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On Elimination of State Constraints in the Construction of Reachable Sets

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Abstract—The paper is devoted to the problem of approximating reachable sets of a nonlinear control system with state constraints given as a solution set of a nonlinear inequality. A state constraint elimination procedure based on the introduction of an auxiliary constraint-free control system is proposed. The equations of the auxiliary system depend on a small parameter. It is shown that the reachable set of the original system can be approximated in the Hausdorff metric by reachable sets of the auxiliary control system as the small parameter tends to zero. Estimates of the convergence rate are given.

Keywords: reachable set, state constraints, penalty function, approximation, Hausdorff metric.

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1. INTRODUCTION

Reachable sets and their analogs play an important role in the solution of various control problems, estimation under uncertainty, and differential games (see [1–5]). In the present paper, we consider an algorithm for constructing reachable sets of a control system with state constraints. Questions of the approximate construction of reachable sets, specifically, for systems with state constraints, were considered in [5–12] and in many other papers. The algorithm proposed in this paper is based on the elimination of state constraints by replacing the original system with an auxiliary system obtained by a modification of the set of velocities of the original system. We add a correcting term to the right-hand side of the system, which directs the velocity vector inside the set of constraints when its boundary is intersected. The right-hand side of the auxiliary system depends on a small parameter defining the domain of action of the correcting term. The reachable domain of this system, which is constructed without consideration of the state constraints, contains the reachable set of the original system with state constraints. As the small parameter vanishes, the reachable sets converge in the Hausdorff metric to the reachable set of the original system.

A method of eliminating state constraints in the construction of reachable sets for differential inclusions was proposed in papers of Kurzhanskii and Filippova [13, 14], where trajectory tubes of the differential inclusion with a convex state constraint

$$\dot{x} \in F(t, x), \quad x(t) \in Y(t), \quad t \in [t_0, \theta], \tag{1.1}$$

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were approximated by solutions of the family of differential inclusions without state constraints

$$\dot{x} \in F_L(t,x) = F(t,x) + L(x - Y(t)), \quad t \in [t_0, \theta],$$

depending on a matrix parameter L. It was established that, under certain quite soft conditions, the intersection of bundles of trajectories of the family over L gives the bundle of trajectories of the original differential inclusion satisfying the state constraint. The intersection over L allows one to obtain an upper estimate of the reachable set. In the general case, the intersection of bundles of trajectories or reachable sets over a parameter does not provide the closeness of the sets in the Hausdorff metric. In [15,16], a method was proposed for eliminating state constraints by restricting the set of velocities of the system near the boundary of the constraints. In this case, the right-hand side of the approximating system depends on a scalar parameter, its trajectories do not intersect this boundary, and the reachable set of the approximating system approximates the reachable set of the system with state constraints from inside. The proof of this fact required imposing a quite rigid condition on the system and the constraints: for any boundary point of the constraints, there must exist a vector of velocity of the control system at this point directed strictly inside the constraints.

In the present paper, under a similar condition, we propose another procedure for eliminating state constraints. This procedure is based on the introduction of an auxiliary control system without constraints, the right-hand side of which depends on a small parameter. Constructing this system, we do not use the operations of intersection of sets; its reachable set contains the reachable set of the original system with state constraints. Together with results [15, 16], this gives the possibility of obtaining two-sided estimates for reachable sets. In this paper, we prove the convergence of reachable sets of the auxiliary control system in the Hausdorff metric to the reachable set of the original system as the small parameter tends to zero. Estimates for the convergence rate are given.

2. DEFINITIONS AND PROBLEM STATEMENT

Consider the nonlinear control system

$$\dot{x} = f(x,u), \quad t_0 \le t \le \theta, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is a state vector and $u \in \mathbb{R}^r$ is a control parameter satisfying the constraint

$$u(t) \in U, \quad t \in [t_0, \theta]. \tag{2.2}$$

Here, U is a compact set in $\mathbb{R}^r$. For controls, we consider measurable functions $u\colon [t_0,\theta] \to U$, and we denote by $\mathcal{U}$ the set of controls.

We will use the following notation. For $x, y \in \mathbb{R}^n$, (x,y) is the scalar product and $\|x\| = (x,x)^{1/2}$ is the Euclidean norm. Denote by $B_r(\bar{x})$ the ball of radius $r > 0$ centered at the point $\bar{x}$: $B_r(\bar{x}) = \{x \in \mathbb{R}^n\colon \|x - \bar{x}\| \le r\}$. For $S \subset \mathbb{R}^n$, we denote by ∂S, $\mathtt{Int}S$, and $\mathtt{co}S$ the boundary, interior, and convex hull of S, respectively; $\nabla g(x)$ is the gradient of $g(x)$ at the point x. Let $h(A,B)$ be the Hausdorff distance between sets $A, B \subset \mathbb{R}^n$, and let $\mathtt{conv}(\mathbb{R}^n)$ be the family of convex compact sets in $\mathbb{R}^n$. We assume that the right-hand side of (2.1) satisfies the following conditions.

