle|\; \widehat{t}(\xi) \in \mathbb{C},\ \xi \in \Lambda \right\}$$

be the space of trigonometric polynomials with complex coefficients and spectrum Λ. For $\mathrm{M} = (\mathrm{M}_1, \dots, \mathrm{M}_n) \in \mathbb{N}_0^n$, we set

$$\Lambda(m;\mathrm{M}) = \mathbb{Z}^k \cap \prod_{\nu=1}^{n} [-\mathrm{M}_\nu, \mathrm{M}_\nu]^{m_\nu}, \qquad \Theta(m;\mathrm{M}) = \prod_{\nu=1}^{n} (2\mathrm{M}_\nu + 1)^{m_\nu}.$$

Obviously,

$$\dim(\mathrm{T}(\Lambda(m;\mathrm{M}))) = \Theta(m;\mathrm{M});$$

for brevity, we will henceforth write $\mathrm{T}(m;\mathrm{M})$ instead of $\mathrm{T}(\Lambda(m;\mathrm{M}))$.

S.M. Nikol'skii's inequality of different metrics. *Let $1 \le p < r \le \infty$. Then the following inequality holds for any polynomial $t \in \mathrm{T}(m;\mathrm{M})$:*

$$\|t|L_r\| \le 2^k \prod_{\nu\in\mathrm{z}_n} \mathrm{M}_\nu^{m_\nu\left(\frac{1}{p}-\frac{1}{r}\right)} \|t|L_p\|. \tag{3.1}$$

For the proof, see, e.g., [16, Ch. 3, Sects. 3.2, 3.3].

To obtain the lower estimates in Theorem 2.1, we will also apply Nikol'skii's duality principle (see, e.g., [14, Ch. 2]): let $\{g_1, \dots, g_N\}$ be a set of elements of a Banach space X, and let X^* be the continuous dual of this space; then the following equality holds for all $f \in X$:

$$\min\left\{ \left\| f - \sum_{j=1}^{N} c_j g_j \middle| X \right\| \;\middle|\; c_j,\ j = 1, \dots, N \right\}$$
$$= \sup\bigl\{ F(f) \;\big|\; F \in X^*\colon\ \|F|X^*\| = 1,\ F(g_j) = 0,\ j = 1, \dots, N \bigr\}.$$

Poisson's summation formula and the Bessel–Macdonald kernel. Let $f\colon \mathbb{R}^k \to \mathbb{C}$ be an arbitrary function. Its periodization $\widetilde{f}\colon \mathbb{T}^k \to \mathbb{C}$ is defined as the (formal) sum of the series

$$\sum_{\xi\in\mathbb{Z}^k} f(x+\xi). \tag{3.2}$$

It is well known (see, e.g., [16, Ch. 8, Sect. 8.1]) that the Bessel–Macdonald kernel

$$G_v(x) = \frac{1}{(4\pi)^{\frac{v}{2}}} \frac{1}{\Gamma\left(\frac{v}{2}\right)} \int_0^\infty e^{\frac{-\pi x x}{\tau}} e^{\frac{-\tau}{4\pi}} \tau^{\frac{-k+v}{2}} \frac{d\tau}{\tau}, \qquad v > 0,$$

belongs to the space $L_1(\mathbb{R}^k)$; therefore, by Poisson's summation formula (see [20, Ch. VII, § 2, Theorem 2.4]), we obtain

$$\widetilde{G}_v(x) = \sum_{\xi \in \mathbb{Z}^k} G_v(x+\xi) \in L_1(\mathbb{T}^k).$$

Since the Fourier transform of the kernel G_v is $\widehat{G}_v(\xi) = (1 + 4\pi^2\xi\xi)^{-v/2}$, $\xi \in \mathbb{R}^k$, the Fourier coefficients of its periodization $\widetilde{G}_v$ are calculated by the formula

$$\widehat{\widetilde{G}}_v(\xi) = (1 + 4\pi^2\xi\xi)^{-\frac{v}{2}}, \qquad \xi \in \mathbb{Z}^k.$$

Moreover, since $G_v(x) \in H_1^v(\mathbb{R}^k)$ [16, Ch. 8, Sect. 8.3], one can easily verify that $\widetilde{G}_v(x) \in H_1^v = B_{1\infty}^{vk}(\mathbb{T}^k)$. Here $H_1^s(\mathbb{R}^k)$ ($s > 0$, $1 \le p \le \infty$) is the Nikol'skii space of functions defined on $\mathbb{R}^k$. Therefore, $\widetilde{G}_v(x)$ satisfies the condition (in this case, $n = 1$, i.e., $m = k$)

$$\big\|\Delta_l^{\eta k}(\widetilde{G}_v; x)\big|L_1\big\| \ll 2^{-vl}, \qquad l \in \mathbb{N}_0. \tag{3.3}$$

Next, we briefly recall the definition of the multiple system of periodized wavelets $\mathcal{W}^m$ (see [2] for details). Let $w^0(t)$ and $w^1(t)$ be Meyer's scaling function and wavelet, respectively (see [15, Ch. 2, Sect. 12, Ch. 3, Sect. 2] as well as [12, Ch. 7]).

Set

$$\mathrm{E}^k = \mathrm{E}^k(0) = \{0,1\}^k, \qquad \mathrm{E}^k(1) = \mathrm{E}^k \setminus \{(0,\dots,0)\},$$

$$\Lambda(k,j) = \mathbb{N}_0^k \cap [0, 2^j - 1]^k, \qquad j \in \mathbb{N}_0,$$

$$\mathrm{E}^m(\alpha) = \big\{\iota \in \mathrm{E}^k \bigm| \iota^\nu \in \mathrm{E}^{m_\nu}(\operatorname{sgn}\alpha_\nu),\ \nu \in \mathrm{z}_n\big\},$$

$$\Lambda(m,\alpha) = \big\{\lambda \in \mathbb{N}_0^k \bigm| \lambda^\nu \in \Lambda(m_\nu, \alpha_\nu),\ \nu \in \mathrm{z}_n\big\}, \qquad \alpha \in \mathbb{N}_0.$$

Define functions $w^\iota\colon \mathbb{R}^k \to \mathbb{R}$ ($\iota = (\iota_1,\dots,\iota_k) \in \mathrm{E}^k$) as follows:

$$w^\iota(x) = w^{\iota_1}(x_1) \times \ldots \times w^{\iota_k}(x_k),$$

and, next,

$$w_{j\lambda}^\iota(x) = 2^{\frac{jk}{2}} w^\iota(2^j x - \lambda), \qquad \lambda \in \mathbb{Z}^k, \quad j \in \mathbb{N}_0.$$

Define functions $\widetilde{w}_{j\lambda}^\iota\colon \mathbb{T}^k \to \mathbb{R}$ as the periodizations of the functions $w_{j\lambda}^\iota$:

$$\widetilde{w}_{j\lambda}^\iota(x) = 2^{\frac{jk}{2}} \widetilde{w}_j^\iota(x - 2^{-j}\lambda), \qquad \lambda \in \Lambda(k,j), \quad j \in \mathbb{N}_0, \quad \iota \in \mathrm{E}^k.$$

Then

$$\mathcal{W}^m \equiv \big\{\widetilde{w}_{\alpha\lambda}^\iota(x) = \widetilde{w}_{\alpha_1\lambda^1}^{\iota^1}(x^1) \times \ldots \times \widetilde{w}_{\alpha_n\lambda^n}^{\iota^n}(x^n) \bigm| \lambda \in \Lambda(m,\alpha),\ \iota \in \mathrm{E}^m(\alpha),\ \alpha \in \mathbb{N}_0^n\big\}$$

is the ("m-fold") system of periodized Meyer wavelets $\mathcal{W}^m$.

Now we introduce operators Δ_α^w ($\alpha \in \mathbb{N}_0^n$): for $f \in L_1$,

$$\Delta_\alpha^w(f;x) = \sum_{\iota \in \mathrm{E}^m(\alpha)} \sum_{\lambda \in \Lambda(m,\alpha)} f_{\alpha\lambda}^\iota \widetilde{w}_{\alpha\lambda}^\iota(x), \qquad f_{\alpha\lambda}^\iota = \int_{\mathbb{T}^k} f(x)\, \widetilde{w}_{\alpha\lambda}^\iota(x)\, dx.$$

First, we formulate a theorem on the representation and characterization by wavelets for the spaces B_{pq}^{sm} from [2] (see also [3]).

Theorem A. *Let* $1 \le p, q \le \infty$ *and* $s \in \mathbb{R}^n_+$. *A function* $f \in L_p$ *belongs to the space* B^{sm}_{pq} *if and only if the function sequence* $(2^{\alpha s}\Delta^w_\alpha(f;x) \mid \alpha \in \mathbb{N}^n_0)$ *belongs to the space* $\ell_q(L_p)$; *in this case, the functional*

$$\big\|(2^{\alpha s}\Delta^w_\alpha(f;x) \mid \alpha \in \mathbb{N}^n_0)\big|\ell_q(L_p)\big\|$$

is a norm of B^{sm}_{pq} *equivalent to the original norm.*

Let us also formulate a Littlewood–Paley type theorem and its analog for representations with respect to the system $\mathcal{W}^m$ (from [2]) that will be applied along with Theorem A in the subsequent sections to the problem of nonlinear trigonometric approximations of functions from the classes $\mathrm{B}^{sm}_{pq}(\mathbb{T}^k)$.

We introduce operators Δ^*_α as follows: for $f \in L_1$,

$$\Delta^*_\alpha(f;x) = \sum_{\xi \in \rho(m,\alpha)} \widehat{f}(\xi)\, e^{2\pi i\, \xi x},$$

$$\rho(m,\alpha) = \big\{\xi \in \mathbb{Z}^k \;\big|\; \lfloor 2^{\alpha_\nu - 1}\rfloor \le |\xi^\nu|_\infty < 2^{\alpha_\nu},\ \nu \in \mathrm{z}_n\big\}, \qquad \alpha \in \mathbb{N}^n_0.$$

Theorem LP. *Let* $1 < p < \infty$. *Then there exist constants* $0 < c_{p,m} < C_{p,m}$ *and* $0 < c^\eta_{p,m} < C^\eta_{p,m}$ *such that*

$$c_{p,m}\|f|L_p\| \le \|(\Delta^*_\alpha(f;x))|L_p(\ell_2)\| \le C_{p,m}\|f|L_p\|, \tag{3.4}$$

$$c^\eta_{p,m}\|f|L_p\| \le \|(\Delta^\eta_\alpha(f;x))|L_p(\ell_2)\| \le C^\eta_{p,m}\|f|L_p\| \tag{3.5}$$

for all $f \in L_p$.

A detailed comment regarding the Littlewood–Paley theorem is given in [3].

Theorem B. *Let* $1 < p < \infty$. *Then there exists a constant* $C = C(w^0, m, p) > 0$ *such that the following inequalities hold for all functions* $f \in L_p(\mathbb{T}^k)$:

$$C^{-1}\|f|L_p\| \le \|(\Delta^w_\alpha(f;x))\,|L_p(\ell_2)\| \le C\|f|L_p\|.$$

The proof of the upper estimates in Theorem 2.1 for $2 \le r \le \infty$ (see Theorem 7.1 below) is based on the following important results of Temlyakov on the so-called incremental algorithm $\mathrm{IA}(\varepsilon)$ with schedule ε (see [28, Ch. 6] as well as [30, Theorems 2.4–2.6]).

Recall the definition of the algorithm $\mathrm{IA}(\varepsilon)$. Let X be a real Banach space with dictionary $\mathcal{D}$; the *dictionary* $\mathcal{D}$ is a set in X that possesses the following properties:

(a) $\|g|X\| = 1$ for all $g \in \mathcal{D}$;

(b) $\operatorname{span}\mathcal{D}$, the linear span of $\mathcal{D}$, is dense in X.

A *symmetrized dictionary* (for the dictionary $\mathcal{D}$) is the set $\mathcal{D}^\pm := \{\pm g \mid g \in \mathcal{D}\}$.

Denote by F_f the norming functional of an element $f \in X$, i.e., a functional on X such that $\|F_f|X^*\| = 1$ and $F_f(f) = \|f|X\|$; its existence follows from the Hahn–Banach theorem. Let $\varepsilon = (\varepsilon_N \mid N \in \mathbb{N})$ be a given number sequence, $\varepsilon_N > 0$, $N \in \mathbb{N}$.

Incremental algorithm $\mathrm{IA}(\varepsilon)$ **with schedule** ε. Let $f^{\mathrm{i},\varepsilon}_0 := f$ and $G^{\mathrm{i},\varepsilon}_0 := 0$; for $N \in \mathbb{N}$, we inductively

(i) define $\varphi^{\mathrm{i},\varepsilon}_N \in \mathcal{D}^\pm$, an arbitrary element such that $F_{f^{\mathrm{i},\varepsilon}_{N-1}}\big(\varphi^{\mathrm{i},\varepsilon}_N - f\big) \ge -\varepsilon_N$;

(ii) set

$$G^{\mathrm{i},\varepsilon}_N := \Big(1 - \frac{1}{N}\Big)G^{\mathrm{i},\varepsilon}_{N-1} + \frac{1}{N}\varphi^{\mathrm{i},\varepsilon}_N;$$

(iii) and define $f^{\mathrm{i},\varepsilon}_N := f - G^{\mathrm{i},\varepsilon}_N$.

Below in Theorems C and D we use a special sequence $\varepsilon = (\varepsilon_N)$ with $\varepsilon_N := v(\frac{r-1}{2})^{1/2} N^{-1/2}$ for $N \in \mathbb{N}$.

Let $\Re\mathfrak{T}^{(k)}$ be the real k-fold trigonometric system,

$$\Re\mathfrak{T}^{(k)} = \big\{\varphi_{\mathrm{z}\xi} \;\big|\; \mathrm{z} \subset \mathrm{z}_k,\ \xi\colon (\xi_\kappa \mid \kappa \in \mathrm{z}) \in \mathbb{N}_0^{\#\mathrm{z}},\ (\xi_\kappa \mid \kappa \in \mathrm{z}') \in \mathbb{N}^{\#\mathrm{z}'}\big\},$$

where[1]

$$\varphi_{\mathrm{z}\xi}(x) = \prod_{\kappa\in\mathrm{z}} \cos(2\pi\xi_\kappa x_\kappa) \prod_{\kappa\in\mathrm{z}'} \sin(2\pi\xi_\kappa x_\kappa).$$

Let, next,

$$\mathrm{T}_{\Re}(m;\mathrm{M}) := \operatorname{span}\big\{\varphi_{\mathrm{z}\xi} \;\big|\; \mathrm{z} \subset \mathrm{z}_k,\ \xi\colon |\xi^\nu|_\infty \le \mathrm{M}_\nu,\ \nu \in \mathrm{z}_n\big\}.$$

Theorem C. *Let $2 \le r < \infty$. For any polynomial $t \in \mathrm{T}_{\Re}(m;\mathrm{M})$, the application of the algorithm* $\mathrm{IA}(\varepsilon)$ *to $f := t/\|t|A\|$ yields, after N iterations, an N-term trigonometric polynomial $G_N^r(t) := G_N^{\mathrm{i},\varepsilon}(f)\|t|A\|$* (with respect to the system $\Re\mathfrak{T}^{(k)}$) *such that*

$$\|t - G_N^r(t)|L_r\| \le C(k)\overline{N}^{-\frac{1}{2}} r^{\frac{1}{2}} \|t|A\|, \qquad N \in \mathbb{N}_0 \quad (\overline{N} = \max\{1, N\}).$$

In addition, there exists a constructive algorithm $G^{\mathrm{i},\infty}$ that is based on $\mathrm{IA}(\varepsilon)$ *and whose application to $f := t/\|t|A\|$ yields, after N iterations, an N-term polynomial $G_N^\infty(t) := G_N^{\mathrm{i},\infty}(f)\|t|A\|$* (with respect to the system $\Re\mathfrak{T}^{(k)}$) *such that*

$$\|t - G_N^\infty(t)|L_\infty\| \le C(k)\overline{N}^{-\frac{1}{2}} (\log\Theta(m;\mathrm{M}))^{\frac{1}{2}} \|t|A\|, \qquad \|G_N^\infty(t)|A\| = \|t|A\|.$$

The following theorem is a corollary to Theorem C (see [30, Theorem 2.6]).