Assumption 1. *The mapping $f(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the conditions*:

(1) *$f(x,u)$ is continuous and locally Lipschitz in x uniformly in $u \in U$*;

(2) *the sublinear growth condition*: *there exists $C > 0$ such that*

$$\|f(x,u)\| \le C(1 + \|x\|), \quad (x,u) \in \mathbb{R}^n \times U;$$

(3) *the set of velocities $F(x) := f(x,U)$ is convex for every x.*

System (2.1) can be represented in the form of the equivalent differential inclusion

$$\dot{x} \in F(x), \quad x(t_0) = x^0, \tag{2.3}$$

where the set-valued mapping $F : \mathbb{R}^n \to \mathtt{conv}(\mathbb{R}^n)$ is locally Lipschitz in the Hausdorff metric. Solutions of (2.3) are absolutely continuous functions $x\colon [t_0, \theta] \to \mathbb{R}^n$ such that $\dot{x}(t) \in F(x(t))$ for almost all t.

The state constraints have the form

$$x(t) \in S, \quad t \in [t_0, \theta], \tag{2.4}$$

where S is a closed set in $\mathbb{R}^n$ containing the vector x^0. In what follows, we assume that

$$S = \{x \in \mathbb{R}^n\colon g(x) \le 0\}, \tag{2.5}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Denote by $x(t, u(\cdot), x^0)$ the solution of system (2.1) with the initial condition $x(t_0) = x^0$. The reachable set (domain) of system (2.1) with state constraint (2.4) at time θ is the set

$$G_0(\theta) = \big\{x \in \mathbb{R}^n\colon \exists u(\cdot) \in \mathcal{U},\ x = x(\theta, u(\cdot), x^0),\ x(t, u(\cdot), x^0) \in S,\ t_0 \le t \le \theta\big\},$$

i.e., the set of all points to which (2.1) can be taken at time θ from the initial state x^0 under constraints (2.2) and (2.4). Denote by $G(\theta)$ the reachable set of (2.1) without state constraints:

$$G(\theta) = \big\{x \in \mathbb{R}^n\colon \exists u(\cdot) \in \mathcal{U},\ x = x(\theta, u(\cdot), x^0)\big\}.$$

Under the conditions of Assumption 1, $G(\theta)$ is compact in $\mathbb{R}^n$, and the trajectories of (2.1) with the initial condition $x(t_0) = x^0$ lie inside some ball $B_R(\bar{x})$, which will be denoted by B_R.

In the present paper, we consider the following problem: construct a control system

$$\dot{x} = f_\varepsilon(x, u), \quad x(t_0) = x^0, \tag{2.6}$$

with the right-hand side depending on a small parameter ε such that:

(1) the mapping $f_\varepsilon(x, u)$ is defined for x from some neighborhood of $S \cap B_R$ and for $u \in U$, is continuous in x and u, and is locally Lipschitz in x uniformly in $u \in U$;

(2) $f_\varepsilon(x, U) = f(x, U)$ for $x \in S \cap B_R$ and $f_\varepsilon(x, U) \subset f(x, U)$;

(3) $G_\varepsilon(\theta) \to G_0(\theta)$ in the Hausdorff metric as $\varepsilon \to 0$, where $G_\varepsilon(\theta)$ is the reachable set of system (2.6) without state constraints.

Thus, the original control system is replaced by a family of control systems without state constraints depending on a parameter ε. The reachable sets of these systems approximate $G_0(\theta)$ as $\varepsilon \to 0$. We call control system (2.6) approximating for system (2.1).

3. APPROXIMATION OF REACHABLE SETS

The further constructions are based on the following condition (see [17–20]).

Assumption 2. *For any $x \in \partial S \cap B_R$,*

$$\min_{u \in U} (\nabla g(x), f(x, u)) < 0. \tag{3.1}$$

Assumption 2 provides the nonemptiness of $G_0(\theta)$. It follows from (3.1) that $\nabla g(x) \neq 0$ for $x \in \partial S \cap B_R$; then,

$$\partial S \cap B_R = \{x \in \mathbb{R}^n \colon g(x) = 0\} \cap B_R.$$

Since the set $\{x \in \mathbb{R}^n \colon g(x) = 0\} \cap B_R$ is compact and the function

$$\eta(x) = \min_{u \in U} (\nabla g(x), f(x, u))$$

is continuous in x, there exists $\sigma > 0$ such that (3.1) holds in the intersection of the σ-neighborhood of the set $\{x \in \mathbb{R}^n \colon g(x) = 0\}$ with B_R. The gradient of $g(x)$ is nonzero at points of this set; hence, there exists $K > 0$ such that

$$d(x) \leq K|g(x)|,$$

where $d(x)$ is the distance from x to the boundary of S (see [16]).

Assertion. *If Assumption* 2 *holds, then there exists* $\sigma > 0$ *such that inequality* (3.1) *holds for all points of the set*

$$S_R^\sigma = \{x \colon 0 \leq g(x) \leq \sigma\} \cap B_R.$$

We will also use the following strengthening of Assumption 2.

Assumption 3. *There exist* $\sigma > 0$ *and a function* $\bar{u} : S_R^\sigma \to U$ *such that* $f(x, \bar{u}(x))$ *satisfies the Lipschitz condition on* S_R^σ *and*

$$(\nabla g(x), f(x, \bar{u}(x))) < 0 \quad \forall x \in S_R^\sigma. \tag{3.2}$$

Under this assumption, we define the right-hand side $f_\varepsilon(x, u)$ of control system (2.6) on the set $\{x \in \mathbb{R}^n \colon g(x) \leq \sigma\} \cap B_R$ as follows. Choose $0 < \varepsilon < \sigma$. Let $h_\varepsilon(\tau) \colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $0 \leq h_\varepsilon(\tau) \leq 1$, $h_\varepsilon(\tau) = 1$ for $\tau < 0$, and $h_\varepsilon(\tau) = 0$ for $\tau > \varepsilon$. Define

$$f_\varepsilon(x, u) = \begin{cases} h_\varepsilon(g(x)) f(x, u) + \big(1 - h_\varepsilon(g(x))\big) f(x, \bar{u}(x)) & \text{for} \quad g(x) > 0, \\ f(x, u) & \text{for} \quad g(x) \leq 0. \end{cases}$$

As $h_\varepsilon(\tau)$, we can use, for example, the linear–quadratic function

$$h_\varepsilon(\tau) = \begin{cases} 1 & \text{for} \quad \tau < 0, \\ 1 - a\tau^2 & \text{for} \quad 0 \leq \tau \leq d\varepsilon, \\ 1 - a(d\varepsilon)^2 - b(\tau - d\varepsilon) & \text{for} \quad d\varepsilon < \tau < (1-d)\varepsilon, \\ a(\tau - \varepsilon)^2 & \text{for} \quad (1-d)\varepsilon \leq \tau \leq \varepsilon, \\ 0 & \text{for} \quad \tau > \varepsilon, \end{cases} \tag{3.3}$$

where $a = 1/(2d(1-d)\varepsilon^2)$, $b = 1/((1-d)\varepsilon)$, and the parameter $0 < d < 1$ is independent of ε.

Theorem 1. *Suppose that the function* $f(x, u)$ *and the constraints of the problem satisfy Assumptions* 1 *and* 3. *Then*:

(1) *for* $0 < \varepsilon < \sigma$, *the mapping* $f_\varepsilon(x, u)$ *is continuous on* $\{x \in \mathbb{R}^n \colon g(x) \leq \sigma\} \cap B_R \times U$ *and Lipschitz in* x *uniformly in* $u \in U$;

(2) *for any* $u(\cdot) \in \mathcal{U}$, *a solution* $x_\varepsilon(t)$ *of system* (2.6) *with the initial condition* $x_\varepsilon(t_0) = x^0$ *can be extended to* $[t_0, \theta]$ *and satisfies the inequality*

$$g(x_\varepsilon(t)) \leq \varepsilon, \quad t \in [t_0, \theta]; \tag{3.4}$$

(3) *$G_0(\theta) \subset G_\varepsilon(\theta)$ for any $0 < \varepsilon < \sigma$, and there exists a constant $L > 0$ such that*

$$h(G_0(\theta), G_\varepsilon(\theta)) \leq L\varepsilon. \tag{3.5}$$

Proof. On the set $S_1 \times U$, where $S_1 = \{x\colon g(x) \leq 0\} \cap B_R$, the function $f_\varepsilon(x,u)$ coincides with $f(x,u)$; hence, it is continuous. For $(x,u) \in S_2 \times U$, where $S_2 = \{x\colon 0 \leq g(x) \leq \sigma\} \cap B_R$, $f_\varepsilon(x,u)$ is continuous as a superposition of continuous functions. It remains to prove the continuity of $f_\varepsilon(x,u)$ at the "gluing" points of these domains, i.e., at x such that $g(x) = 0$. Since these points belong to each of the sets S_1 and S_2, the continuity of $f_\varepsilon(x,u)$ at them is proved by means of obvious arguments. To prove the Lipschitz condition for $f_\varepsilon(x,u)$, note that there exist constants $L_1, L_2 > 0$ independent of u and such that $\forall i = 1,2$