Theorem D. *Let $2 \le r < \infty$. There exist constructive approximation methods $G_N^r(\cdot)$ and $G_N^\infty(\cdot)$ based on greedy-type algorithms that lead to N-term polynomials with respect to the system $\mathfrak{T}^{(k)}$ with the following properties: for $f \in A$,*

$$\|f - G_N^r(f)|L_r\| \le c_1(k)\overline{N}^{-\frac{1}{2}} r^{\frac{1}{2}} \|f|A\|, \qquad \|G_N^r(f)|A\| \le c_2(k)\|f|A\|;$$

and for $f \in \mathrm{T}(m;\mathrm{M})$,

$$\|f - G_N^\infty(f)|L_\infty\| \le c_3(k)\overline{N}^{-\frac{1}{2}} (\log\Theta(m;\mathrm{M}))^{\frac{1}{2}} \|f|A\|, \qquad \|G_N^\infty(f)|A\| \le c_4(k)\|f|A\|.$$

4. AUXILIARY LEMMAS

For $\alpha \in \mathbb{N}_0^n$, we set

$$\Pi(m,\alpha) = \Lambda(m; (2^{\alpha_1}, \dots, 2^{\alpha_n})), \qquad \vartheta(m,\alpha) = \Theta(m; (2^{\alpha_1}, \dots, 2^{\alpha_n}));$$

next,

$$\Lambda_a(u) = \bigcup_{u < \alpha m \le u + a} \rho(m,\alpha), \qquad \Lambda(u) = \bigcup_{\alpha m \le u} \rho(m,\alpha).$$

It is clear that

$$\dim(\mathrm{T}(\Pi(m,\alpha))) = \vartheta(m,\alpha).$$

[1]We assume that $\prod_{\kappa\in\varnothing} \varphi_\kappa(x_\kappa) \equiv 1,\ \mathbb{N}_0^{\#\varnothing} = \mathbb{N}^{\#\varnothing} = \varnothing,$ and $\mathrm{z}' = \mathrm{z}_k \setminus \mathrm{z}$.

For a finite set $\Lambda \subset \mathbb{Z}^k$, we define the corresponding Dirichlet kernel

$$\mathcal{D}_\Lambda(x) = \sum_{\xi\in\Lambda} e^{2\pi i\xi x};$$

then it is obvious that

$$\mathcal{D}_{\Lambda(u)}(x) = \sum_{\alpha m\le u} \mathcal{D}_{\rho(m,\alpha)}(x), \qquad \Delta^*_\alpha(f;x) = \mathcal{D}_{\rho(m,\alpha)} * f(x);$$

moreover, the estimates for the one-dimensional Dirichlet kernels yield ($1 < p < \infty$)

$$\|\mathcal{D}_{\rho(m,\alpha)}|L_p\| \ll 2^{(1-\frac{1}{p})\alpha m}. \tag{4.1}$$

Below, we will need the following simple facts on the operators Δ^η_α, Δ^*_α, and Δ^w_α.

It follows from the properties of η that the spectrum of the trigonometric polynomial $\Delta^\eta_\alpha(f;x)$, $\alpha \in \mathbb{N}^n_0$, for a function $f \in L_1$ lies in the "corridor" $P^\eta(m,\alpha) = \bigotimes_{\nu\in \mathrm{z}_n} P^\eta(m_\nu,\alpha_\nu)$, i.e.,

$$\Delta^\eta_\alpha(f;x) \in \mathrm{T}(P^\eta(m,\alpha)), \tag{4.2}$$

where

$$P^\eta(m_\nu,\alpha_\nu) = \bigl\{\xi^\nu \in \mathbb{Z}^{m_\nu} \bigm| 2^{\alpha_\nu-1} < |\xi^\nu|_\infty < 3\cdot 2^{\alpha_\nu-1}\bigr\}, \qquad \alpha_\nu \in \mathbb{N},$$
$$P^\eta(m_\nu,0) = \bigl\{\xi^\nu \in \mathbb{Z}^{m_\nu} \bigm| |\xi^\nu|_\infty \le 1\bigr\}.$$

The spectrum of the trigonometric polynomial $\widetilde{w}^\iota_{\alpha\lambda}(x)$ is contained in the "corridor" (see [2] as well as [12, Ch. 7])

$$P^w(m,\alpha) = \bigotimes_{\nu\in \mathrm{z}_n} P^w(m_\nu,\alpha_\nu),$$

where for $\alpha_\nu \ge 1$

$$P^w(m_\nu,\alpha_\nu) = \left\{\xi^\nu \in \mathbb{Z}^{m_\nu} \Bigm| \left\lfloor\frac{2^{\alpha_\nu}}{3}\right\rfloor < |\xi^\nu|_\infty \le \left\lfloor\frac{2^{\alpha_\nu+2}}{3}\right\rfloor\right\}, \qquad P^w(m_\nu,0) = \bigl\{\xi^\nu \in \mathbb{Z}^{m_\nu} \bigm| |\xi^\nu|_\infty \le 1\bigr\};$$

hence, for a function $f \in L_1$,

$$\Delta^w_\alpha(f;x) \in \mathrm{T}(P^w(m,\alpha)), \qquad \alpha \in \mathbb{N}^n_0. \tag{4.3}$$

It follows easily from (4.2) and (4.3) (see [2]) that

$$\Delta^w_\alpha(f;x) = \sum_{\alpha'\in\mathbb{N}^n_0\colon |\alpha-\alpha'|_\infty\le 2} \Delta^w_\alpha \circ \Delta^\eta_{\alpha'}(f;x). \tag{4.4}$$

Moreover, it is easy to see that if the composition $\Delta^\eta_\beta \circ \Delta^*_\alpha$, where $\alpha,\beta \in \mathbb{N}^n_0$, is a nonzero operator, then the inequalities $\alpha^- \le \beta \le \alpha$ must be satisfied (henceforth, for $\alpha \in \mathbb{N}^n_0$, we define $\alpha^- = (\alpha^-_1,\dots,\alpha^-_n)$ and $\alpha^+ = (\alpha^+_1,\dots,\alpha^+_n)$, where $\alpha^-_\nu = (\alpha_\nu - 1)_+$ and $\alpha^+_\nu = \alpha_\nu + 1$, $\nu \in \mathrm{z}_n$).

It is clear that similar facts hold for all compositions $\Delta^\varphi_\alpha \circ \Delta^\psi_{\alpha'}$ with $\varphi,\psi \in \{w,\eta,*\}$.

We will need the following version of Temlyakov's well-known lemma [23, Ch. 1, Lemma 3.1]. For $a = (a_1,\dots,a_n) \in \mathbb{R}^n_+$, let

$$P_\alpha(m,a) = \bigl\{\xi \in \mathbb{Z}^k \bigm| |\xi^\nu|_\infty \le 2^{\alpha_\nu} a_\nu,\ \nu \in \mathrm{z}_n\bigr\}.$$

Lemma T1. *Let $1\le p<r<\infty$ and $1\le q\le\infty$. Then there exists a constant $C(p,r,a,m)>0$ such that for $f\in L_p$,*

$$f(x)=\sum_{\alpha\in\mathbb{N}_0^n} t_\alpha(x),\qquad t_\alpha\in \mathrm{T}(P_\alpha(m,a)),$$

the following inequality holds:

$$\|(t_\alpha)|L_r(\ell_q)\|\le C(p,r,a,m)\big\|\big(2^{\alpha m(\frac{1}{p}-\frac{1}{r})}t_\alpha(x)\big)\big|\ell_r(L_p)\big\|. \tag{4.5}$$

The proof of Lemma T1 is given in [3]. The following lemma is an analog of another lemma of Temlyakov [23, Ch. 1, Lemma 3.1′].

Lemma T2. *Let $1<p<r\le\infty$ and $1\le q\le 2$. Then there exists a constant $c(p,r,m)>0$ such that the following inequalities hold for $f\in L_p$*:

$$\|(\Delta_\alpha^w(f;x))|L_p(\ell_q)\|\ge c(p,r,m)\big\|\big(2^{\alpha m(\frac{1}{r}-\frac{1}{p})}\Delta_\alpha^w(f;x)\big)\big|\ell_p(L_r)\big\|, \tag{4.6}$$

$$\|(\Delta_\alpha(f;x))|L_p(\ell_q)\|\ge c(p,r,m)\big\|\big(2^{\alpha m(\frac{1}{r}-\frac{1}{p})}\Delta_\alpha(f;x)\big)\big|\ell_p(L_r)\big\|. \tag{4.7}$$

Proof. Let us prove (4.6). Applying successively Jensen's inequality,[2] Theorem B, Lemma T1 together with the orthonormality of the system $\mathcal{W}^m$, and the dual description of the norm in the spaces L_p and ℓ_p, we obtain ($\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{r}+\frac{1}{r'}=1$)

$$\|(\Delta_\alpha^w(f;x))|L_p(\ell_q)\|\gg\|(\Delta_\alpha^w(f;x))|L_p(\ell_2)\|\asymp\|f|L_p\|=\sup\big\{\langle f,\varphi\rangle\;\big|\;\|\varphi|L_{p'}\|=1\big\}$$

$$=\sup\left\{\sum_{\alpha\in\mathbb{N}_0^n}\big\langle\Delta_\alpha^w(f;x),\Delta_\alpha^w(\varphi;x)\big\rangle\;\middle|\;\|\varphi|L_{p'}\|=1\right\}$$

$$\gg\sup\left\{\sum_{\alpha\in\mathbb{N}_0^n}\big\langle\Delta_\alpha^w(f;x),\Delta_\alpha^w(\varphi;x)\big\rangle\;\middle|\;\varphi,(c_\alpha)\colon\|\Delta_\alpha^w(f;x)|L_{r'}\|\le c_\alpha,\;\big\|\big(2^{\alpha m(\frac{1}{r'}-\frac{1}{p'})}c_\alpha\big)\big|\ell_{p'}\big\|\le 1\right\}$$

$$=\sup\left\{\sum_{\alpha\in\mathbb{N}_0^n}c_\alpha\|\Delta_\alpha^w(f;x)|L_r\|\;\middle|\;(c_\alpha)\colon\big\|\big(2^{\alpha m(\frac{1}{p}-\frac{1}{r})}c_\alpha\big)\big|\ell_{p'}\big\|\le 1\right\}=\big\|\big(2^{\alpha m(\frac{1}{r}-\frac{1}{p})}\Delta_\alpha^w(f;x)\big)\big|\ell_p(L_r)\big\|.$$

The proof of inequality (4.7) is analogous; one should just apply the Littlewood–Paley theorem (relation (3.4)) instead of Theorem B. Note that (4.7) is a direct generalization of Lemma 3.1′ from [23] to the case of an arbitrary m. □

Below, when proving upper estimates for sparse trigonometric approximations and estimating the dimensions of the corresponding subspaces or the spectra of trigonometric polynomials, we will systematically use the following lemma, which is a modification of Lemmas B–D from [23] to our case.

Lemma A. *Let $\beta,\gamma\in\mathbb{R}_+^n$ be such that $\beta_\nu=\gamma_\nu$ for $\nu\in\mathrm{z}_\omega$ and $\beta_\nu>\gamma_\nu$ for $\nu\in\mathrm{z}_n\setminus\mathrm{z}_\omega$, and let $L>0$. Then the following relations are valid*:

$$\mathcal{I}_L^{\beta,\gamma}(u)\equiv\sum_{\alpha\in\mathbb{N}_0^n\colon\alpha\gamma>u}2^{-L\alpha\beta}\asymp 2^{-Lu}u^{\omega-1}\qquad \textit{as}\quad u\to+\infty, \tag{4.8}$$

$$\mathcal{J}_L^{\gamma,\beta}(u)\equiv\sum_{\alpha\in\mathbb{N}_0^n\colon\alpha\beta\le u}2^{L\alpha\gamma}\asymp 2^{Lu}u^{\omega-1}\qquad \textit{as}\quad u\to+\infty. \tag{4.9}$$

[2] Let $1\le p<r\le\infty$; then Jensen's inequality $\|(c_j)|\ell_r(\mathbb{J})\|\le\|(c_j)|\ell_p(\mathbb{J})\|$ is valid for any number sequence $(c_j)=(c_j\mid j\in\mathbb{J})$.

This is Lemma 5.1 from [2] (its proof is given in [3], where it is named Lemma A).

Next, let

$$\theta(u;a) \equiv \theta(u;a;m) = \{\alpha \in \mathbb{N}_0^n \mid u < \alpha m \le u + a\}, \qquad u, a \in \mathbb{N}_+.$$

Applying Lemma A (see [3]), we can easily show that there exists an $a \in \mathbb{N}$ such that

$$\sum_{\alpha\in\theta(u;a)} 2^{\alpha m} \asymp 2^u u^{n-1}. \tag{4.10}$$

Fix such an $a \in \mathbb{N}$ and introduce the notation

$$\theta(u) \equiv \theta(u;a), \qquad u \in \mathbb{N}. \tag{4.11}$$

It is clear (see [3]) that $\#\theta(u) \asymp u^{n-1}$, $u \in \mathbb{N}$.

Applying Lemma A, we can easily obtain the following estimates (see [3]):

$$\#\Lambda_a(u) \asymp 2^u u^{n-1}, \qquad \#\Lambda(u) \asymp 2^u u^{n-1}.$$

One of the key ingredients in the constructive method for proving upper estimates in [30, 31] is Theorem 2.2 from [23, Ch. 1], which relates the norms in the spaces A and L_r $(1 < r \le 2)$ of a trigonometric polynomial with harmonics in a hyperbolic cross. The following lemma is an analog of this theorem for the polynomials in $\mathrm{T}(\Lambda(u))$.

Lemma T3. *Let $1 < r \le 2$. Then there exists a constant $c(r,m) > 0$ such that the following inequality holds for any polynomial $t \in \mathrm{T}(\Lambda(u))$:*

$$\|t|A\| := \sum_{\xi\in\Lambda(u)} |\widehat{t}(\xi)| \le c(r,m)\cdot 2^{\frac{1}{r}u} u^{(n-1)(1-\frac{1}{r})} \|t|L_r\|.$$

Proof. Let t be an arbitrary polynomial from $\mathrm{T}(\Lambda(u))$. Using t, we define a polynomial t_1 by setting (for $z \in \mathbb{C}$, denote its complex conjugate by $\overline{z}$ and its sign by $\operatorname{sgn}(z)$, $\operatorname{sgn}(z) = z/|z|$ if $z \ne 0$)

$$t_1(x) = \sum_{\xi\in\Lambda(u)} \operatorname{sgn}(\overline{\widehat{t}(\xi)}) e^{2\pi i \xi x}.$$

Then it is clear that $(\frac{1}{r} + \frac{1}{r'} = 1)$

$$\|t|A\| = \sum_{\xi\in\Lambda(u)} |\widehat{t}(\xi)| = \langle t, t_1\rangle \le \|t|L_r\| \cdot \|t|L_{r'}\|.$$

On the other hand, by Lemmas T1 and A, we obtain

$$\|t|L_{r'}\| \ll \left\{ \sum_{\alpha m\le u} 2^{\alpha m(\frac{1}{2}-\frac{1}{r'})r'} \|t|L_{r'}\| \cdot \|\Delta^*(t_1)|L_2\|^{r'} \right\}^{\frac{1}{r'}} \ll \left\{ \sum_{\alpha m \le u} 2^{\alpha m(\frac{1}{2}-\frac{1}{r'})r'} \cdot 2^{\alpha m \frac{1}{2} r'} \right\}^{\frac{1}{r'}}$$
$$\asymp 2^{\frac{1}{r}u} u^{(n-1)(1-\frac{1}{r})}.$$

Therefore,

$$\|t|A\| \ll 2^{\frac{1}{r}u} u^{(n-1)(1-\frac{1}{r})} \|t|L_r\|.$$

The lemma is proved. $\square$

It is easy to see from the proof of Lemma T3 that for any polynomial $t \in \mathrm{T}(P^w(m,\alpha))$ we have the estimate

$$\|t|A\| \ll 2^{\frac{1}{r}\alpha m} \|t|L_r\| \tag{4.12}$$

$(1 < r \le 2)$, while the Nikol'skii inequality (3.1) yields

$$\|t|L_r\| \ll 2^{\alpha m(1-\frac{1}{r})} \|t|L_1\|;$$

hence, the following estimate is also valid:

$$\|t|A\| \ll 2^{\alpha m}\|t|L_1\|. \tag{4.13}$$

For a finite set $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n \in \mathbb{Z}^k$, let

$$d(\Lambda^\nu) = \max\{|\lambda^\nu - \xi^\nu|_\infty\colon\ \lambda^\nu, \xi^\nu \in \Lambda^\nu\}, \qquad \overline{d}(\Lambda^\nu) = \max\{1, d(\Lambda^\nu)\}.$$

Lemma M. *Let $1 \le p \le \infty$ and $\frac{1}{2}m < \varkappa = (\varkappa_1, \ldots, \varkappa_n) \in \mathbb{R}^n_+$. Then there exists a constant $C = C(p, m, \varkappa) > 0$ such that for any finite set $\Lambda \subset \mathbb{Z}^k$ of the form $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n \ne \varnothing$, where $\Lambda^\nu \subset \mathbb{Z}^{m_\nu}$, the inequality*

$$\left\|\sum_{\lambda\in\Lambda} \mathfrak{m}(\lambda)\widehat{t}(\lambda)e^{2\pi i\lambda x}\right| L_p\right\| \le C \prod_{\nu\in z_n} \left\|\mathfrak{m}^\nu(\overline{d}(\Lambda^\nu)x^\nu)\middle|\mathcal{H}^{\varkappa_\nu}(\mathbb{R}^{m_\nu})\right\| \cdot \|t(x)|L_p\|$$

holds for all functions $\mathfrak{m}(x) = \mathfrak{m}^1(x^1)\ldots\mathfrak{m}^n(x^n)$ with $\mathfrak{m}^\nu(x^\nu) \in \mathcal{H}^{\varkappa_\nu}(\mathbb{R}^{m_\nu})$ and all polynomials $t(x) \in T(\Lambda)$.

Here $\mathcal{H}^s(\mathbb{R}^k)$ ($s \in \mathbb{R}$) is the space of Bessel potentials (see [33, Ch. 2]). In the case of $1 = n \le k$ and $\mathbb{I} = \mathbb{T}$ or $\mathbb{R}$, as well as in the case of $n = k = 2$ and $\mathbb{I} = \mathbb{R}$, Lemma M and its nonperiodic analog were proved in [33, Sects. 1.5.2, 1.6.3] and [19, Sects. 3.3.4, 3.4.1, 1.8.3, 1.10.3]. Combining the arguments used there, one can easily obtain the proof in the general ("m-fold") case.

Moreover, we will often use the following numerical lemma (see, e.g., [23, Ch. 4] for a proof).

Lemma B. *Let $1 \le p \le r \le \infty$, $J \in \mathbb{N}$, and $b_1 \ge b_2 \ge \ldots \ge b_J \ge 0 = b_j$ ($j \in \mathbb{N}$, $j \ge J+1$). Then the following inequality is valid for all $I \in \mathbb{N}$:*[3]

$$\left(\sum_{j=I}^{J} b_j^r\right)^{\frac{1}{r}} \le I^{\frac{1}{r}-\frac{1}{p}} \left(\sum_{j=1}^{J} b_j^p\right)^{\frac{1}{p}}.$$

Under the hypotheses of Lemma B, the following elementary inequality is valid, which will also be useful below: for all $I, K \in \mathbb{N}$ such that $I \le K$, we have

$$b_K \le I^{-\frac{1}{p}} \left(\sum_{i=1}^{I} b_i^p\right)^{\frac{1}{p}}. \tag{4.14}$$

Below, for simplicity (in particular, when proving the upper estimates in Theorem 2.1), we will restrict the analysis to the "isotropic" situation, when $\omega = n$, i.e.,

$$\varsigma = \frac{s_1}{m_1} = \ldots = \frac{s_n}{m_n}.$$

Note that when we prove the lower estimates, this assumption does not restrict the generality.

5. UPPER ESTIMATES: PRELIMINARY REMARKS

Clearly, it suffices to prove all the upper estimates only for numbers $N \in \mathbb{N}$ such that $N \asymp 2^{au}(au)^{n-1}(\log(au))^b$, $u \in \mathbb{N}$, where a is from (4.10) and $b \in \{0,1\}$.

For $f \in L_1$, we define a function sequence ($\theta(-a) \equiv \{0\} \subset \mathbb{R}^n$)

$$(f_l(x)) = (f_l(x) \mid l \in \mathbb{N}_0), \qquad f_l(x) = \sum_{\alpha\in\theta(a(l-1))} \Delta^w_\alpha(f; x), \quad l \in \mathbb{N}_0; \tag{5.1}$$

[3] For $I > J$, we assume that $\sum_{j=I}^{J} c_j \equiv 0$ for any sequence $(c_j \mid j \in \mathbb{N})$.

in addition, we define a "hyperbolic" partial sum

$$S_u^{wm}(f;x) = \sum_{\alpha m \le u} \Delta_\alpha^w(f;x), \qquad u \in \mathbb{R}_+,$$

of the Fourier series of the function f with respect to the system $\mathcal{W}^m$. We will need the following estimates obtained in [2] (particular cases of Theorem 4.1 from [2]): let $1 \le p \le r \le 2$ and $1 \le q \le \infty$; then for $f \in B_{pq}^{sm}$ it holds that

$$\|f - S_u^{wm}(f)|L_r\| \ll 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u} u^{(n-1)(\frac{1}{r} - \frac{1}{q})_+} \|f|B_{pq}^{sm}\|. \tag{5.2}$$

Now, by analogy with [30, 32], we introduce auxiliary function spaces. Let $\varsigma \in \mathbb{R}_+$, $\epsilon \in \mathbb{R}$, and $1 \le p, q \le \infty$. Then

$$H_{pq}^{\varsigma\epsilon} \equiv H_{pq}^{\varsigma\epsilon}[\mathcal{W}^m, a](\mathbb{T}^k) := \{f \in L_1 \mid \|f|H_{pq}^{\varsigma\epsilon}\| < \infty\},$$

$$\|f|H_{pq}^{\varsigma\epsilon}\| = \sup\left\{2^{\varsigma a l}(\overline{al})^{(1-n)\epsilon}\left(\sum_{\alpha \in \theta(a(l-1))} \|\Delta_\alpha^w(f;x)|L_p\|^q\right)^{\frac{1}{q}} \,\middle|\, l \in \mathbb{N}_0\right\}$$

(here and everywhere below, $\overline{K} = \max\{1, K\}$ for $K \in \mathbb{R}$), and we set

$$\mathrm{H}_{pq}^{\varsigma\epsilon} := \{f \in H_{pq}^{\varsigma\epsilon} \mid \|f|H_{pq}^{\varsigma\epsilon}\| \le 1\}.$$

It is clear that the embeddings $\mathrm{B}_{pq}^{sm} \hookrightarrow \mathrm{H}_{pq}^{\varsigma\epsilon}$ (with $\epsilon \ge 0$) hold and the inequalities

$$\sigma_N(\mathrm{B}_{pq}^{sm}, \mathfrak{T}^{(k)}, L_r) \ll \sigma_N(\mathrm{H}_{pq}^{\varsigma\epsilon}, \mathfrak{T}^{(k)}, L_r) \qquad \text{for} \quad \epsilon \ge 0 \tag{5.3}$$

are satisfied. Next, let $f \in \operatorname{span}\{w_{\alpha\lambda}^\iota \mid \alpha \in \theta(a(l-1)),\ \lambda \in \Lambda(m,\alpha),\ \iota \in \mathrm{E}^m(\alpha)\}$, i.e.,

$$f(x) \equiv f_l(x) := \sum_{\alpha \in \theta(a(l-1))} \Delta_\alpha^w(f;x).$$

Then by Theorem A we have

$$\|f_l|B_{pq}^{sm}\| \asymp (\overline{al})^{(n-1)\epsilon}\|f_l|H_{pq}^{\varsigma\epsilon}\|, \tag{5.4}$$

which, in view of (5.2) and the equality $S_{a(l-1)}^{wm}(f;x) = 0$, implies the following estimate for the norm:

$$\|f_l|L_r\| \ll 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})al}(al)^{(n-1)(\epsilon + (\frac{1}{r} - \frac{1}{q})_+)}\|f_l|H_{pq}^{\varsigma\epsilon}\| \qquad \text{for} \quad 1 \le p \le r \le 2. \tag{5.5}$$

Moreover, by the definition of the norm $\|\cdot|H_{pq}^{\varsigma\epsilon}\|$, for $f \in H_{pq}^{\varsigma\epsilon}$ we obviously have

$$\|f_l|H_{pq}^{\varsigma\epsilon}\| \le \|f|H_{pq}^{\varsigma\epsilon}\|, \qquad l \in \mathbb{N}_0. \tag{5.6}$$

Set (see (4.3))

$$\mathcal{P}[u] \equiv \mathcal{P}[u; m, a] := \bigcup_{\alpha m \le au} P^w(m,\alpha) \ (\subset \mathbb{Z}^k), \qquad u \in \mathbb{N}.$$

Then, by Lemma A we find

$$\delta(u,m) := \#\mathcal{P}[u] \le \sum_{\alpha m \le au} \#P^w(m,\alpha) \le 3^k \sum_{\alpha m \le u} 2^{\alpha m} \asymp 2^u u^{n-1}. \tag{5.7}$$

Take an $L \in \mathbb{N}$ (depending on $u \in \mathbb{N}$) such that

$$2^{a(u+L)} \asymp 2^{au}(au)^{n-1} \tag{5.8}$$

and fix a number $\varkappa \in (0,1)$. Next, we define three families of numbers ($l \in \mathrm{z}_L$)

$$N_1(l) := \lfloor 2^{-(\varkappa+1)al}(a(u+l-1))^{n-1} \rfloor, \qquad N_2(l) := \lfloor 2^{\varkappa a(L-l)} \rfloor, \qquad N_3(l) := 2^{a(L-l)}. \tag{5.9}$$

Then, for $j = 1, 2, 3$ and any sets $\theta[l;j] \subset \theta(a(u+l-1))$ of cardinality

$$\#\theta[l;j] = \mathtt{N}_j(l) := \min\{N_j(l), \#\theta(a(u+l-1))\}, \qquad l \in \mathrm{z}_L, \tag{5.10}$$

we put

$$\mathcal{P}_j[u] \equiv \mathcal{P}_j[u; \mathtt{N}_j(1), \dots, \mathtt{N}_j(L)] := \mathcal{P}[u] \cup \bigcup_{l \in \mathrm{z}_L} \bigcup_{\alpha \in \theta[l;j]} P^w(m, \alpha).$$

The cardinalities of these sets can be estimated as

$$\begin{aligned} \delta_j(u,m,L) &:= \#\mathcal{P}_j[u] \le \#\mathcal{P}[u] + \sum_{l \in \mathrm{z}_L} \sum_{\alpha \in \theta[l;j]} \#P^w(m,\alpha) \\ &\ll 2^{au}(au)^{n-1} + \sum_{l \in \mathrm{z}_L} \sum_{\alpha \in \theta[l;j]} 2^{\alpha m} \le 2^{au}(au)^{n-1} + \sum_{l \in \mathrm{z}_L} 2^{a(u+l)} N_j(l). \end{aligned} \tag{5.11}$$

Substituting the values of the numbers $N_j(l)$, $l \in \mathrm{z}_L$, from (5.9) ($j = 1, 2, 3$) into (5.11), we obtain the following estimates:

$$\delta_1(u,m,L) \le 2^{au}(au)^{n-1} + 2^{au} \sum_{l=1}^{L} 2^{-\varkappa al}(a(u+l-1))^{n-1} \asymp 2^{au}(au)^{n-1}, \tag{5.12}$$

$$\begin{aligned} \delta_2(u,m,L) &\ll 2^{au}(au)^{n-1} + 2^{au} \sum_{l=1}^{L} 2^{\varkappa aL} \cdot 2^{a(1-\varkappa)l} \asymp 2^{au}(au)^{n-1} + 2^{au} \cdot 2^{\varkappa aL} \cdot 2^{a(1-\varkappa)L} \\ &= 2^{au}(au)^{n-1} + 2^{a(u+L)} \asymp 2^{au}(au)^{n-1} \end{aligned} \tag{5.13}$$

(here and in the following estimate, we took into account the choice of the number L), and

$$\delta_3(u,m,L) \ll 2^{au}(au)^{n-1} + 2^{au} \sum_{l \in \mathrm{z}_L} 2^{aL} = 2^{au}(au)^{n-1} + 2^{a(u+L)} L \asymp 2^{au}(au)^{n-1} \log(au). \tag{5.14}$$

6. UPPER ESTIMATES: CASE I

First, we prove the upper estimates for the best N-term trigonometric approximations of the classes $\mathrm{H}^{\varsigma\epsilon}_{pq}$ from the following theorem.

Theorem 6.1. *Let $1 \le p \le r \le 2$, $1 < r$, $1 \le q \le \infty$, and $\varsigma > \frac{1}{p} - \frac{1}{r}$. Then the following relations are valid:*

$$\begin{aligned} \sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq}, \mathfrak{T}^{(k)}, L_r\big) &\asymp N^{\varsigma - \frac{1}{p} + \frac{1}{r}} (\log^{n-1} N)^{\epsilon + (\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q})_+} \qquad \textit{if} \quad \varsigma \ne \frac{1}{p} - \frac{2}{r} + \frac{1}{q}, \\ \sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq}, \mathfrak{T}^{(k)}, L_r\big) &\asymp N^{\varsigma - \frac{1}{p} + \frac{1}{r}} (\log^{n-1} N)^{\epsilon} (\log\log N)^{\frac{1}{q}} \qquad \textit{if} \quad \varsigma = \frac{1}{p} - \frac{2}{r} + \frac{1}{q}. \end{aligned} \tag{6.1}$$

Proof of the upper estimates in Theorem 6.1. A. Let first $q \geq r$ and hence $\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q} > 0$. Choose $N \in \mathbb{N}$ from the conditions $N \geq \delta(u,m)$ and $N \asymp 2^{au}(au)^{n-1}$ ($\delta(u,m)$ is defined in (5.7)).

Let $f \in \mathrm{H}^{\varsigma\epsilon}_{pq}$. Then the trigonometric polynomial

$$t_N(f;x) := S^{wm}_{au}(f;x) \tag{6.2}$$

has at most N harmonics in view of (5.7), the choice of N, and the inclusion (4.3).

Let us show that the polynomial $t_N(f;x)$ approximates the function f in the metric of L_r within the required error bound. Indeed, applying the Minkowski inequality, estimate (5.5), and inequality (5.6), we obtain

$$\begin{aligned}\|f - t_N(f)\| &\leq \sum_{l=1}^{\infty} \|f_{u+l}|L_r\| \ll \sum_{l=1}^{\infty} 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})a(u+l)}(a(u+l))^{(n-1)(\epsilon + \frac{1}{r} - \frac{1}{q})}\|f_{u+l}|H^{\varsigma\epsilon}_{pq}\| \\ &\leq 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})au}\|f|H^{\varsigma\epsilon}_{pq}\| \sum_{l=1}^{\infty} 2^{-(\epsilon - \frac{1}{p} + \frac{1}{r})al}(a(u+l))^{(n-1)(\varsigma + \frac{1}{r} - \frac{1}{q})} \\ &\ll 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})au}(au)^{(n-1)(\epsilon + \frac{1}{r} - \frac{1}{q})} \asymp N^{-\varsigma + \frac{1}{p} - \frac{1}{r}}(\log^{n-1} N)^{\epsilon + (\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q})_+}.\end{aligned} \tag{6.3}$$

Hence,

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \ll N^{-\varsigma + \frac{1}{p} - \frac{1}{r}}(\log^{n-1} N)^{\epsilon + (\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q})_+}. \tag{6.4}$$

B. Let now $1 \leq \max\{p,q\} < r$. Consider separately the following cases:

(i) $\tau \equiv \varsigma - \frac{1}{p} + \frac{1}{r} > \frac{1}{q} - \frac{1}{r}$,

(ii) $\tau < \frac{1}{q} - \frac{1}{r}$, and

(iii) $\tau = \frac{1}{q} - \frac{1}{r}$.