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \leq L_i\|x - y\| \quad \forall x,y \in S_i,\ \forall u \in U.$$

For $x, y \in S_1$, the inequality follows from the fact that the function $f(x,u)$, which coincides with $f_\varepsilon(x,u)$ on $S_1 \times U$, is Lipschitz in x. On $S_2 \times U$, the inequality holds since $f_\varepsilon(x,u)$ is a superposition of functions Lipschitz in x. Naturally, the constant L_2 depends on ε. Let $x \in S_1$ and $y \in S_2$. Let us connect x and y with a line segment. The function g takes values of different signs at the end-points of the segment; hence, there exists a point z on the segment such that $g(z) = 0$. Taking into account that $z \in S_i$, $i = 1,2$, we obtain

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \leq |f_\varepsilon(x,u) - f_\varepsilon(z,u)| + |f_\varepsilon(z,u) - f_\varepsilon(y,u)|$$
$$\leq L_1\|x - z\| + L_2\|y - z\| \leq \max\{L_1, L_2\}(\|x - z\| + \|y - z\|) = \max\{L_1, L_2\}\|x - y\| \quad \forall u \in U.$$

Consider a solution $x_\varepsilon(t)$ of (2.6) corresponding to a control $u(\cdot) \in \mathcal{U}$. Since $f_\varepsilon(x,u)$ is a convex combination of the vectors $f(x,u)$ and $f(x,\bar{u}(x))$ belonging to the convex set $f(x,U)$, we have $\dot{x}_\varepsilon(t) \in f(x_\varepsilon(t), U)$ for a.a. t. Then, by Filippov's lemma (see [21]), there exists a control $u_\varepsilon(\cdot) \in \mathcal{U}$ such that

$$\dot{x}_\varepsilon(t) = f(x_\varepsilon(t), u_\varepsilon(t));$$

i.e., any trajectory of the auxiliary system is a trajectory of (2.1) generated by some control different from $u(\cdot)$. It remains to prove that this trajectory does not leave the set $\{x \in \mathbb{R}^n\colon g(x) \leq \sigma\} \cap B_R$ on which the right-hand side $f_\varepsilon(x,u)$ of system (2.6), is defined. Let γ^* be the maximum among the numbers γ not exceeding θ such that the solution $x_\varepsilon(t)$ is defined on the interval $[t_0, \gamma]$. Let us prove that the inequality $g(x_\varepsilon(t)) \leq \varepsilon$ holds at all points $[t_0, \gamma^*]$. Assume, by contradiction, that $g(x_\varepsilon(\hat{t})) > \varepsilon$ for some $\hat{t} \in [t_0, \gamma^*]$. Define $\delta = (g(x_\varepsilon(\hat{t})) - \varepsilon)/2$; then, $g(x_\varepsilon(\hat{t})) > \varepsilon + \delta$. Let

$$t^* = \min\{t : t \in [t_0, \gamma^*], g(x_\varepsilon(t)) = \varepsilon + \delta\}.$$

Then, $g(x_\varepsilon(t^*)) = \varepsilon + \delta$ and, by the continuity of $g(x_\varepsilon(t))$, there exists $\beta > 0$ such that $g(x_\varepsilon(t)) > \varepsilon$ for $t^* - \beta \leq t \leq t^*$. Consequently, $h_\varepsilon(x_\varepsilon(t)) = 0$ for these values of t, and

$$\frac{d}{dt} g(x_\varepsilon(t)) = (\nabla g(x_\varepsilon(t)), f\bigl(x_\varepsilon(t), \bar{u}(x_\varepsilon(t))\bigr) < 0.$$

Hence, $g(x_\varepsilon(t)) \geq \varepsilon + \delta$ for $t^* - \beta \leq t \leq t^*$, which contradicts the definition of t^*. Thus, $\gamma^* = \theta$ and the inequality $g(x_\varepsilon(t)) \leq \varepsilon$ holds on the interval $[t_0, \theta]$.