(i) Let first $\tau > \frac{1}{q} - \frac{1}{r}$. We fix a number $\varkappa > 0$ such that $\tau > (\varkappa + 1)\big(\frac{1}{q} - \frac{1}{r}\big)$. Then, depending on $u \in \mathbb{N}$, we choose an $L \in \mathbb{N}$ satisfying (5.8) and take the numbers $N_1(l)$ and $\mathtt{N}_1(l)$, $l \in \mathrm{z}_L$, from (5.9) and (5.10). Finally, we choose $N \in \mathbb{N}$ from the conditions $N \geq \delta_1(u,m,L)$ and $N \asymp 2^{au}(au)^{n-1}$ ($\delta_1(u,m,L)$ is defined in (5.12)). Let $f \in \mathrm{H}^{\varsigma\epsilon}_{pq}$. Then the polynomial

$$t_N(f;x) := S^{wm}_{au}(f;x) + \sum_{l=1}^{L} t_l(f;x) := S^{wm}_{au}(f;x) + \sum_{l=1}^{L} \sum_{\alpha \in \theta[l;f]} \Delta^w_\alpha(f;x)$$

has at most N harmonics in view of (5.12), the choice of N, and (4.3); here, as $\theta[l;1]$ in (5.10), we take the set $\theta[l;f] \subset \theta(a(u+l-1))$ of those α that correspond to the maximum values of $2^{\alpha m(\frac{1}{p} - \frac{1}{r})}\|\Delta^w_\alpha(f;x)|L_p\|$, i.e.,

$$2^{\alpha m(\frac{1}{p} - \frac{1}{r})}\|\Delta^w_\alpha(f;x)|L_p\| \geq 2^{\alpha' m(\frac{1}{p} - \frac{1}{r})}\|\Delta^w_{\alpha'}(f;x)|L_p\|$$

for all $\alpha \in \theta[l;f]$ and $\alpha' \in \theta'[l;f] := \theta(a(u+l-1)) \setminus \theta[l;f]$.

Now, let us show that the polynomial $t_N(f;x)$ yields the required approximation error for the function f in the metric of L_r. By the Minkowski inequality, we have

$$\|f - t_N(f)|L_r\| \leq \left\|\sum_{l=1}^{L}(f_{u+l} - t_l(f))\middle|L_r\right\| + \sum_{l=u+L+1}^{\infty}\|f_l|L_r\| =: \mathfrak{J}^{(1)}_1(p,r) + \mathfrak{J}_2(p,r). \tag{6.5}$$

First, we estimate $\mathfrak{J}_1^{(1)}(p,r)$. Successively applying Lemmas T1 and B, taking into account the definition of $\theta(a(u+l-1))$, and substituting the numbers $N_1(l)$, $l\in \mathrm{z}_L$, according to the choice of the numbers $\varkappa$, L, and N and the definition of the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$ we obtain[4]

$$\begin{aligned}
\mathfrak{J}_1^{(1)}(p,r)^r &\ll \sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]} 2^{\alpha m(\frac1p-\frac1r)r}\|\Delta^w_\alpha(f;x)|L_p\|^r \\
&\le \sum_{l=1}^{L}\overline{\mathtt{N}}_1(l)^{(\frac1r-\frac1q)r}\Bigg\{\sum_{\alpha\in\theta(a(u+l-1))} 2^{\alpha m(\frac1p-\frac1r)q}\|\Delta^w_\alpha(f;x)|L_p\|^q\Bigg\}^{\frac rq} \\
&\le \sum_{l=1}^{L} N_1(l)^{(\frac1r-\frac1q)r}\cdot 2^{a(u+l)(\frac1p-\frac1r)r}\Bigg\{\sum_{\alpha\in\theta(a(u+l-1))}\|\Delta^w_\alpha(f;x)|L_p\|^q\Bigg\}^{\frac rq} \\
&\le \sum_{l=1}^{L}\big(2^{-(\varkappa+1)al}(a(u+l))^{n-1}\big)^{(\frac1r-\frac1q)r}\cdot 2^{-\tau a(u+l)r}(a(u+l))^{(n-1)\epsilon r} \\
&= 2^{-\tau aur}\sum_{l=1}^{L}2^{-al(\tau-(1+\varkappa)(\frac1r-\frac1q))r}(a(u+l))^{(n-1)(\epsilon+\frac1r-\frac1q)r} \\
&\asymp 2^{-\tau aur}(au)^{(n-1)(\epsilon+\frac1r-\frac1q)r}\asymp \Big(N^{-\varsigma+\frac1p-\frac1r}(\log^{n-1}N)^{\varsigma-\frac1p+\frac2r-\frac1q}\Big)^r. \qquad (6.6)
\end{aligned}$$

Now we proceed to the estimate for $\mathfrak{J}_2(p,r)$. Using (5.5) and (5.6), we find

$$\begin{aligned}
\mathfrak{J}_2(p,r) &\ll \sum_{l=u+L-1}^{\infty}2^{-\tau al}(al)^{(n-1)\epsilon}\|f_l|H^{\varsigma\epsilon}_{pq}\| \ll \sum_{l=u+L-1}^{\infty}2^{-\tau al}(al)^{(n-1)\epsilon} \\
&\asymp 2^{-\tau a(u+L)}(a(u+L))^{(n-1)\epsilon}\asymp N^{-\tau}(\log^{n-1}N)^{\epsilon}\asymp N^{-\varsigma+\frac1p-\frac1r}(\log^{n-1}N)^{\epsilon}. \qquad (6.7)
\end{aligned}$$

Thus, (6.5)–(6.7) yield the estimate

$$\|f-t_N(f)|L_r\|\ll N^{-\varsigma+\frac1p-\frac1r}(\log^{n-1}N)^{\epsilon+\varsigma-\frac1p+\frac2r-\frac1q}.$$

This implies that

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_r\big)\ll N^{-\varsigma+\frac1p-\frac1r}(\log^{n-1}N)^{\epsilon+\varsigma-\frac1p+\frac2r-\frac1q}. \qquad (6.8)$$

(ii) Now, let $0<\tau<\frac1q-\frac1r$. Fix a number $\varkappa\in(0,1)$ such that $\tau=\varsigma-\frac1p+\frac1r<\varkappa\big(\frac1q-\frac1r\big)$. Then, depending on $u\in\mathbb{N}$, we choose an $L\in\mathbb{N}$ satisfying (5.8) and take the numbers $N_2(l)$ and $\mathtt{N}_2(l)$, $l\in\mathrm{z}_L$, from (5.9) and (5.10). Finally, we choose $N\in\mathbb{N}$ from the conditions $N\ge\delta_2(u,m,L)$ and $N\asymp 2^{au}(au)^{n-1}$ ($\delta_2(u,m,L)$ is defined in (5.13)).

Let $f\in\mathrm{H}^{\varsigma\epsilon}_{pq}$; as $\theta[l;2]$ in (5.10), we take $\theta[l;f]\subset\theta(a(u+l-1))$, just as in case (i) above ($l\in\mathrm{z}_L$). Then the polynomial

$$t_N(f;x):=S^{wm}_{au}(f;x)+\sum_{l=1}^{L}t_l(f;x):=S^{wm}_{au}(f;x)+\sum_{l=1}^{L}\sum_{\alpha\in\theta[l;f]}\Delta^w_\alpha(f;x)$$

has at most N harmonics in view of (5.13), the choice of N, and (4.3).

[4]Henceforth we assume that $\sum_{\alpha\in\varnothing}c_\alpha\equiv 0$ for any sequence $(c_\alpha\mid\alpha\in\mathbb{N}_0^n)$.

Now, let us estimate the error of approximation of the function f by the polynomial $t_N(f;x)$ in the metric of L_r. We have

$$\|f - t_N(f)|L_r\| \le \left\| \sum_{l=1}^{L} (f_{u+l} - t_l(f)) \Big| L_r \right\| + \sum_{l=u+L+1}^{\infty} \|f_l|L_r\| =: \mathfrak{I}_1^{(2)}(p,r) + \mathfrak{I}_2(p,r). \tag{6.9}$$

The necessary upper estimate for the "tail" $\mathfrak{I}_2(p,r)$ is already obtained in (6.7). Therefore, we proceed to estimate $\mathfrak{I}_1^{(2)}(p,r)$. Applying Lemma T1 and reasoning as above, substituting the numbers $N_2(l)$, $l \in \mathrm{z}_L$, and taking into account the choice of $\varkappa$, N, and L, we obtain

$$\begin{aligned}
\mathfrak{I}_1^{(2)}(p,r)^r &\ll \sum_{l=1}^{L} N_2(l)^{\left(\frac{1}{r}-\frac{1}{q}\right)r} \cdot 2^{a(u+l)\left(\frac{1}{p}-\frac{1}{r}\right)r} \left\{ \sum_{\alpha \in \theta(a(u+l-1))} \|\Delta_\alpha^w(f;x)|L_p\|^q \right\}^{\frac{r}{q}} \\
&\ll \sum_{l=1}^{L} 2^{a(u+l)\left(\frac{1}{p}-\frac{1}{r}\right)r} N_2(l)^{\left(\frac{1}{r}-\frac{1}{q}\right)r} \left\{ \sum_{\alpha \in \theta(a(u+l-1))} \|\Delta_\alpha^w(f;x)|L_p\|^q \right\}^{\frac{r}{q}} \\
&\le \sum_{l=1}^{L} 2^{-\tau a(u+l)r} \cdot 2^{\varkappa a(L-l)\left(\frac{1}{r}-\frac{1}{q}\right)r} (a(u+l))^{(n-1)\epsilon r} \\
&= 2^{-\tau aur} \cdot 2^{\varkappa aL\left(\frac{1}{r}-\frac{1}{q}\right)r} \sum_{l=1}^{L} 2^{al\left(\varkappa\left(\frac{1}{q}-\frac{1}{r}\right)-\tau\right)r} (a(u+l))^{(n-1)\epsilon r} \\
&\asymp 2^{-\tau aur} \cdot 2^{\varkappa aL\left(\frac{1}{r}-\frac{1}{q}\right)r} \cdot 2^{aL\left(\varkappa\left(\frac{1}{q}-\frac{1}{r}\right)-\tau\right)r} (a(u+L))^{(n-1)\epsilon r} \\
&= 2^{-\tau a(u+L)r} (a(u+L))^{(n-1)\epsilon r} \asymp N^{-\tau r} (\log^{n-1} N)^{\epsilon r}.
\end{aligned} \tag{6.10}$$

Combining estimates (6.9), (6.7), and (6.10), we find

$$\|f - t_N(f)|L_r\| \ll N^{-\varsigma + \frac{1}{p} - \frac{1}{r}} (\log^{n-1} N)^{\epsilon}.$$

Hence we obtain the desired estimate for the class $\mathrm{H}_{pq}^{\varsigma\epsilon}$:

$$\sigma_N\big(\mathrm{H}_{pq}^{\varsigma\epsilon}, \mathfrak{T}^{(k)}, L_r\big) \ll N^{-\varsigma + \frac{1}{p} - \frac{1}{r}} (\log^{n-1} N)^{\epsilon}. \tag{6.11}$$

(iii) Let, finally, $0 < \tau = \frac{1}{q} - \frac{1}{r}$. Depending on $u \in \mathbb{N}$, we choose a number $L \in \mathbb{N}$ satisfying (5.8) and then take the numbers $N_3(l)$ and $\mathtt{N}_3(l)$, $l \in \mathrm{z}_L$, from (5.9) and (5.10). After that, we choose $N \in \mathbb{N}$ from the conditions $N \ge \delta_3(u,m,L)$ and $N \asymp 2^{au}(au)^{n-1}\log(au)$ ($\delta_3(u,m,L)$ is defined in (5.14)).

Let $f \in \mathrm{H}_{pq}^{\varsigma\epsilon}$; as $\theta[l;3]$ in (5.10), we take $\theta[l;f] \subset \theta(a(u+l-1))$ as above. Then the polynomial

$$t_N(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} t_l(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} \sum_{\alpha \in \theta[l;f]} \Delta_\alpha^w(f;x)$$

has at most N harmonics in view of (5.14), the choice of N, and (4.3).

Now, we estimate the error of approximation of the function f by the polynomial $t_N(f;x)$ in the metric of L_r. Again, we have

$$\|f - t_N(f)|L_r\| \le \left\| \sum_{l=1}^{L} (f_{u+l} - t_l(f)) \Big| L_r \right\| + \sum_{l=u+L+1}^{\infty} \|f_l|L_r\| =: \mathfrak{I}_1^{(3)}(p,r) + \mathfrak{I}_2(p,r). \tag{6.12}$$

Let us estimate $\mathfrak{J}_1^{(3)}(p,r)$. Again, the successive application of Lemmas T1 and B followed by the substitution of the numbers $N_3(l)$, $l \in \mathrm{z}_L$ (in view of the choice of L and N and the equality $\tau = \frac{1}{q} - \frac{1}{r}$), yields

$$\begin{aligned}
\mathfrak{J}_1^{(3)}(p,r)^r &\ll \sum_{l=1}^{L} N_3(l)^{\left(\frac{1}{r}-\frac{1}{q}\right)r} \cdot 2^{a(u+l)\left(\frac{1}{p}-\frac{1}{r}\right)r} \Bigg\{ \sum_{\alpha\in\theta(a(u+l-1))} \|\Delta_\alpha^w(f;x)|L_p\|^q \Bigg\}^{\frac{r}{q}} \\
&\le \sum_{l=1}^{L} N_3(l)^{\left(\frac{1}{r}-\frac{1}{q}\right)r} \cdot 2^{a(u+l)\left(\frac{1}{p}-\frac{1}{r}\right)r} \cdot 2^{-a(u+l)\varsigma r}(a(u+l))^{(n-1)\epsilon r} \\
&\le \sum_{l=1}^{L} 2^{-\tau a(u+l)r} \cdot 2^{a(L-l)\left(\frac{1}{r}-\frac{1}{q}\right)r}(a(u+l))^{(n-1)\epsilon r} \\
&= 2^{-\tau a u r} \cdot 2^{aL\left(\frac{1}{r}-\frac{1}{q}\right)r} \sum_{l=1}^{L} 2^{-al\left(\frac{1}{r}-\frac{1}{q}+\tau\right)r}(a(u+l))^{(n-1)\epsilon r} \le 2^{-\tau a(u+L)r}(a(u+l))^{(n-1)\epsilon r} L \\
&\asymp \left(2^{au}(au)^{n-1}\right)^{-\tau r}(au)^{(n-1)\epsilon r}\log(au) \asymp N^{-\tau r}(\log^{n-1} N)^{\epsilon r}(\log\log N)^{\frac{r}{q}}, \qquad (6.13)
\end{aligned}$$

because $N \asymp 2^{au}(au)^{n-1}\log(au)$ and, hence, $\log\log N \asymp \log(au)$.

Combining estimates (6.12), (6.7), and (6.13), we obtain

$$\|f - t_N(f)|L_r\| \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\frac{1}{q}}.$$

This implies the required estimate for the class $\mathrm{H}_{pq}^{\varsigma\epsilon}$ in the present case as well:

$$\sigma_N\left(\mathrm{H}_{pq}^{\varsigma\epsilon}, \mathfrak{T}^{(k)}, L_r\right) \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\frac{1}{q}}. \qquad (6.14)$$

C. Finally, let $1 \le q < p = r \le 2$. Just as in part B, consider separately the cases

(i) $\varsigma > \frac{1}{q} - \frac{1}{r}$,

(ii) $\varsigma < \frac{1}{q} - \frac{1}{r}$, and

(iii) $\varsigma = \frac{1}{q} - \frac{1}{r}$.

(i) Let first $\varsigma > \frac{1}{q} - \frac{1}{p}$. Fix a number $\varkappa > 0$ such that $\varsigma > (1+\varkappa)\left(\frac{1}{q} - \frac{1}{p}\right)$. Then, depending on $u \in \mathbb{N}$, we choose an $L \in \mathbb{N}$, the numbers $N_1(l)$ and $\mathtt{N}_1(l)$, $l \in \mathrm{z}_L$, from (5.9) and (5.10), and $N \in \mathbb{N}$ as in part B(i).

Let $f \in \mathrm{H}_{pq}^{\varsigma\epsilon}$. As $\theta[l;1]$ in (5.10), we take the set $\theta[l;f] \subset \theta(a(u+l-1))$ of those α that correspond to the maximum values of $\|\Delta_\alpha^w(f;x)|L_p\|$, i.e.,

$$\|\Delta_\alpha^w(f;x)|L_p\| \ge \|\Delta_{\alpha'}^w(f;x)|L_p\|, \qquad \alpha \in \theta[l;f], \quad \alpha' \in \theta'[l;f] = \theta(a(u+l-1)) \setminus \theta[l;f].$$

Then the polynomial

$$t_N(f;x) := t_N(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} t_l(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} \sum_{\alpha\in\theta[l;f]} \Delta_\alpha^w(f;x)$$

has at most N harmonics in view of (5.12).

Let us show that the polynomial $t_N(f;x)$ provides the necessary error of approximation of the function f in the metric of L_p. By the Minkowski inequality,

$$\|f - t_N(f)|L_p\| \le \left\| \sum_{l=1}^{L} (f_{u+l} - t_l(f)) \Big| L_p \right\| + \left\| \sum_{l=u+L+1}^{\infty} f_l \Big| L_p \right\| =: \mathfrak{J}_1^{(1)}(p,p) + \mathfrak{J}_2(p,p). \qquad (6.15)$$

First, we estimate $\mathfrak{J}_1^{(1)}(p,p)$. Successively applying Theorem B, Jensen's inequality, and (4.14), substituting the numbers $N_1(l)$, $l\in \mathrm{z}_L$, and taking into account the definitions of the set $\theta(a(u+l-1))$ and the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$ and the choice of the numbers $\varkappa$, L, and N, we obtain

$$\begin{aligned}\mathfrak{J}_1^{(1)}(p,p)^p &\ll \sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]}\|\Delta^w_\alpha(f;x)|L_p\|^p = \sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]}\|\Delta^w_\alpha(f;x)|L_p\|^{p-q}\|\Delta^w_\alpha(f;x)|L_p\|^q\\ &\le \sum_{l=1}^{L}\mathtt{N}_1(l)^{1-\frac{p}{q}}\Bigg[\sum_{\alpha\in\theta[l;f]}\|\Delta^w_\alpha(f;x)|L_p\|^q\Bigg]^{\frac{p}{q}-1}\sum_{\alpha\in\theta'[l;f]}\|\Delta^w_\alpha(f;x)|L_p\|^q\\ &\le \sum_{l=1}^{L}N_1(l)^{1-\frac{p}{q}}\Bigg[\sum_{\alpha\in\theta(a(u+l-1))}\|\Delta^w_\alpha(f;x)|L_p\|^q\Bigg]^{\frac{p}{q}}\\ &\le \sum_{l=1}^{L}\big(2^{-\varsigma a(u+l)}(a(u+l))^{(n-1)\epsilon}\big)^p\big(2^{-(1+\varkappa)al}(a(u+l))^{n-1}\big)^{1-\frac{p}{q}}\\ &= 2^{-\varsigma aup}\sum_{l=1}^{L}2^{-al\left(\varsigma+(1+\varkappa)\left(\frac1p-\frac1q\right)\right)p}(a(u+l))^{(n-1)\left(\epsilon+\frac1p-\frac1q\right)p}\\ &\ll \Big(2^{-\varsigma au}(au)^{(n-1)\left(\epsilon+\frac1p-\frac1q\right)}\Big)^p \asymp \Big(N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\epsilon+\frac1p-\frac1q}\Big)^p. \end{aligned}\tag{6.16}$$

Let us proceed to the estimate for $\mathfrak{J}_2(p,p)$. By Theorem B and Jensen's inequality (since $q<p\le 2$), taking into account the definition of the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$ and the choice of the number L, we find

$$\begin{aligned}\mathfrak{J}_2(p,p)^p &\ll \sum_{\alpha m>a(u+L)}\|\Delta^w_\alpha(f;x)|L_p\|^p \le \Bigg\{\sum_{l=L+1}^{\infty}\sum_{\alpha\in\theta(a(u+l-1))}\|\Delta^w_\alpha(f;x)|L_p\|^q\Bigg\}^{\frac{p}{q}}\\ &\le \Bigg\{\sum_{l=L+1}^{\infty}2^{-\varsigma a(u+l)q}(a(u+l))^{(n-1)\epsilon q}\Bigg\}^{\frac{p}{q}} \asymp \big(2^{-\varsigma a(u+L)}(a(u+L))^{(n-1)\epsilon}\big)^p\\ &\asymp \big((2^{au}(au)^{n-1})^{-\varsigma}(a(u+L))^{(n-1)\epsilon}\big)^p \asymp \big(N^{-\varsigma}(\log^{n-1}N)^{\epsilon}\big)^p. \end{aligned}\tag{6.17}$$

Now, substituting estimates (6.16) and (6.17) into (6.15), we arrive at

$$\|f-t_N(f)|L_p\| \ll N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\epsilon+\frac1p-\frac1q}.$$

This implies the required estimate for the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$:

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_r\big) \ll N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\epsilon+\frac1p-\frac1q}.\tag{6.18}$$

(ii) Let now $\varsigma<\frac1q-\frac1p$. Fix a number $\varkappa\in(0,1)$ such that $\varsigma<\varkappa\big(\frac1q-\frac1p\big)$. Depending on $u\in\mathbb{N}$, we choose an $L\in\mathbb{N}$, the numbers $N_2(l)$ and $\mathtt{N}_2(l)$, $l\in\mathrm{z}_L$, from (5.9) and (5.10), and $N\in\mathbb{N}$ as in part B(ii).

Let $f\in\mathrm{H}^{\varsigma\epsilon}_{pq}$. As $\theta[l;2]$ in (5.10), we take $\theta[l;f]\subset\theta(a(u+l-1))$, $l\in\mathrm{z}_L$, as in case (i). Then the polynomial

$$t_N(f;x) := S^{wm}_{au}(f;x)+\sum_{l=1}^{L}t_l(f;x) := S^{wm}_{au}(f;x)+\sum_{l=1}^{L}\sum_{\alpha\in\theta[l;f]}\Delta^w_\alpha(f;x)$$

has at most N harmonics in view of (5.13).

Now, let us estimate the error of approximation of the function f by the polynomial $t_N(f;x)$ in the metric of L_p. We have

$$\|f - t_N(f)|L_p\| \le \Bigg\|\sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]}\Delta_\alpha^w(f;x)\Bigg|L_p\Bigg\| + \mathfrak{J}_2(p,p) =: \mathfrak{J}_1^{(2)}(p,p) + \mathfrak{J}_2(p,p). \tag{6.19}$$

The necessary estimate for the "tail" $\mathfrak{J}_2(p,p)$ is already established in (6.17). Therefore, it remains to derive an appropriate estimate for $\mathfrak{J}_1^{(2)}(p,p)$. Arguing as in case (i), substituting the values of $N_2(l)$, $l \in \mathrm{z}_L$, and taking into account the choice of $\varkappa$, L, and N, we obtain

$$\begin{aligned}
\mathfrak{J}_1^{(2)}(p,p)^p &\ll \sum_{l=1}^{L} N_2(l)^{1-\frac{p}{q}}\Bigg[\sum_{\alpha\in\theta[l;f]}\|\Delta_\alpha^w(f;x)|L_p\|^q\Bigg]^{\frac{p}{q}-1}\sum_{\alpha\in\theta'[l;f]}\|\Delta_\alpha^w(f;x)|L_p\|^q \\
&\le \sum_{l=1}^{L} 2^{\varkappa a(L-l)(\frac{1}{p}-\frac{1}{q})p}\big(2^{-\varsigma a(u+l)}(a(u+l))^{(n-1)\epsilon}\big)^p \\
&\asymp 2^{-\varsigma aup}\cdot 2^{aL\varkappa(\frac{1}{p}-\frac{1}{q})p}\sum_{l=1}^{L} 2^{al(-\varsigma+\varkappa(\frac{1}{q}-\frac{1}{p}))p}(a(u+l))^{(n-1)\epsilon p} \\
&= \big(2^{-\varsigma a(u+L)}(a(u+L))^{(n-1)\epsilon}\big)^p \asymp \big(N^{-\varsigma}(\log^{n-1}N)^\epsilon\big)^p.
\end{aligned} \tag{6.20}$$

Thus, substituting estimates (6.17) and (6.20) into (6.19), we arrive at

$$\|f - t_N(f)|L_p\| \ll N^{-\varsigma}(\log^{n-1}N)^\epsilon,$$

which implies the required upper estimate for the class $\mathrm{H}_{pq}^{\varsigma\epsilon}$:

$$\sigma_N\big(\mathrm{H}_{pq}^{\varsigma\epsilon}, \mathfrak{T}^{(k)}, L_r\big) \ll N^{-\varsigma}(\log^{n-1}N)^\epsilon. \tag{6.21}$$

(iii) Finally, let $\varsigma = \frac{1}{q} - \frac{1}{p}$. Depending on $u \in \mathbb{N}$, we choose a number $L \in \mathbb{N}$, the numbers $N_3(l)$ and $\mathtt{N}_3(l)$, $l \in \mathrm{z}_L$, from (5.9) and (5.10), and $N \in \mathbb{N}$ as in part B(iii).

Let $f \in \mathrm{H}_{pq}^{\varsigma\epsilon}$. As $\theta[l;3]$ in (5.10), we take $\theta[l;f] \subset \theta(a(u+l-1))$ ($l \in \mathrm{z}_L$) as in case (i). Then the polynomial

$$t_N(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} t_l(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L}\sum_{\alpha\in\theta[l;f]}\Delta_\alpha^w(f;x)$$

has at most N harmonics (see (5.14)).

Now, let us proceed to estimating the error of approximation of the function f by the polynomial $t_N(f;x)$ in the metric of L_p:

$$\|f - t_N(f)|L_p\| \le \Bigg\|\sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]}\Delta_\alpha^w(f;x)\Bigg|L_p\Bigg\| + \mathfrak{J}_2(p,p) =: \mathfrak{J}_1^{(3)}(p,p) + \mathfrak{J}_2(p,p). \tag{6.22}$$

In view of (6.17), it remains to derive the required estimate for $\mathfrak{J}_1^{(3)}(p,p)$. Arguing as above (substituting the numbers $N_3(l)$, $l \in \mathrm{z}_L$, and taking account of the equality $\varsigma = \frac{1}{q} - \frac{1}{p}$), we get

$$\begin{aligned}
\mathfrak{J}_1^{(3)}(p,p)^p &\ll \sum_{l=1}^{L} N_3(l)^{1-\frac{p}{q}}\Bigg[\sum_{\alpha\in\theta(a(u+l-1))}\|\Delta_\alpha^w(f;x)|L_p\|^q\Bigg]^{\frac{p}{q}} \\
&\le \sum_{l=1}^{L} 2^{a(L-l)(\frac{1}{p}-\frac{1}{q})p}\big(2^{-\varsigma a(u+l)}(a(u+l))^{(n-1)\epsilon}\big)^p
\end{aligned}$$

$$\ll 2^{-\varsigma a(u+L)}(a(u+l))^{(n-1)\epsilon}L^{\frac{1}{p}} \asymp (2^{au}(au)^{n-1})^{-\varsigma}(au)^{(n-1)\epsilon}(\log(au))^{\frac{1}{p}}$$

$$\asymp N^{-\varsigma}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\varsigma+\frac{1}{p}} = N^{-\varsigma}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\frac{1}{q}}, \qquad (6.23)$$

because $N \asymp 2^{au}(au)^{n-1}\log(au)$ and $L \asymp \log(au)$.

Substituting estimates (6.12) and (6.7) into (6.13), we find

$$\|f - t_N(f)|L_p\| \ll N^{-\varsigma}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\frac{1}{q}},$$

which implies the required upper estimate for the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$:

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq}, \mathfrak{T}^{(k)}, L_p\big) \ll N^{-\varsigma}(\log^{n-1} N)^{\epsilon}(\log\log N)^{\frac{1}{q}}. \qquad (6.24)$$

Thus, the upper estimates in Theorem 6.1 are completely proved. Appropriate lower estimates will be established in Section 9. □

Theorem 6.1 and inequality (5.3) imply the required upper estimates for the best N-term trigonometric approximations of the classes B^{sm}_{pq} in case I of Theorem 2.1, except for the only situation when $\varsigma = \frac{1}{p} - \frac{2}{r} + \frac{1}{q}$. In this case, estimates (6.14) and (6.24) obtained for the classes $\mathrm{H}^{\varsigma\epsilon}_{pq}$ turn out to be rougher than it is required for the classes B^{sm}_{pq}: (6.14) and (6.24) contain a redundant factor of $(\log\log N)^{\frac{1}{q}}$.

Therefore, to achieve the required approximation error in the case of $\varsigma = \frac{1}{p} - \frac{2}{r} + \frac{1}{q}$, we correct the construction of an approximating polynomial for $f \in \mathrm{B}^{sm}_{pq}$ while keeping in mind the difference between the definitions of the classes B^{sm}_{pq} and $\mathrm{H}^{\varsigma\epsilon}_{pq}$.

Proof of the upper estimate in Theorem 2.1, case I. Let first $1 \le p < r \le 2$. Let also $f \in \mathrm{B}^{sm}_{pq}$ and $L \in \mathbb{N}$ be such that $2^{a(u+L)} \asymp 2^{au}(au)^{n-1}$. Define a number

$$N'_3(l) := \left\lfloor 2^{a(L-l)} \sum_{\alpha\in\theta(a(u+l-1))} 2^{\alpha sq}\|\Delta^w_\alpha(f;x)|L_p\|^q \right\rfloor + 1, \qquad l \in \mathrm{z}_L. \qquad (6.25)$$

Then, for arbitrary sets $\theta[l] \subset \theta(a(u+l-1))$ with $\#\theta[l] = \mathtt{N}'_3(l)$,

$$\mathtt{N}'_3(l) = \min\{N'_3(l), \#\theta(a(u+l-1))\}, \qquad l \in \mathrm{z}_L, \qquad (6.26)$$

by Lemma A together with the definition of $\theta(a(u+l-1))$, Theorem A, and the inclusion $f \in \mathrm{B}^{sm}_{pq}$, we obtain the estimate

$$\delta'_3(u,m,L) \equiv \delta'_3(u,m,L;f) := \sum_{\alpha m\le au} 3^k\cdot 2^{\alpha m} + \sum_{l=1}^{L}\sum_{\alpha\in\theta[l]} 3^k\cdot 2^{\alpha m}$$

$$\le 3^k\left(\sum_{\alpha m\le au} 2^{\alpha m} + \sum_{l=1}^{L} 2^{a(u+l)}N'_3(l)\right)$$

$$\asymp 2^{au}(au)^{n-1} + 2^{a(u+L)}\sum_{l=1}^{L}\sum_{\alpha\in\theta(a(u+l-1))} 2^{\alpha sq}\|\Delta^w_\alpha(f;x)|L_p\|^q$$

$$\ll 2^{au}(au)^{n-1} + 2^{a(u+L)} \asymp 2^{au}(au)^{n-1}. \qquad (6.27)$$

Now, we choose $N \in \mathbb{N}$ from the conditions $N \geq \delta_3'(u,m,L)$ and $N \asymp 2^{au}(au)^{n-1}$. Then the trigonometric polynomial

$$t_N(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} t_l(f;x) := S_{au}^{wm}(f;x) + \sum_{l=1}^{L} \sum_{\alpha\in\theta[l;f]} \Delta_\alpha^w(f;x),$$

where the set $\theta[l;f] \subset \theta(a(u+l-1))$ with $\#\theta[l] = \mathtt{N}_3'(l)$ is chosen as in part B(i) ($l \in \mathrm{z}_L$), has at most N harmonics in view of (6.27) and (4.3).