To prove the concluding part of the theorem, note that, if $g(x) \leq 0$, we have $f_\varepsilon(x,u) = f(x,u)$ $\forall u \in U$; hence, $G_0(\theta) \subset G_\varepsilon(\theta)$. It follows from the NFT theorems [17–19] about approximation of trajectories of a control system by trajectories satisfying state constraints (neighboring feasible trajectories) that, under Assumption 2, there exists a constant L with the following property. For any trajectory of system (2.1) with the initial condition $x(t_0) = x^0$, there exists a trajectory $\hat{x}(\cdot)$, $\hat{x}(t_0) = x^0$, satisfying the state constraints and such that

$$\|x(\cdot) - \hat{x}(\cdot)\|_{C[t_0,\theta]} \le L \max_{t_0 \le t \le \theta} \max\{g(x(t)), 0\}.$$

Since any trajectory $x_\varepsilon(t)$ of system (2.6) is a trajectory of the original system, we find from the latter inequality that, for any $x_\varepsilon(\theta) \in G_\varepsilon(\theta)$, there exists $\hat{x}(\theta) \in G_0(\theta)$ such that

$$\|x_\varepsilon(\theta) - \hat{x}(\theta)\| \le L \max_{t_0 \le t \le \theta} \max\{g(x_\varepsilon(t)), 0\} \le L\varepsilon.$$

This, in view of the inclusion $G_0(\theta) \subset G_\varepsilon(\theta)$, implies the statement of the theorem.

Remark 1. Estimate (3.5) is uniform for all θ from a bounded set. If, instead of reachable sets at time θ, we consider reachable sets by time θ

$$\bar{G}_0(\theta) = \bigcup_{0 \le \tau \le \theta} G_0(\tau), \ \ \bar{G}_\varepsilon(\theta) = \bigcup_{0 \le \tau \le \theta} G_\varepsilon(\tau),$$

then estimate (3.5) also holds for these sets.

Remark 2. Using variants of the NFT theorems for the nonstationary case (see [17–19]), we can extend the above results to nonstationary systems with state constraints depending on time.

Remark 3. In Theorem 1, the convexity assumption for the set of velocities $f(x, U)$ can be eliminated. Let us briefly describe the scheme of the proof in this case. Along with system (2.1), we consider the control system with a control w

$$\dot{x} = \bar{f}(x, w(t)), \quad x(t_0) = x^0, \quad w(t) \in W, \tag{3.6}$$

which is obtained in the standard way by convexifying the set of velocities of the original system: $\bar{f}(x, W) = \mathtt{co} f(x, U)$. The set of trajectories of system (2.1) is contained in the set of trajectories of system (3.6) and is dense in this set in the uniform metric. Let $g(t_0) < 0$, and let Assumption 2 hold. Using the results of [20], we can show that any trajectory of (3.6) satisfying the state constraint can be arbitrarily exactly approximated in the uniform metric by trajectories of (2.1) also satisfying the state constraint. Based on this fact and applying Theorem 1 to system (3.6), it is easy to obtain an estimate of form (3.5) for system (2.1) without the assumption of convexity of $f(x, U)$.

4. LINEAR IN CONTROL SYSTEMS WITH ELLIPSOIDAL CONSTRAINTS ON THE CONTROL

Consider the linear in control system

$$\dot{x} = f(x, u) = f_1(x) + f_2(x)u, \quad u(t) \in U, \quad x(t_0) = x^0,$$

where $f_1(x)$ and $f_2(x)$ are continuously differentiable mappings, under the assumption that the constraints on the control u are given by a nondegenerate ellipsoid in $\mathbb{R}^r$:

$$U = \{u \in \mathbb{R}^r \colon (u - \hat{u})^\top Q(u - \hat{u}) \le 1\};$$

here, Q is a positive definite symmetric matrix and $\hat{u} \in \mathbb{R}^n$ is the center of the ellipsoid. We will also assume that $\nabla g(x)$ satisfies the Lipschitz condition.

Assumption 2 for this system can be written in the form

$$(\nabla g(x), f_1(x)) + \nabla^\top g(x) f_2(x)\hat{u} + \min_{v\in V} \nabla^\top g(x) f_2(x) v < 0 \tag{4.1}$$

for $x \in S_R^\sigma$. Here, $V = \{v\colon v^\top Q v \le 1\}$ is the ellipsoid centered at zero.

Let

$$a(x) = (\nabla g(x), f_1(x)) + \nabla^\top g(x) f_2(x)\hat{u}, \quad b^\top(x) = \nabla^\top g(x) f_2(x).$$

Then,

$$\min_{v\in V} b^\top(x) v = \min_{(w,w)\le 1} b^\top(x) Q^{-1/2} w = -\|Q^{-1/2} b(x)\| = -\sqrt{b^\top(x) Q^{-1} b(x)},$$

where $Q^{-1/2} = (Q^{-1})^{1/2}$ is the square root of the positive definite matrix Q^{-1}. In view of the introduced notation, condition (4.1) takes the form

$$a(x) + \min_{v\in V} b^\top(x) v = a(x) - \sqrt{b^\top(x) Q^{-1} b(x)} < 0. \tag{4.2}$$

The minimum in (4.1) is attained at the vector

$$v(x) = \frac{Q^{-1} b(x)}{\sqrt{b^\top(x) Q^{-1} b(x)}}. \tag{4.3}$$

Thus, the function $\bar{u}(x) = v(x) + \hat{u}$ provides the inequality $(\nabla g(x), f(x, \bar{u}(x))) < 0$. However, in general, this function is not Lipschitz and may even have discontinuities at points x such that $b(x) = 0$. Let us show that, modifying formula (4.3), we can make the corresponding function $\bar{u}(x)$ satisfy the Lipschitz condition.