Let us proceed to estimating the error of approximation of the function f by the polynomial $t_N(f;x)$ in the metric of L_r:

$$\|f - t_N(f)|L_r\| \leq \left\| \sum_{l=1}^{L}(f_{u+l} - t_l(f)) \Big| L_r \right\| + \sum_{l=u+L+1}^{\infty} \|f_l|L_r\| =: \mathfrak{J}_1^{(3+)}(p,r) + \mathfrak{J}_2(p,r); \qquad (6.28)$$

in this case, due to the choice of L, the estimate for the "tail" $\mathfrak{J}_2(p,r)$ is already proved in (6.7). Hence, it remains to estimate $\mathfrak{J}_1^{(3+)}(p,r)$. Successively applying Lemmas T1 and B, Theorem A and taking into account the condition $\tau = \frac{1}{q} - \frac{1}{r}$ and the choice of the number L, we obtain

$$\begin{aligned}
\mathfrak{J}_1^{(3+)}(p,r)^r &\ll \sum_{l=1}^{L} \sum_{\alpha\in\theta'[l;f]} 2^{\alpha m(\frac{1}{p}-\frac{1}{r})r} \|\Delta_\alpha^w(f;x)|L_p\|^r \\
&\leq \sum_{l=1}^{L} \mathtt{N}_3'(l)^{(\frac{1}{r}-\frac{1}{q})r} \left\{ \sum_{\alpha\in\theta(a(u+l-1))} 2^{-\tau\alpha m q} \cdot 2^{\alpha s q} \|\Delta_\alpha^w(f;x)|L_p\|^q \right\}^{\frac{r}{q}} \\
&\leq \sum_{l=1}^{L} N_3'(l)^{(\frac{1}{r}-\frac{1}{q})r} \cdot 2^{-\tau a(u+l)r} \left\{ \sum_{\alpha\in\theta(a(u+l-1))} 2^{\alpha s q} \|\Delta_\alpha^w(f;x)|L_p\|^q \right\}^{\frac{r}{q}} \\
&\leq \sum_{l=1}^{L} 2^{-\tau a(u+l)r} \cdot 2^{a(L-l)(\frac{1}{r}-\frac{1}{q})r} \sum_{\alpha\in\theta(au+a(l-1))} 2^{\alpha s q} \|\Delta_\alpha^w(f;x)|L_p\|^q \\
&\ll 2^{-\tau a(u+L)r} \|f|B_{pq}^{sm}\|^q \ll 2^{-\tau a(u+L)r} \asymp (2^{au}(au)^{n-1})^{-\tau r} \asymp N^{-\tau r}. \qquad (6.29)
\end{aligned}$$

Thus, combining estimates (6.28), (6.7), and (6.29), we see that

$$\|f - t_N(f)|L_r\| \ll N^{-\varsigma + \frac{1}{p} - \frac{1}{r}},$$

which implies the required estimate for the class B_{pq}^{sm}:

$$\sigma_N\big(\mathrm{B}_{pq}^{sm}, \mathfrak{T}^{(k)}, L_r\big) \ll N^{-\varsigma + \frac{1}{p} - \frac{1}{r}}.$$

Now, let $1 < p = r \leq 2$. Let $f \in \mathrm{B}_{pq}^{sm}$ and $L \in \mathbb{N}$, $N_3'(l)$ from (6.25) ($l \in \mathrm{z}_L$), and $N \in \mathbb{N}$ be chosen as in the previous case, and let the sets $\theta[l;f] \subset \theta(a(u+l-1))$ with $\#\theta[l] = N_3'(l)$ be defined by the condition

$$2^{\alpha s}\|\Delta_\alpha^w(f;x)|L_p\| \geq 2^{\alpha' s}\|\Delta_\beta^w(f;x)|L_p\|, \qquad \alpha \in \theta[l;f], \quad \alpha' \in \theta'[l;f] := \theta(a(u+l-1)) \setminus \theta[l;f].$$

Then the polynomial

$$t_N(f;x) := \sum_{\alpha m \leq au} \Delta_\alpha^w(f;x) + \sum_{l=1}^{L} \sum_{\alpha\in\theta[l;f]} \Delta_\alpha^w(f;x)$$

has at most N harmonics in view of (6.27).

Now, let us show that this polynomial $t_N(f;x)$ gives the required approximation error for the function f in the metric of L_p. We have

$$\|f - t_N(f)|L_p\| \le \left\|\sum_{l=1}^{L}\sum_{\alpha\in\theta'[l;f]}\Delta_\alpha^w(f;x)\Big|L_p\right\| + \mathfrak{J}_2(p,p) =: \mathfrak{J}_1^{(3+)}(p,p) + \mathfrak{J}_2(p,p); \tag{6.30}$$

here, in view of the choice of L and (6.17), it remains to prove the required estimate for $\mathfrak{J}_1^{(3+)}(p,p)$. Arguing as in part C of the proof of the upper estimates in Theorem 6.1 above, we obtain

$$\begin{aligned}
\mathfrak{J}_1^{(3+)}(p,p)^p &\ll 2^{-\varsigma aup}\sum_{l=1}^{L}2^{-\varsigma alp}\left[2^{a(L-l)}\sum_{\alpha\in\theta(a(u+l-1))}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q\right]^{1-\frac{p}{q}}\\
&\quad\times\left[\sum_{\alpha\in\theta[l;f]}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q\right]^{\frac{p}{q}-1}\sum_{\alpha\in\theta'[l;f]}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q\\
&\le \left(2^{-\varsigma au}\cdot 2^{aL\left(\frac{1}{p}-\frac{1}{q}\right)}\right)^p\sum_{l=1}^{L}2^{-\varsigma alp}\cdot 2^{al\left(\frac{1}{q}-\frac{1}{p}\right)p}\sum_{\alpha\in\theta(a(u+l-1))}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q\\
&= 2^{-\varsigma a(u+L)p}\sum_{l=1}^{L}\sum_{\alpha\in\theta(a(u+l-1))}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q\\
&\le 2^{-\varsigma a(u+L)p}\sum_{\alpha\in\mathbb{N}_0^n}2^{\alpha sq}\|\Delta_\alpha^w(f;x)|L_p\|^q \ll 2^{-\varsigma a(u+L)p}\|f|\mathrm{B}_{pq}^{sm}\|^q\\
&\le 2^{-\varsigma a(u+L)p} \asymp (2^{au}(au)^{n-1})^{-\varsigma p} \asymp N^{-\varsigma p}.
\end{aligned}\tag{6.31}$$

Substituting the estimates contained in (6.31) and (6.17) into (6.30), we arrive at the inequality

$$\|f - t_N(f)|L_p\| \ll N^{-\varsigma}.$$

This implies the required upper estimate for the class B_{pq}^{sm}:

$$\sigma_N\big(\mathrm{B}_{pq}^{sm},\mathfrak{T}^{(k)},L_p\big) \ll N^{-\varsigma}. \tag{6.32}$$

Now, the upper estimates in case I of Theorem 2.1 are completely proved. ☐

7. UPPER ESTIMATES: CASES II–IV

Just as in Section 6, we will prove the upper estimates in the cases II–IV considered here for the wider classes $\mathrm{H}_{pq}^{\varsigma\epsilon}$ instead of B_{pq}^{sm}.

Theorem 7.1. *The following estimates hold for the best N-term trigonometric approximations*:

II. *Let $1 \le p \le 2 \le r < \infty$, $1 \le q \le \infty$, and $\varsigma > \frac{1}{p}$. Then*

$$\sigma_N\big(\mathrm{H}_{pq}^{\varsigma\epsilon},\mathfrak{T}^{(k)},L_r\big) \asymp N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{\omega-1}N)^{\varsigma+\epsilon-\frac{1}{p}+1-\frac{1}{q}}.$$

III. *Let $2 \le p \le r < \infty$, $1 \le q \le \infty$, and $\varsigma > \frac{1}{2}$. Then*

$$\sigma_N\big(\mathrm{H}_{pq}^{\varsigma\epsilon},\mathfrak{T}^{(k)},L_r\big) \asymp N^{-\varsigma}(\log^{\omega-1}N)^{\varsigma+\epsilon+\frac{1}{2}-\frac{1}{q}}.$$

IV. *Let $1 \le p \le \infty$, $1 \le q \le \infty$, and $\varsigma > \frac{1}{p_*}$. Then*

$$\sigma_N\big(\mathrm{H}_{pq}^{\varsigma\epsilon},\mathfrak{T}^{(k)},L_\infty\big) \ll N^{-\varsigma+\frac{1}{p_*}-\frac{1}{2}}(\log^{\omega-1}N)^{\varsigma+\epsilon-\frac{1}{p_*}+1-\frac{1}{q}}(\log N)^{\frac{1}{2}}.$$

Consider another auxiliary class of functions ($\varsigma > 0$, $\epsilon \in \mathbb{R}$)

$$\mathrm{W}_A^{\varsigma\epsilon} := \{ f \in L_1 \mid 2^{\varsigma al}(a\bar{l})^{-(n-1)\epsilon} \|f_l|A\| \le 1,\ l \in \mathbb{N}_0 \}$$

(the sequence (f_l) is defined for $f \in L_1$ in Section 5, see (5.1)).

The following statement is valid.

Lemma 7.1. *Let $2 \le r < \infty$, $\varsigma > 0$, $\epsilon \in \mathbb{R}$, and $\varkappa \in (0, \varsigma)$. Then there exist constructive methods $A_N(\cdot, r, \varkappa)$ and $A_N(\cdot, \infty, \varkappa)$ of N-term trigonometric approximation that are based on the algorithm* IA(ϵ) *and provide the following estimates for $f \in \mathrm{W}_A^{\varsigma\epsilon}$:*

$$\|f - A_N(f, r, \varkappa)|L_r\| \ll N^{-\varsigma-\frac{1}{2}}(\log^{n-1} N)^{\varsigma+\epsilon},$$

$$\|f - A_N(f, \infty, \varkappa)|L_\infty\| \ll N^{-\varsigma-\frac{1}{2}}(\log^{n-1} N)^{\varsigma+\epsilon}(\log N)^{\frac{1}{2}}.$$

Proof. Clearly, it suffices to derive the required estimates for numbers $N \in \mathbb{N}$ of the form $N \asymp 2^{au}(au)^{n-1}$, $u \in \mathbb{N}$. Let $f \in \mathrm{W}_a^{\varsigma\epsilon}$. By Theorem D and the definition of $\mathrm{W}_A^{\varsigma\epsilon}$, for all $l \in \mathbb{N}_0$ and any $N(l) \in \mathbb{N}_0$ we have (see also (4.3))

$$\|f_l - G^r_{N(l)}(f_l)|L_r\| \ll (\overline{N}(l))^{-\frac{1}{2}}\|f_l|A\| \le (\overline{N}(l))^{-\frac{1}{2}} \cdot 2^{-\varsigma al}(al)^{(n-1)\epsilon}, \tag{7.1}$$

$$\|f_l - G^\infty_{N(l)}(f_l)|L_\infty\| \ll (\overline{N}(l))^{-\frac{1}{2}}(aln)^{\frac{1}{2}}\|f_l|A\| \le (\overline{N}(l))^{-\frac{1}{2}} \cdot 2^{-\varsigma al}(al)^{(n-1)(\epsilon+\frac{1}{2})}. \tag{7.2}$$

Now, we fix $\varkappa \in (0, \varsigma)$ and define the numbers (depending on u)

$$N(l) := \lfloor 2^{a(u-\varkappa(l-u))}(al)^{n-1} \rfloor, \qquad l = u+1, u+2, \dots. \tag{7.3}$$

Consider the polynomial

$$A_N(f, r, \varkappa; x) := S^{wm}_{au}(f; x) + \sum_{l=u+1}^{\infty} G^r_{N(l)}(f_l; x).$$

By construction, the number N of harmonics of such a polynomial can be estimated (see (5.6) and (7.2)) as

$$N \ll 2^{au}(au)^{n-1} + 2^{au}\sum_{l=u+1}^{\infty} 2^{\mu a(l-u)}(al)^{n-1} \asymp 2^{au}(au)^{n-1}.$$

Let us estimate the error of approximation of the function f by the polynomial $A_N(f, r, \varkappa)$ in the metric of L_r. In view of (7.1), for $2 \le r < \infty$ we have

$$\begin{aligned}
\|f - A_N(f, r, \varkappa)|L_r\| &\le \sum_{l=u+1}^{\infty} \|f_l - G^r_{N(l)}(f_l)|L_r\| \\
&\ll \sum_{l=u+1}^{\infty} (\overline{N}(l))^{-\frac{1}{2}} \cdot 2^{-\varsigma al}(al)^{(n-1)\epsilon} \ll \sum_{l=u+1}^{\infty} 2^{-\frac{1}{2}a(u-\varkappa(l-u))}(al)^{-\frac{1}{2}(n-1)} \cdot 2^{-\varsigma al}(al)^{(n-1)\epsilon} \\
&= 2^{-\frac{1}{2}au(1+\varkappa)} \sum_{l=u+1}^{\infty} 2^{-(\varsigma-\frac{1}{2}\varkappa)al}(al)^{(n-1)(\epsilon-\frac{1}{2})} \ll 2^{-\frac{1}{2}au(1+\varkappa)} \cdot 2^{-(\varsigma-\frac{1}{2}\varkappa)au}(au)^{(n-1)(\epsilon-\frac{1}{2})} \\
&= 2^{-(\varsigma+\frac{1}{2})au}(au)^{(n-1)(\epsilon-\frac{1}{2})} \asymp N^{-\varsigma-\frac{1}{2}}(\log^{n-1} N)^{\varsigma+\epsilon}.
\end{aligned}$$

Similarly, using (7.2) instead of (7.1), we obtain the required estimate for the uniform approximation of f by the polynomial $A_N(f, \infty, \varkappa)$.

The lemma is proved. □

Proof of the upper estimates in Theorem 7.1. II, IV ($p \le 2$). First, consider the case of $1 \le p \le 2 \le r < \infty$. Let $f \in \mathrm{H}^{\varsigma\epsilon}_{pq}$. Then, successively applying inequality (4.12) (or (4.13)) and Hölder's inequality and taking into account the definition of the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$, we have

$$\|f_l|A\| \le \sum_{\alpha\in\theta(a(l-1))} \|\Delta^w_\alpha(f;x)|A\| \ll \sum_{\alpha\in\theta(a(l-1))} 2^{\alpha m\frac{1}{p}}\|\Delta^w_\alpha(f;x)|L_p\|$$

$$\ll 2^{al\frac{1}{p}}(al)^{(n-1)(1-\frac{1}{q})}\cdot 2^{-al\varsigma}(al)^{(n-1)\epsilon} = 2^{-al(\varsigma-\frac{1}{p})}(al)^{(n-1)(\epsilon+1-\frac{1}{q})}.$$

Hence, by Lemma 7.1 (with $\varsigma-\frac{1}{p}$ and $\epsilon+1-\frac{1}{q}$ instead of ς and ϵ), we obtain

$$\|f-A_N(f,r,\varkappa)|L_r\| \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{n-1}N)^{\varsigma+\epsilon-\frac{1}{p}+1-\frac{1}{q}},$$

$$\|f-A_N(f,\infty,\varkappa)|L_\infty\| \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{n-1}N)^{\varsigma+\epsilon-\frac{1}{p}+1-\frac{1}{q}}(\log N)^{\frac{1}{2}}.$$

Then it follows that

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_r\big) \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{n-1}N)^{\varsigma+\epsilon-\frac{1}{p}+1-\frac{1}{q}},$$

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_\infty\big) \ll N^{-\varsigma+\frac{1}{p}-\frac{1}{2}}(\log^{n-1}N)^{\varsigma+\epsilon-\frac{1}{p}+1-\frac{1}{q}}(\log N)^{\frac{1}{2}}.$$

III, IV ($p > 2$). Now, let $2 < p \le r \le \infty$. Then the norm inequality $\|\cdot|L_2\| \le \|\cdot|L_p\|$ implies the elementary embedding $\mathrm{H}^{\varsigma\epsilon}_{pq} \hookrightarrow \mathrm{H}^{\varsigma\epsilon}_{2\,q}$, and in view of what has already been proved above we obtain (for $r < \infty$)

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_r\big) \ll N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\epsilon+\frac{1}{2}-\frac{1}{q}},$$

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq},\mathfrak{T}^{(k)},L_\infty\big) \ll N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\epsilon+\frac{1}{2}-\frac{1}{q}}(\log N)^{\frac{1}{2}},$$

as required. $\square$

Thus, all upper estimates in Theorem 7.1, and hence all the remaining upper estimates in Theorem 2.1 (in view of (5.3)), are completely proved. It remains to prove the lower estimates (in Theorems 2.1, 6.1, and 7.1).

8. LOWER ESTIMATES: CASE I

First, we make a general remark that follows from Lemma M. Let $1 \le p, q \le \infty$, $s \in \mathbb{R}^n_+$, and $f \in L_p$. If $(2^{\alpha s}\Delta^*_\alpha(f;x)) \in \ell_q(L_p)$, then $f \in B^{sm}_{pq}$ and

$$\|f|B^{sm}_{pq}\| \ll \|(2^{\alpha s}\Delta^*_\alpha(f;x))|\ell_q(L_p)\| \tag{8.1}$$

(see [3] for a proof).

It is also clear that when deriving the lower estimates (in all cases I–IV), we can restrict ourselves to a sequence of numbers $N \in \mathbb{N}$ of the form $N \asymp 2^{au}(au)^{n-1}(\log(au))^b$, $u \in \mathbb{N}$, where $b \in \{0,1\}$.

Proof of the lower estimate in Theorem 2.1, case I. A. Let first $1 < p \le r \le 2$. Consider the function

$$f_1(x;u) = \mathcal{D}_{\Lambda_a(au)}(x) = \sum_{\alpha\in\theta(au)} \mathcal{D}_{\rho(m,\alpha)}(x).$$

Let us estimate the norm of f_1 in the space B^{sm}_{pq}. By Lemma A, (4.1), and (7.1), we have (for $q < \infty$)

$$\|f_1(\cdot,u)|B^{sm}_{pq}\|^q \ll \sum_{\alpha\in\theta(au)} 2^{\alpha sq}\|\mathcal{D}_{\rho(m,\alpha)}|L_p\|^q \ll \sum_{\alpha\in\theta(au)} 2^{\alpha m\varsigma q}\cdot 2^{\alpha m(1-\frac{1}{p})q} \ll 2^{au(\varsigma+1-\frac{1}{p})q}(au)^{n-1};$$

similarly we obtain

$$\|f_1(\cdot,u)|B^{sm}_{p\infty}\| \ll 2^{au\left(\varsigma+1-\frac{1}{p}\right)}.$$

Therefore, the function

$$g_1(x) \equiv g_1(x;u) := cf_1(x;u)\cdot 2^{-au\left(\varsigma+1-\frac{1}{p}\right)}(au)^{(1-n)\frac{1}{q}}$$

belongs to the class B^{sm}_{pq} (with a constant independent of u), $1\le q\le\infty$.

Now, take an arbitrary spectrum $\Lambda\in\mathbb{Z}^k$, $\#\Lambda=N$, where $N\in\mathbb{N}$ is chosen in such a way that $4N\le\#\Lambda_a(au)$ and $N\asymp 2^{au}(au)^{n-1}$.

Denote by $\mathfrak{A}$ the set of $\alpha\in\theta(au)$ for which

$$\#(\Lambda\cap\rho(m,\alpha)) \le \frac{1}{2}\#\rho(m,\alpha).$$

Due to the choice of N, we have $\#\mathfrak{A}\ge\frac{1}{2}\#\theta(au)$.

Then, by Lemma T2, we obtain the following lower estimate for any polynomial $t\in\mathrm{T}(\Lambda)$:

$$\begin{aligned}\|g_1-t|L_r\|^r &\gg \sum_{\alpha\in\theta(au)} 2^{\alpha m\left(\frac{1}{2}-\frac{1}{r}\right)r}\|\Delta^*_\alpha(g_1-t,x)|L_2\|^r\\ &\gg 2^{au\left(\frac{1}{2}-\frac{1}{r}\right)r}\cdot 2^{-au\left(\varsigma+1-\frac{1}{r}\right)r}(au)^{-(n-1)\frac{r}{q}}\sum_{\alpha\in\mathfrak{A}}\left[\sum_{\xi\in\rho(m,\alpha)\setminus\Lambda}1\right]^{\frac{r}{2}}\\ &\gg 2^{au\left(\frac{1}{2}-\frac{1}{r}\right)r}\cdot 2^{-au\left(\varsigma+1-\frac{1}{r}\right)r}(au)^{-(n-1)\frac{r}{q}}\cdot 2^{au\frac{r}{2}}(au)^{n-1}\\ &= \left(2^{-au\left(\varsigma-\frac{1}{p}+\frac{1}{r}\right)}(au)^{(n-1)\left(\frac{1}{r}-\frac{1}{q}\right)}\right)^r,\end{aligned}$$

i.e.,

$$\|g_1-t|L_r\| \gg N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1}N)^{\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q}}.$$

Therefore,

$$\sigma_N\big(\mathrm{B}^{sm}_{pq},\mathfrak{T}^{(k)},L_r\big) \gg N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1}N)^{\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q}}.$$

Thus, the required lower estimate is proved provided that $\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q}\ge 0$.

If $\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q}<0$, then the desired lower estimate follows from the analysis of the isotropic case $n=1$ ($\Rightarrow m=k$, $s\in\mathbb{R}^1_+$, $\varsigma=\frac{s}{k}$):

$$\sigma_N\big(\mathrm{B}^{sk}_{pq},\mathfrak{T}^{(k)},L_r\big) \gg N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}. \tag{8.2}$$

Recall that estimate (8.2) was established in [9] (see estimate (2.2)). Estimate (8.2) can also be proved by means of the lower estimate for the best N-term trigonometric approximations of appropriately chosen cubic smooth means

$$\sum_{j=1}^{J}\Delta^{\eta k}_j\big(\widetilde{G}_{s+k\left(1-\frac{1}{p}\right)};x\big)$$

of the periodized Bessel–Macdonald kernel.

B. Now, let $1=p<r\le 2$. Applying the periodized Bessel–Macdonald kernels, we construct a function in the class B^{sm}_{1q} that is poorly approximated by N-term trigonometric polynomials.

Since $\widetilde{G}_\delta \in \mathrm{B}^\delta_{1\infty}(\mathbb{T}^k)$ (see Section 3), for the function

$$\widetilde{G}^{(m)}_s(x) := \prod_{\nu\in\mathrm{z}_n} \widetilde{G}_{s_\nu}(x^\nu),$$

in view of the easily verifiable equality

$$\Delta^{\eta m}_\alpha(\widetilde{G}^{(m)}_s, x) = \prod_{\nu\in\mathrm{z}_n} \Delta^{\eta m_\nu}_{\alpha_\nu}(\widetilde{G}_{s_\nu}, x^\nu), \qquad \alpha\in\mathbb{N}^n_0,$$

it follows from (3.3) that

$$2^{\alpha s}\|\Delta^{\eta m}_\alpha(\widetilde{G}^{(m)}_s, x)|L_1\| \ll 1, \qquad \alpha\in\mathbb{N}^n_0, \tag{8.3}$$

i.e., $\widetilde{G}^{(m)}_s \in \mathrm{B}^{sm}_{1\infty}$. Taking into account the remarks on the composition of the operators Δ^η_α and $\Delta^\eta_{\alpha'}$ (see Section 4) and Lemma M, one can easily show that the function

$$f_2(x) = f_2(x;\widetilde{G}^{(m)}_s;u) = \sum_{\alpha\in\theta(au)} \Delta^{\eta m}_\alpha(\widetilde{G}^{(m)}_s;x)$$

satisfies the following norm estimate in the space B^{sm}_{1q} $(1\le q<\infty)$:

$$\|f_2|B^{sm}_{1q}\|^q = \sum_{\alpha'\in\mathbb{N}^n_0} 2^{\alpha' sq}\left\|\sum_{\alpha\in\theta(au)} \Delta^\eta_{\alpha'}\circ\Delta^\eta_\alpha(\widetilde{G}^{(m)}_s;x)\middle|L_1\right\|^q \ll \sum_{\alpha\in\theta(au)} 2^{\alpha sq}\|\Delta^\eta_\alpha(\widetilde{G}^{(m)}_s;x)|L_1\|^q$$
$$\ll \#\theta(au) \asymp (au)^{n-1};$$

its norm in $B^{sm}_{1\infty}$ is estimated in a similar way: $\|f_2|B^{sm}_{1\infty}\|\ll 1$. Thus, the function

$$g_2(x) = g_2(x;\widetilde{G}^{(m)}_s;u) = c(au)^{-(n-1)\frac{1}{q}} f_2(x;\widetilde{G}^{(m)}_s;u), \qquad 1\le q\le\infty,$$

belongs to the class B^{sm}_{1q} (with a constant independent of u).

Now, arguing as in part A and retaining the notation adopted there, we obtain the following chain of inequalities for any polynomial t consisting at most N harmonics:

$$\|g_2 - t|L_r\|^r \gg \sum_{\alpha\in\theta(au)} 2^{\alpha m(\frac12-\frac1r)r}\|\Delta_\alpha(g_2-t,x)|L_2\|^r$$
$$\gg 2^{aur(\frac12-\frac1r)}(au)^{-(n-1)\frac{r}{q}}\sum_{\alpha\in\mathfrak{A}}\left[\sum_{\xi\in\rho(m,\alpha)\colon\,\widehat{t}(\xi)=0}\left(\sum_{\alpha'\in\mathbb{Z}^k}\widehat{\eta}_{\alpha'}(\xi)\right)^2\prod_{\nu\in\mathrm{z}_n}(1+\xi^\nu\xi^\nu)^{-s_\nu}\right]^{\frac r2}$$
$$\gg 2^{aur(\frac12-\frac1r)}(au)^{-(n-1)\frac{r}{q}}\sum_{\alpha\in\mathfrak{A}} 2^{-\alpha m\varsigma r}\cdot 2^{\alpha m\frac12 r}$$
$$\asymp 2^{aur(\frac12-\frac1r)}(au)^{-(n-1)\frac{r}{q}}\cdot 2^{-aur\varsigma}\cdot 2^{au\frac{r}{2}}(au)^{n-1} = 2^{-aur(\varsigma-1+\frac1r)}(au)^{(n-1)(\frac1r-\frac1q)r};$$

therefore,

$$\sigma_N\big(g_2,\mathfrak{T}^{(k)},L_r\big) \gg N^{-\varsigma+1-\frac1r}(\log^{n-1}N)^{\varsigma-1+\frac2r-\frac1q}.$$

This implies the required lower estimate for the class B^{sm}_{1q}:

$$\sigma_N\big(\mathrm{B}^{sm}_{1q},\mathfrak{T}^{(k)},L_r\big) \gg N^{-\varsigma+1-\frac1r}(\log^{n-1}N)^{\varsigma-1+\frac2r-\frac1q},$$

again provided that $\varsigma-1+\frac2r-\frac1q\ge 0$. In the case of $\varsigma-1+\frac2r-\frac1q<0$, the necessary lower estimate follows from the above-mentioned result (2.2) from [9] for the isotropic class B^{sk}_{1q}. $\square$

9. LOWER ESTIMATES: CASES II AND III

To derive the lower estimates in case III, we apply Nikolskii's duality relation (3.3). Namely, from the definition of the best N-term trigonometric approximation of a function $f \in L_r$ $(1 < r < \infty)$, by virtue of relation (3.3) we deduce the equality

$$\sigma_N\big(f, \mathfrak{T}^{(k)}, L_r\big) = \inf_{\Lambda \subset \mathbb{Z}^k,\ \#\Lambda = N} \sup\big\{\langle f, g\rangle \colon\ \|g|L_{r'}\| = 1,\ \widehat{g}(\xi) = 0,\ \xi \in \Lambda\big\}.$$

Therefore, to obtain the lower estimate

$$\sigma_N\big(\mathrm{F}, \mathfrak{T}^{(k)}, L_r\big) \gg a_N,$$

it suffices to show that for every N there exists a function f_0 in the class F and for an arbitrary spectrum $\Lambda \subset \mathbb{Z}^k$, $\#\Lambda = N$, there exists a function $\varphi_\Lambda(x) = \varphi_\Lambda(x; f_0)$ with $\|\varphi_\Lambda|L_{r'}\| = 1$ and $\widehat{\varphi}_\Lambda(\xi) = 0$, $\xi \in \Lambda$, such that

$$\langle f_0, \varphi_\Lambda\rangle \gg a_N \tag{9.1}$$

with a constant independent of N.

Proof of the lower estimates in Theorem 2.1, cases II and III. A. Let first $p > 2$. In this case, to construct a function f_0 from the class B^{sm}_{pq}, we need the well-known Rudin–Shapiro trigonometric polynomials $\mathcal{R}_j(z)$, $j \in \mathbb{N}$ (see, e.g., [12, Ch. 4]). For every $j \in \mathbb{N}$, this polynomial has the form

$$\mathcal{R}_j(z) = \sum_{\zeta = 2^{j-1}}^{2^j - 1} \widehat{\mathcal{R}}(\zeta) e^{2\pi i \zeta z}$$

with coefficients $\widehat{\mathcal{R}}(\zeta) \in \{-1, 1\}$, $\zeta = 2^{j-1}, \dots, 2^j - 1$, and its uniform norm is estimated as

$$\|\mathcal{R}_j|L_\infty\| \le 2^{\frac{1}{2}(j+1)}.$$

Consider the function

$$f_3(x) \equiv f_3(x; u) := \sum_{\alpha \in \theta(au)} \mathcal{R}_\alpha(x) := \sum_{\alpha \in \theta(au)} \prod_{\nu \in \mathrm{z}_n} \prod_{\kappa \in \mathrm{k}_\nu} \mathcal{R}_{\alpha_\nu}(x_\kappa).$$

It is clear that

$$\Delta^*_\alpha(f_3, x) = \mathcal{R}_\alpha(x) \qquad \text{if} \quad \alpha \in \theta(au),$$

$$\Delta^*_\alpha(f_3, x) \equiv 0 \qquad \text{if} \quad \alpha \in \mathbb{N}_0^n \setminus \theta(au).$$

Let us estimate the norm of the function f_3 in the space $B^{sm}_{\infty q}$. According to (8.1) and the properties of the Rudin–Shapiro polynomials, we obtain $(q < \infty)$

$$\|f_3|B^{sm}_{\infty q}\|^q \ll \sum_{\alpha \in \theta(au)} 2^{\alpha s q} \|\mathcal{R}_\alpha|L_\infty\|^q \ll \sum_{\alpha \in \theta(au)} 2^{\alpha s q} \cdot 2^{\alpha m \frac{1}{2} q} \ll 2^{au\left(\varsigma + \frac{1}{2}\right)q} (au)^{n-1};$$

similarly,

$$\|f_3|B^{sm}_{\infty\infty}\| \ll 2^{au\left(\varsigma + \frac{1}{2}\right)}.$$

Therefore, the function

$$g_3(x) \equiv g_3(x; u) = c \cdot 2^{-au\left(\varsigma + \frac{1}{2}\right)} (au)^{(1-n)\frac{1}{q}} f_3(x; u)$$

belongs to the class $\mathrm{B}^{sm}_{\infty q}$ (with a constant independent of u).

One can see from the properties of the Rudin–Shapiro polynomials and the construction of the polynomial $f_3(x;u)$ that the Fourier coefficients $\widehat{f}_3(\xi,u)$ of the latter take values $+1$ or -1 for $\xi\in\rho(m,\alpha)\cap\mathbb{N}^k =: \rho_+(m,\alpha)$, $\alpha\in\theta(au)$, and 0 for all the other ξ.

Again, take an arbitrary spectrum $\Lambda\subset\mathbb{Z}^k$, $\#\Lambda=N$, where N depends on u and is chosen so that $2^{k+2}N\le\#\Lambda_a(au)$ and $N\asymp 2^{au}(au)^{n-1}$. Denote by $\mathfrak{A}$ the set of those $\alpha\in\theta(au)$ for which

$$\#(\Lambda\cap\rho_+(m,\alpha))\le\frac{1}{2}\rho_+(m,\alpha).$$

In view of the choice of N, we have $\#\mathfrak{A}\ge\frac{1}{2}\theta(au)$.

Now, consider the function

$$\psi_\Lambda(x):=\psi_\Lambda(x;u):=f_3(x)-\sum_{\xi\in\Lambda}\widehat{f}_3(\xi)e^{2\pi i\xi x}.$$

It is clear that

$$\widehat{\psi}_\Lambda(\xi)=0\qquad\text{for all}\quad\xi\in\Lambda.$$

By the Parseval identity, we have

$$\|\psi_\Lambda|L_2\|^2=\sum_{\xi\in\mathbb{Z}^k}|\widehat{\psi}_\Lambda(\xi)|^2=\sum_{\alpha\in\theta(au)}\ \sum_{\xi\in\rho_+(m,\alpha)}|\widehat{\psi}_\Lambda(\xi)|^2.$$

Hence, on the one hand, we obtain the upper estimate

$$\|\psi_\Lambda|L_2\|^2\le\sum_{\alpha\in\theta(au)}\ \sum_{\xi\in\rho_+(m,\alpha)}1\asymp a^{au}(au)^{n-1}$$

and, on the other hand, the lower estimate

$$\begin{aligned}\|\psi_\Lambda|L_2\|^2&\ge\sum_{\alpha\in\mathfrak{A}}\ \sum_{\xi\in\rho_+(m,\alpha)}|\widehat{\psi}_\Lambda(\xi)|^2=\sum_{\alpha\in\mathfrak{A}}\ \sum_{\xi\in\rho_+(m,\alpha)\setminus\Lambda}1\ge\frac{1}{2}\sum_{\alpha\in\mathfrak{A}}\rho_+(m,\alpha)=\frac{1}{2}\sum_{\alpha\in\mathfrak{A}}2^{\alpha m}\\&\asymp 2^{au}\#\mathfrak{A}\asymp 2^{au}(au)^{n-1}.\end{aligned}$$

Therefore,

$$\|\psi_\Lambda|L_2\|\asymp 2^{\frac{1}{2}au}(au)^{(n-1)\frac{1}{2}}.$$

Now, we introduce the function

$$\varphi_\Lambda(x):=\frac{1}{\|\psi_\Lambda|L_2\|}\psi_\Lambda(x).$$

It follows from the above constructions and estimates that

$$\begin{aligned}\langle g_3,\varphi_\Lambda\rangle&\gg 2^{-\frac{1}{2}au}(au)^{-(n-1)\frac{1}{2}}\cdot 2^{-au(\varsigma+\frac{1}{2})}(au)^{-(n-1)\frac{1}{q}}\langle f_3,\psi_\Lambda\rangle\\&=2^{-(\varsigma+1)au}(au)^{-(n-1)(\frac{1}{2}+\frac{1}{q})}\|\psi_\Lambda|L_2\|^2\asymp 2^{-\varsigma au}(au)^{(n-1)(\frac{1}{2}-\frac{1}{q})}\asymp N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\frac{1}{2}-\frac{1}{q}}.\end{aligned}$$

Hence, in view of the arbitrariness of the spectrum Λ and relation (9.1), we obtain

$$\sigma_N\bigl(\mathrm{B}^{sm}_{\infty q},\mathfrak{T}^{(k)},L_2\bigr)\gg N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\frac{1}{2}-\frac{1}{q}}.\tag{9.2}$$

Using the elementary embedding $B^{sm}_{\infty q}\hookrightarrow B^{sm}_{pq}$ and the inequality $\|\cdot|L_2\|\le\|\cdot|L_r\|$, we then finally derive the following estimate for the class B^{sm}_{pq} in case III:

$$\sigma_N\bigl(\mathrm{B}^{sm}_{pq},\mathfrak{T}^{(k)},L_r\bigr)\gg N^{-\varsigma}(\log^{n-1}N)^{\varsigma+\frac{1}{2}-\frac{1}{q}}.$$

B. Let, finally, $1 \le p \le 2$. Then, by virtue of the obvious inequality

$$\sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \ge \sigma_N\big(\mathrm{B}^{sm}_{pq}, \mathfrak{T}^{(k)}, L_2\big),$$

the required lower estimate follows from the already analyzed case I with $r = 2$.