Theorem 2. *Let condition* (4.2), *which is equivalent to* (4.3), *hold on the set* S_R^σ. *Then, there exists a Lipschitz function* $\bar{u}(x)$ *such that*

$$(\nabla g(x), f_1(x) + f_2(x)\bar{u}(x)) < 0 \quad \forall x \in S_R^\sigma. \tag{4.4}$$

Proof. As follows from (4.3), $a(x) < 0$ at the points where $b(x) = 0$. Let $w = Q^{1/2} v$. Then, $a(x) + b^\top(x) v = a(x) + b_1^\top(x) w$, where $b_1(x) = Q^{-1/2} b(x)$, and the ellipsoid V becomes the ball $\{w\colon (w,w) \le 1\}$. Consider a nonnegative function $p(x)$ defined on S_R^σ by the equality

$$p(x) = \begin{cases} \dfrac{a(x) + \sqrt{a^2(x) + (b_1^\top(x) b_1(x))^2}}{b_1^\top(x) b_1(x)} & \text{for} \quad b_1(x) \ne 0, \\ 0 & \text{for} \quad b_1(x) = 0. \end{cases}$$

Since $a(x) < 0$ for $b_1(x) = 0$, the function $p(x)$ is continuously differentiable [22]. The Sontag control[2] $\bar{w}(x) = -p(x) b_1(x)$ satisfies the inequality $a(x) + b_1^\top(x)\bar{w}(x) < 0$; hence, for $\bar{u}(x) = Q^{-1/2}\bar{w}(x) + \hat{u}$, we have $(\nabla g(x), f(x, \bar{u}(x))) < 0$. Obviously, $\bar{u}(x)$ is a Lipschitz function; however, its values do not necessarily belong to U. To provide the condition $\bar{u}(x) \in U$, we proceed as follows.

[2]The formula for $\bar{w}$ coincides with the formula for Sontag's regulator stabilizing a control system, if we substitute $g(x)$ instead of the control Lyapunov function into the latter formula.

Let $\pi(w)$ be the operator of metric projection on the unit Euclidean ball in $\mathbb{R}^r$; $\pi(w)$ satisfies the Lipschitz condition with the constant equal to one. Let us modify Sontag's control, setting

$$\bar{w}(x) = \pi(-p(x)b_1(x)) = \begin{cases} -p(x)b_1(x) & \text{for } \|p(x)b_1(x)\| \le 1, \\ \dfrac{-p(x)b_1(x)}{\|p(x)b_1(x)\|} = -\dfrac{b_1(x)}{\|b_1(x)\|} & \text{for } \|p(x)b_1(x)\| > 1. \end{cases}$$

For $\|p(x)b_1(x)\| \le 1$, we have $\bar{w}(x) = -p(x)b_1(x)$ and, consequently, $a(x) + b_1^\top(x)\bar{w}(x) < 0$. For $\|p(x)b_1(x)\| > 1$, by (4.2), we have

$$a(x) + b_1^\top(x)\bar{w}(x) = a(x) - \|b_1(x)\| = a(x) - \sqrt{b^\top(x)Q^{-1}b(x)} < 0.$$

Thus, $\bar{u}(x) = Q^{-1/2}\pi(-p(x)Q^{-1/2}f_2^\top(x)\nabla g(x)) + \hat{u}$ is the required control. The theorem is proved.

5. EXAMPLES

As the first example, consider the linear control system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x(0) = (0,0), \quad |u| \le 1, \quad 0 \le t \le 2, \tag{5.1}$$

with state constraint $|x_2| \le 1$. The function $g(x) = |x_2| - 1$ is nondifferentiable at $x_2 = 0$, which is not an obstacle for the application of the proposed method, since, in fact, the differentiability of $g(x)$ is required only in a neighborhood of the boundary of S, i.e., at x_2 close to 1 and -1.