Thus, all the lower estimates in Theorem 2.1 are completely proved. □

Now, let us discuss the lower estimates for the best N-term trigonometric approximations of the classes $\mathrm{H}^{\varsigma\epsilon}_{pq}$.

Proof of the lower estimates in Theorem 6.1 for $\varsigma \ne \frac{1}{p} - \frac{2}{r} + \frac{1}{q}$ and in Theorem 7.1. Consider the functions

$$h_j(x) := h_j(x; u) := (au)^{(n-1)\epsilon} g_j(x; u), \qquad j = 1, 2, 3.$$

It is clear that in the situations analyzed in parts A and B of the proof of the lower estimates in case I of Theorem 2.1 (Section 8) and in part A of the same proof in cases II and III (above in the current section), the previous considerations imply the estimates

$$\sigma_N\big(h_1, \mathfrak{T}^{(k)}, L_r\big) \gg N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1} N)^{\epsilon+(\varsigma-\frac{1}{p}+\frac{2}{r}-\frac{1}{q})_+},$$

$$\sigma_N\big(h_2, \mathfrak{T}^{(k)}, L_r\big) \gg N^{-\varsigma+1-\frac{1}{r}}(\log^{n-1} N)^{\epsilon+(\varsigma-1+\frac{2}{r}-\frac{1}{q})_+},$$

$$\sigma_N\big(h_3, \mathfrak{T}^{(k)}, L_r\big) \gg N^{-\varsigma}(\log^{n-1} N)^{\epsilon+\varsigma+\frac{1}{2}-\frac{1}{q}}.$$

Next, by Theorem A and the remark on the compositions $\Delta^w_\alpha \circ \Delta^*_{\alpha'}$ and $\Delta^w_\alpha \circ \Delta^\eta_{\alpha'}$ (see Section 4), the inclusion $g_j \in \mathrm{B}^{sm}_{pq}$ readily implies that $h_j \in \mathrm{H}^{\varsigma\epsilon}_{pq}$, $j = 1, 2, 3$.

Thus, it follows that in all the cases considered in Theorems 6.1 and 7.1 except for the case when $\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q} = 0$ in Theorem 6.1, the upper estimates established there are order sharp.

In particular, Theorem 7.1 is completely proved. □

It remains to consider the lower estimates for the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$ in the case of $1 \le p \le r \le 2$, $r > 1$, and $\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q} = 0$.

Proof of the lower estimate in Theorem 6.1 for $\varsigma = \frac{1}{p} - \frac{2}{r} + \frac{1}{q}$. Depending on $u \in \mathbb{N}$, we choose a number $L \in \mathbb{N}$ satisfying (5.8) and then, for every $l \in \mathrm{z}_L$, take an arbitrary set $\theta[l] \subset \theta(a(u + l - 1))$ with

$$\#\theta[l] := \big\lfloor 2^{-al}\#\theta(a(u + l - 1))\big\rfloor + 1;$$

it is clear that $\#\theta[l] \asymp 2^{a(L-l)}$, $l \in \mathrm{z}_L$.

Let first $1 < p \le r \le 2$. Consider the function f_4 defined by the formula

$$f_4(x) := f_4(x; u; L) := \sum_{l=1}^{L} \sum_{\alpha\in\theta[l]} \mathcal{D}_{\rho(m,\alpha)}(x).$$

Let us estimate the norm of f_4 in the space $H^{\varsigma\epsilon}_{pq}$. By Theorem A and relation (8.3), taking into account the remark on the compositions $\Delta^w_\alpha \circ \Delta^*_{\alpha'}$ (Section 4) and (4.1), we obtain ($q < \infty$)

$$\|f_4|H^{\varsigma\epsilon}_{pq}\|^q = \sup_{l\in\mathbb{N}_0^n}\bigg\{\big(2^{\varsigma a l}(a\bar{l})^{(1-n)\epsilon}\big)^q \sum_{\alpha\in\theta(a(l-1))} \|\Delta^*_\alpha(f_4; x)|L_p\|^q\bigg\}$$

$$\ll \max_{l\in\mathrm{z}_L}\bigg\{\big(2^{\varsigma a(u+l)}(a(u + l))^{(1-n)\epsilon}\big)^q \sum_{\alpha\in\theta[l]} \|\mathcal{D}_{\rho(m,\alpha)}|L_p\|^q\bigg\}$$

$$\ll \max_{l\in z_L}\left\{\left(2^{\varsigma a(u+l)}(a(u+l))^{(1-n)\epsilon}\right)^q \sum_{\alpha\in\theta[l]} 2^{\alpha m\left(1-\frac{1}{p}\right)q}\right\}$$

$$\asymp \max_{l\in z_L}\left\{\left(2^{\varsigma a(u+l)}(a(u+l))^{(1-n)\epsilon}\right)^q \cdot 2^{a(u+l)\left(1-\frac{1}{p}\right)q}\#\theta[l]\right\}$$

$$\asymp \max_{l\in z_L}\left\{\left(2^{\left(\varsigma+1-\frac{1}{p}\right)a(u+l)}(a(u+l))^{(1-n)\epsilon}\right)^q \cdot 2^{-al}(a(u+l))^{n-1}\right\}$$

$$\asymp 2^{au}\max_{l\in z_L}\left\{2^{\left(\varsigma+1-\frac{1}{p}-\frac{1}{q}\right)a(u+l)q}(a(u+l))^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}\right\} =: \mathcal{K}(r).$$

If $r=2$, then $\varsigma+1-\frac{1}{p}-\frac{1}{q}=0$; therefore, in view of the choice of L, we obtain

$$\mathcal{K}(2) = 2^{au}\max\left\{(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}, (a(u+L))^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}\right\} \asymp 2^{au}(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}.$$

If $1<r<2$, then $\varsigma+1-\frac{1}{p}-\frac{1}{q}=1-\frac{2}{r}<0$; therefore, we find

$$\mathcal{K}(r) \asymp 2^{au}\cdot 2^{\left(\varsigma+1-\frac{1}{p}-\frac{1}{q}\right)auq}(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)q} = 2^{\left(\varsigma+1-\frac{1}{p}\right)auq}(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}.$$

Hence, for all $1<r\le 2$, we have

$$\mathcal{K}(r) \asymp 2^{\left(\varsigma+1-\frac{1}{p}\right)auq}(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)q}.$$

Thus, we obtain the norm estimate

$$\|f_4|H^{\varsigma\epsilon}_{pq}\| \ll 2^{au\left(\varsigma+1-\frac{1}{p}\right)}(au)^{(n-1)\left(\frac{1}{q}-\epsilon\right)};$$

similarly we get

$$\|f_4|H^{\varsigma\epsilon}_{p\infty}\| \ll 2^{au\left(\varsigma+1-\frac{1}{p}\right)}(au)^{-(n-1)\epsilon}.$$

Therefore, the function

$$h_4(x) := h_4(x;u) := c\cdot 2^{-au\left(\varsigma+1-\frac{1}{p}\right)}(au)^{(n-1)\left(\epsilon-\frac{1}{q}\right)}f_4(x;u)$$

belongs to the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$ $(1\le q\le\infty)$ (with a constant independent of u).

Now, we choose $N\in\mathbb{N}$ from the conditions

$$c_1N \le \sum_{l=1}^{L}\sum_{\alpha\in\theta[l]}\#\rho(m,\alpha), \qquad N\asymp 2^{au}(au)^{n-1}\log(au).$$

For $\Lambda\subset\mathbb{Z}^k$ with $\#\Lambda=N$, we introduce the notation

$$\mathrm{z}^\star = \mathrm{z}^\star(\Lambda) = \left\{l\in z_L \;\middle|\; \#\left(\Lambda\cap\bigcup_{\alpha\in\theta[l]}\rho(m,\alpha)\right)\le c_2\cdot 2^{a(u+l)}\right\},$$

$$\theta^\star[l] = \left\{\alpha\in\theta[l] \;\middle|\; \#(\Lambda\cap\rho(m,\alpha))\le c_3\#\rho(m,\alpha)\right\}, \qquad l\in \mathrm{z}^\star.$$

One can easily verify that the positive constants c_1, c_2, and c_3 can be chosen (independently of u) so that $\#\mathrm{z}^\star\ge c_4L$ and $\theta^\star[l]\ge c_5\theta[l]$ with constants $c_4,c_5\in(0,1)$ that are also independent of u.

It remains to prove the lower estimate for the error of approximation of f_4 by an arbitrary polynomial $t \in \mathrm{T}(\Lambda)$ in L_r. By Lemma T2 and the Parseval identity, taking into account the choice of $\mathrm{z}^\star$ and $\theta^\star[l]$, we obtain

$$\|f_4 - t|L_r\|^r \gg \sum_{l\in \mathrm{z}^\star} \sum_{\alpha\in\theta^\star[l]} 2^{\alpha m(\frac{1}{2}-\frac{1}{r})r} \|\Delta^*_\alpha(f_4 - t, x)|L_2\|^r \gg \sum_{l\in \mathrm{z}^\star} 2^{a(u+l)(\frac{1}{2}-\frac{1}{r})r} \sum_{\alpha\in\theta^\star[l]} 2^{\alpha m\frac{r}{2}}$$

$$\asymp \sum_{l\in \mathrm{z}^\star} 2^{a(u+l)(1-\frac{1}{r})r} \#\theta^\star[l] \asymp 2^{au} \sum_{l\in \mathrm{z}^\star} 2^{a(u+l)(1-\frac{2}{r})r} (a(u+l))^{n-1} =: \mathcal{E}(r). \tag{9.3}$$

If $r = 2$, then the choice of L yields

$$\mathcal{E}(2) = 2^{au} \sum_{l\in \mathrm{z}^\star} (a(u+l))^{n-1} \asymp 2^{au}(au)^{n-1}L \asymp 2^{au}(au)^{n-1}\log(au).$$

If $1 < r < 2$, then

$$\mathcal{E}(r) \gg 2^{au} \cdot 2^{a(u+L)(1-\frac{2}{r})r}(a(u+L))^{n-1}L \asymp 2^{au(1-\frac{1}{r})r}(au)^{n-1}\log(au).$$

This estimate and (9.3), in view of the arbitrariness of the spectrum Λ, imply the following lower estimate for $1 < r \le 2$:

$$\sigma_N\big(h_4, \mathfrak{T}^{(k)}, L_r\big) \gg 2^{-au(\varsigma+1-\frac{1}{p})}(au)^{(n-1)(\epsilon-\frac{1}{q})} \cdot 2^{au(1-\frac{1}{r})}(au)^{(n-1)\frac{1}{r}}(\log(au))^{\frac{1}{r}}$$

$$= 2^{-au(\varsigma-\frac{1}{p}+\frac{1}{r})}(au)^{(n-1)(\epsilon-\frac{1}{q}+\frac{1}{r})}(\log(au))^{\frac{1}{r}} \asymp N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1}N)^{\epsilon}(\log\log N)^{\frac{1}{q}}.$$

Hence, we arrive at the required lower estimate for the class $\mathrm{H}^{\varsigma\epsilon}_{pq}$:

$$\sigma_N\big(\mathrm{H}^{\varsigma\epsilon}_{pq}, \mathfrak{T}^{(k)}, L_r\big) \gg N^{-\varsigma+\frac{1}{p}-\frac{1}{r}}(\log^{n-1}N)^{\epsilon}(\log\log N)^{\frac{1}{q}}. \tag{9.4}$$

In the case of $1 = p < r \le 2$, we consider functions f_5 and h_5 that are defined by analogy with the functions f_4 and h_4 except that the Dirichlet kernels $\mathcal{D}_{\rho(m,\alpha)}(x)$ should be replaced by appropriate "sections" of the Bessel–Macdonald kernels $\Delta^\eta_\alpha(\widetilde{G}_s; x)$. Just as above, combining the arguments related to the estimates for f_2 and g_2 from part B of the proof of the lower estimate in case I of Theorem 2.1 (Section 8) and for f_4, one can easily verify that the function h_5 belongs to the class $\mathrm{H}^{\varsigma\epsilon}_{1q}$ and

$$\sigma_N\big(h_5, \mathfrak{T}^{(k)}, L_r\big) \gg N^{-\varsigma+1-\frac{1}{r}}(\log^{n-1}N)^{\epsilon}(\log\log N)^{\frac{1}{q}},$$

which implies the required estimate (9.4) for $p = 1$.

Thus, Theorem 6.1 is also completely proved. □

In conclusion, we make a few remarks.

Remark 9.1. In case IV, the lower estimate

$$\sigma_N\big(B^{sm}_{pq}, \mathfrak{T}^{(k)}, L_\infty\big) \gg N^{-\varsigma+\frac{1}{p_*}-\frac{1}{2}}(\log^{\omega-1}N)^{\varsigma-\frac{1}{p_*}+1-\frac{1}{q}}$$

holds, which follows from the estimates established in Theorem 2.1: for $1 \le p \le 2$, from case I with $r = 2$; for $2 < p < \infty$, from case III with $r = p$; and for $p = \infty$, from (9.2).

Remark 9.2. The classes $\mathrm{H}^{\varsigma\epsilon}_{pq}$ and $\mathrm{W}^{\varsigma\epsilon}_{A}$ are analogs of the classes $\mathrm{H}^{\varsigma}_{pq}$ and $\mathrm{W}^{\varsigma\epsilon}_{A}$ (which are $\mathrm{H}^{\varsigma 0}_{pq}[\Delta^*]$ and $\mathrm{W}^{\varsigma\epsilon}_{A}[\Delta^*]$ in our notation, respectively) considered by Temlyakov in [30] (see also the classes $W^{a,b}_p$ from [32]); however, instead of the sequence

$$\left(f_l(x) := \sum_{|\alpha|=l} \Delta^*_\alpha(f; x) \;\middle|\; l \in \mathbb{N}_0\right),$$

we use the sequence (5.1); in other words, we use the system $\mathcal{W}^m$ instead of $\mathfrak{T}^{(k)}$, which, in particular, has allowed us to analyze the cases when $p \in \{1, \infty\}$. The classes $\mathrm{H}^{\varsigma\epsilon}_{pq}$, which play an auxiliary role (just as $\mathrm{H}^{\varsigma\epsilon}_{pq}$ in [30]) in the proof of the upper estimates for the classes B^{sm}_{pq}, are nevertheless of some independent interest: in the "limit" case I with $\varsigma - \frac{1}{p} + \frac{2}{r} - \frac{1}{q} = 0$, there is a difference in the decrease rates of (6.1) and (2.1).

Remark 9.3. Here we have not touched upon the results on the classes $\mathrm{MW}^s_p(\mathbb{T}^k)$ with bounded mixed derivative at all. These results are closely related to those mentioned in Remarks 2.1(a)–2.1(c), and we are going to discuss them in a forthcoming paper devoted to the best N-term trigonometric approximations of the Lizorkin–Triebel classes L^{sm}_{pq}, because the scale of Lizorkin–Triebel classes naturally includes the classes MW^s_p.

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