Let us choose the function $h_\varepsilon(\tau)$ in form (3.3). For the control $\bar{u}(x)$ providing the inequality $g'(x_2)\bar{u}(x) < 0$ in a neighborhood of the lines $|x_2| = 1$, we can take the function

$$\bar{u}_(x) = \begin{cases} -1 & \text{for } x_2 \ge 1, \\ -x_2 & \text{for } -1 < x_2 < 1, \\ 1 & \text{for } x_2 \le -1. \end{cases}$$

The approximating system takes the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = p_\varepsilon(x_2, u), \tag{5.2}$$

where $p_\varepsilon(x_2,u) = h_\varepsilon(|x_2|-1)\,u + (1-h_\varepsilon(|x_2|-1))\,\bar{u}(x)$. Note that the first equation in system (5.1) is unchanged, since it does not contain u.

It is known that controls taking trajectories of a system to the boundary of the reachable set satisfy Pontryagin's maximum principle (see, for example, [23]). Let us write the relations of the maximum principle. The Hamiltonian of the system has the form

$$H(x,\psi) = \psi_1 x_2 + \psi_2\, p_\varepsilon(x_2,u).$$

From the maximum principle, $u(t) = \operatorname{sgn} \psi_2(t)$, where $\psi_2(t)$ is found from the adjoint system

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\partial H}{\partial x_1} = 0, \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x_2} = -\psi_1 - \psi_2\, p'_\varepsilon(x_2,u), \end{aligned}$$

and $p'_\varepsilon(x_2,u)$ is the derivative of p_ε with respect to x_2.

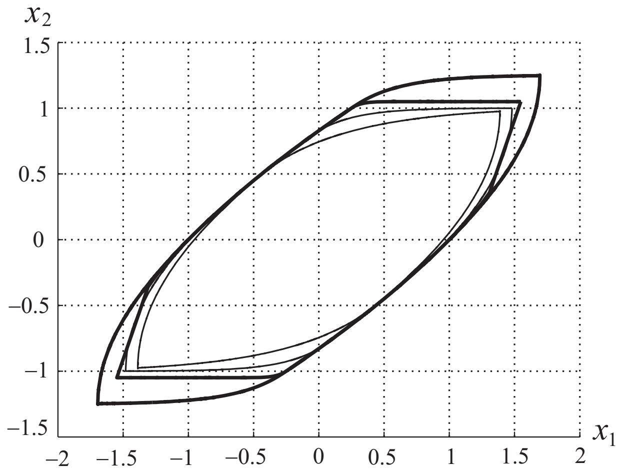

Fig 1. Interior and exterior approximating sets of system (5.5).

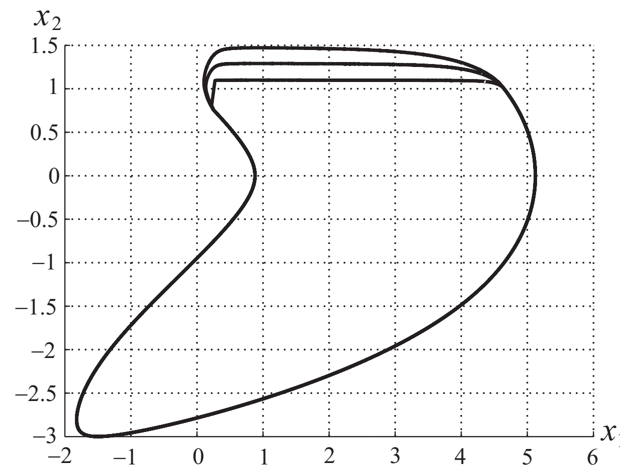

Fig 2. Exterior approximating sets for different values of ε.

Thus, we obtain the fourth-order nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= p_\varepsilon(x_2, \operatorname{sgn}\,\psi_2),\\ \dot{\psi}_1 &= 0,\\ \dot{\psi}_2 &= -\psi_1 - \psi_2\, p'_\varepsilon(x_2, \operatorname{sgn}\,\psi_2).\end{aligned}$$

Integrating this system under the initial condition $x_1(0) = 0,\ x_2(0) = 0,\ \psi_1(0) = \sin\alpha,\ \psi_2(0) = \cos\alpha$, where α runs from 0 to 2π, we obtain a planar family of points $(x_1(2,\alpha), x_2(2,\alpha))$, which contains all boundary points of the reachable set $G_\varepsilon(2)$ of system (5.2). Figure 1 shows the results of a numerical simulation according to the above algorithm. Here, the boundaries of the approximating sets are shown. The bold line depicts the boundaries of $G_\varepsilon(2)$ for different ε, and the thin line depicts the boundaries of the interior approximating sets obtained by the algorithm from [24].

In the second example, the control system has the form

$$\dot{x}_1 = 1 - p\,x_2^2 + u_1, \quad \dot{x}_2 = u_2, \quad x(0) = (0,0), \quad 0 \le t \le 3, \tag{5.3}$$

where $p > 0$ and the constraints are given by the conditions $u_1^2 + u_2^2 \le 1$ and $x_2 \le 1$. For $\bar{u}(x)$, we can take $\bar{u}(x) \equiv (0,-1)\ \forall x$. Then, the auxiliary system can be written in the form

$$\dot{x}_1 = 1 - p\,x_2^2 + h_\varepsilon(x_2 - 1)u_1, \quad \dot{x}_2 = h_\varepsilon(x_2 - 1)(1 + u_2) - 1.$$

The algorithm for constructing the boundaries of the reachable sets is similar to that in the first example. The constructed boundaries of approximations of the reachable sets $G_\varepsilon(3)$ for $p = 0.5$ and different values of ε are shown in Fig. 2; note that the lower parts of the boundaries coincide.

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REFERENCES

1. N. N. Krasovskii, *Game Problems on the Encounter of Motions* (Nauka, Moscow, 1970) [in Russian].
2. A. B. Kurzhanskii, *Control and Observation under Uncertainty* (Nauka, Moscow, 1977) [in Russian].
3. F. L. Chernous'ko, *Estimation of the Phase State of Dynamic Systems* (Nauka, Moscow, 1988) [in Russian].
4. N. N. Krasovski and A. I. Subbotin, *Game-Theoretical Control Problems* (Springer, New York, 1988).
5. A. B. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control* (Birkhäuser, Boston, 1997).
6. A. V. Lotov, "A numerical method for constructing sets of attainability for linear controlled systems with phase constraints," USSR Comp. Math. Math. Phys. **15** (1), 63–74 (1975).
7. A. R. Matviichuk and V. N. Ushakov, "On the construction of resolving controls in control problems with phase constraints," J. Comput. Syst. Sci. Int. **45** (1), 1–16 (2006).
8. A. B. Kurzhanski, I. M. Mitchell, and P. Varaiya, "Optimization techniques for state-constrained control and obstacle problems," J. Optim. Theory Appl. **128** (3), 499–521 (2006).
9. R. Baier, I. A. Chahma, and F. Lempio, "Stability and convergence of Euler's method for state-constrained differential inclusions," SIAM J. Optim. **18** (3), 1004–1026 (2007).
10. N. Bonneuil, "Computing reachable sets as capture-viability kernels in reverse time," Appl. Math. **3**, 1593–1597 (2012).
11. E. K. Kostousova, "External and internal estimation of attainability domains by means of parallelotopes," Vychisl. Tekhnologii **3** (2), 11–20 (1998).
12. M. I. Gusev, "External estimates of the reachability sets of nonlinear controlled systems," Autom. Remote Control **73** (3), 450–461 (2012).
13. A. B. Kurzhanskii and T. F. Filippova, "Description of a set of viable trajectories of a differential inclusion," Dokl. Akad. Nauk SSSR **289** (1), 38–41 (1986).
14. A. B. Kurzhanskii and T. F. Filippova, "Description of the bundle of viable trajectories of a control system," Differents. Uravneniya **23** (8), 1303–1315 (1987).
15. M. I. Gusev, "On the penalty function method in the problem of constructing reachable sets for control systems with state constraints," Trudy Inst. Mat. Mekh. UrO RAN **19** (1), 8186 (2013).
16. M. I. Gusev, "Internal approximations of reachable sets of control systems with state constraints," Proc. Steklov Inst. Math. **287** (Suppl. 1), S77–S92 (2014).
17. F. Forcellini and F. Rampazzo, "On nonconvex differential inclusions whose state is constrained in the closure of an open set," Differential Integral Equations **12** (4), 471–497 (1999).
18. H. Frankowska and R. B. Vinter, "Existence of neighboring feasible trajectories: Applications to dynamic programming for state-constrained optimal control problems," J. Optim. Theory Appl. **104** (1), 21–40 (2000).
19. P. Bettiol, A. Bressan, and R. Vinter, "Trajectories satisfying a state constraint: $W^{(1,1)}$ estimates and counterexamples," SIAM J. Control Optim. **48** (7), 4664–4679 (2010).
20. R. J. Stern, "Characterization of the state constrained minimal time function," SIAM J. Control Optim. **43** (2), 697–707 (2004).
21. A. D. Ioffe and V. M. Tikhomirov, *Theory of Extremal Problems* (Nauka, Moscow, 1974) [in Russian].
22. E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," System and Control Letters **13** (2), 117–123 (1989).
23. E. B. Lee and L. Markus, *Foundations of Optimal Control Theory* (Wiley, New York, 1967; Nauka, Moscow, 1972).
24. M. I. Gusev, "On the reachibility problem under phase constraints," in *Proceedings of the 12th All-Russia Meeting on Control Problems, Moscow, Russia, 2014* (Inst. Probl. Upravl., Moscow, 2014), pp. 610–621.

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